This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.
+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.
+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.
+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

101
A64

[redacted]

[redacted]

PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY.

VOL. XXXI.

APRIL, 1899, TO DECEMBER, 1899.

LONDON:
FRANCIS HODGSON, 89 FARRINGDON STREET, E.C.

1900.

LIBRARY OF THE
LELAND STANFORD JR. UNIVERSITY.

a. 46507

LONDON:
PRINTED BY C. F. HODGSON & SON,
2 NEWTON STREET, HIGH HOLBORN, W.C.

OCT 30 1900

PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY.

VOL. XXXI.

THIRTY-FIFTH SESSION, 1898–99

(since the Formation of the Society, January 16th, 1865).

Thursday, April 13th, 1899.

Lt.-Col. A. J. C. CUNNINGHAM, R.E., Vice-President, in the Chair.

Ten members present.

The following were elected members of the Society:—Prof. Benjamin F. Finkel, Drury College, Springfield, Missouri, U.S.A.; Henry Thomas Kelsey, M.A. Camb., The Grammar School, Leeds; Prof. Edgar Odell Lovett, M.A., Ph.D., University of Virginia, U.S.A.; Arthur Lionel Pedder, M.A., Fellow and Tutor, Magdalen College, Oxford; and Arthur John Wade-Gery, B.A., Assistant Lecturer in Mathematics, University College, Cardiff.

The Chairman briefly alluded to the recent decease of Prof. Sophus Lie, who was elected an honorary member of the Society, May 9th, 1878.

Mr. Kempe having taken the Chair, Lt.-Col. Cunningham read a note on "Conformal Division." The Chairman, Major MacMahon, and Messrs. Lawrence and Western took part in a discussion of the Note.

The following papers were communicated in abstract:—

Note on the Characteristic Invariants of an Asymmetric Optical System: T. J. I'A. Bromwich.

Concerning the Four Known Simple Linear Groups of Order 25920, with an Introduction to the Hyper-Abelian Linear Groups: Dr. L. E. Dickson.

(1) On the Direct Determination of Stress in an Elastic Solid, with application to the Theory of Plates; (2) On the Stress in a Rotating Lamina; (3) The Uniform Torsion and Flexure of Incomplete Tores, with application to Helical Springs Mr. J. H. Michell.

The Theorem of Residuation, Noether's Theorem, and the Riemann-Roch Theorem: Dr. F. S. Macaulay.

Impromptu communications were made by Messrs. Hargreaves, Heppel, Roseveare, Western, and Lt.-Col. Cunningham.

The following is Mr. Roseveare's communication:—

Notes on an Elementary Proposition in Geometrical Conics.

1. The following proposition is an easy deduction from Taylor's proof of the diameter property of a general conic:—If O is the middle point of a chord of any conic, and OG is drawn at right angles to the chord, meeting the axis at G, and if OI is perpendicular to the directrix, then $SG = e^2 . OI$."

2. Hence, if N is the projection of O on the axis, and if the suffixes 1, 2 refer to two chords, $G_1G_2 = e^2 . N_1N_2$. Now, if the direction of the axis is known, N_1N_2 is known; and, if three points on the conic are given, the relation $G_1G_2 : G_2G_3 :: N_1N_2 : N_2N_3$, *i.e.*, in a constant ratio, enables us to draw the axis.

3. Hence, if 1, 2, 3, 4, 5 are points on a conic, and a, b, c are the centres of the circles 123, 124, 125, it can be proved that $ab : bc$ in the ratio of the projections of the chords 34, 45 on the line which is inclined to the axis at the same angle as the line abc.

4. Hence, when five points are given, the direction of the axis of the conic through them can be found by an easy geometrical construction; and the axis, focus, &c., follow without difficulty.

The following presents were made to the Library:—

Heusch, F. de.—"Cours d'Analyse," "Calcul Differentiel," 8vo; Bruxelles, 1898.

"The Nautical Almanac for 1902," 8vo; Edinburgh, 1899.

"Comptes Rendus et Mémoires de la Société des Naturalistes à l'Université Impériale de Varsovie" [(Russian.) Biology, 1897; Physics and Chemistry, 1898]; Warsaw, 1898.

Frolov, M.—"La Théorie des Parallèles démontrée rigoureusement," 8vo; Paris, &c., 1899.

"Mittheilungen der Mathematischen Gesellschaft," Bd. III., Heft 9; Hamburg.

"Periodico di Matematica," Serie 2, Vol. I., Fasc. 5; "Supplemento al Periodico di Matematica," Anno 2, Fasc. 5, 6; Livorno, 1899.

"Wiadomosci Matematyczne," Tom III., Zeszyt 1, 2; Warsaw, 1899.

"Memorias de la Real Academia de Ciencias," "Sobre o Desenvolvimento das Funcções en Série," F. Gomes Teixeira, Tomo XVIII., P. 1; Madrid.

"Memoirs of the National Academy of Sciences," Vol. VIII.; Washington, 1898.

Queen's College, Galway, "Calendar, 1898–1899," 8vo; Dublin, 1899.

"Educational Times," April, 1899.

"Indian Engineering," Vol. XXV., Nos. 7–11, Feb. 18—March 18, 1899.

The following off-prints have been presented to the Library by Dr. L. E. Dickson:—

"On the Inscription of Regular Polygons" (*Annals of Mathematics*, 1894).

"A Quadratic Cremona Transformation defined by a Conic" (*Rendiconti del Circolo Matematico di Palermo*, July, 1895; the *American Mathematical Monthly*, 1895).

"Analytic Functions suitable to represent Substitutions" (*American Journal of Mathematics*, Vol. XVIII., 3; 1895).

From the *Bulletin of the American Mathematical Society*:—

"Systems of Continuous and Discontinuous Simple Groups" (Vol. III., pp. 265–273).

"Higher Irreducible Congruences" (Vol. III., pp. 381–389).

"Orthogonal Group in a Galois Field" (Vol. IV., pp. 196–200).

"Systems of Simple Groups derived from the Orthogonal Group" (Vol. IV., pp. 382–389).

"The Structure of the Hypo-Abelian Groups" (Vol. IV., pp. 495–510).

"Concerning a Linear Homogeneous Group in $C_{m,q}$ Variables Isomorphic to the General Linear Homogeneous Group in m Variables" (Vol. V., pp. 120–135).

"A Triply Infinite System of Simple Groups" (*Quarterly Journal of Mathematics*, No. 114, pp. 169–178).

"The First Hypo-Abelian Group Generalized" (*Quarterly Journal of Mathematics*, No. 117, pp. 1–16).

"The Analytic Representation of Substitutions on a Power of a Prime Number of Letters, with a Discussion of the Linear Group," with corrections (*Annals of Mathematics*, 1897, pp. 65–143).

"The Quadratic Cremona Transformation" (*Proceedings of California Academy of Sciences*, February, 1898—with this are stitched up "On Rational Quadratic Transformations," and "On Curvilinear Asymptotes," M. W. Haskell).

"Systems of Simple Groups derived from the Orthogonal Group" (*Proceedings of California Academy of Sciences*, March, 1898).

The following exchanges were received:—

"Proceedings of the Royal Society," Vol. LXIV., Nos. 409, 410: 1899.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XXIII., St. 3; Leipzig, 1899.

"Rendiconti del Circolo Matematico di Palermo," Tomo XIII., Fasc. 1, 2; 1899.

"Bulletin of the American Mathematical Society," 2nd Series, Vol. V., No. 6; New York, March, 1899.

"Monatshefte für Mathematik und Physik," Jahrgang X., Pt. 2; Wien, 1898.

"Bulletin des Sciences Mathématiques," Tome XXIII., Février, 1899; Paris.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Serie 3, Vol. V., Fasc. 2, 3; Napoli, 1899.

"Atti della reale Accademia dei Lincei—Rendiconti," Sem. 1, Vol. VIII., Fasc. 4, 5; Roma, 1899.

"Berichte über die Verhandlungen der Königl. Sächs. Gesellschaft der Wissenschaften zu Leipzig," I., 1899.

"Nyt Tidsskrift for Matematik," A, Aargang IX., Nr. 6–8; Copenhagen, 1898.

"Jahrbuch über die Fortschritte der Mathematik," Bd. XXVII., Jahrgang, 1896, Heft 3; Berlin, 1899.

Note on the Characteristic Invariants of an Asymmetric Optical System. *By* T. J. I'A. Bromwich.

Received March 15th, 1899. Communicated April 13th, 1899. Received, in revised form, May 17th, 1899.

1. The object of this note is to apply the method of the characteristic function, as treated by Maxwell and Larmor,* to deduce the invariants of an asymmetric optical system. These have been very fully worked out by Prof. Sampson in a recent communication to the Society,† using a continuation of methods due to Gauss.

The direct application of the usual form for the reduced path from one point to another leads to very complicated algebra in calculating the invariants, and it appears advantageous to use a modified form. The form I adopt is found by applying the Hamiltonian method of reciprocation, as employed in the general equations of dynamics; or, rather, Routh's modified form of the transformation, in which some of the coordinates are not transformed. The following is a short account of the theory.

---

* *Proc. Lond. Math. Soc.*, Vols. IV., XX., XXIII.

† *Ibid.*, Vol. XXIX.

2. Suppose V represents the reduced path from a point (x, y, z) in a medium of refractive index μ_1 to (x_2, y_2, z_2) in a second medium of index μ_2. By the ordinary theory of the characteristic function the direction-cosines of the ray from the first point to the second are given by

$$\mu_1 l_1 = -\frac{\partial V}{\partial x_1}, \quad \mu_2 l_2 = +\frac{\partial V}{\partial x_2};$$

$$\mu_1 m_1 = -\frac{\partial V}{\partial y_1}, \quad \mu_2 m_2 = +\frac{\partial V}{\partial y_2};$$

$$\mu_1 n_1 = -\frac{\partial V}{\partial z_1}, \quad \mu_2 n_2 = +\frac{\partial V}{\partial z_2}.$$

It will be convenient to use the notation

$$(\alpha, \beta, \gamma) = \mu\,(l, m, n),$$

suffixes being attached as required. Then let

$$U = (\alpha_2 x_2 + \beta_2 y_2) - (\alpha_1 x_1 + \beta_1 y_1) - V,$$

and U is to be expressed in terms of $\alpha_1 \beta_1 z_1$, $\alpha_2 \beta_2 z_2$. Forming the complete differential of U, we see that

$$dU = (x_2 d\alpha_2 + y_2 d\beta_2) - (x_1 d\alpha_1 + y_1 d\beta_1) - \left(\frac{\partial V}{\partial z_1} dz_1 + \frac{\partial V}{\partial z_2} dz_2\right),$$

the coefficients of dx_1, dy_1, dx_2, dy_2 being all zero by definition of α_1, β_1, α_2, β_2. Hence we find the equations

$$x_1 = -\frac{\partial U}{\partial \alpha_1}, \quad x_2 = +\frac{\partial U}{\partial \alpha_2};$$

$$y_1 = -\frac{\partial U}{\partial \beta_1}, \quad y_2 = +\frac{\partial U}{\partial \beta_2};$$

$$\gamma_1 = +\frac{\partial U}{\partial z_1}, \quad \gamma_2 = -\frac{\partial U}{\partial z_2}.$$

To ensure perfect symmetry we should have used the complete Hamiltonian transformation, but the above form has been employed with a view to the special application required in this paper.

We can at once calculate the effect of a change of origins, provided that the new points are not outside the media μ_1, μ_2. If this condition is satisfied, the directions of the ray are not affected by the change; so, supposing z_1, z_2 changed to $z_1 + \zeta_1$, $z_2 + \zeta_2$, respectively, we shall have that the new value of U is

$$U + \gamma_1 \zeta_1 - \gamma_2 \zeta_2.$$

This is obtained by applying Taylor's theorem and observing that γ_1, γ_2 are independent of z_1, z_2.

3. I now proceed to the consideration of the special case of thin pencils; it is known that, if each z be measured along the central ray of the pencil, the approximate value of V may be arranged in the form

$$V = V_0 + V_2,$$

where suffixes indicate the order of the terms in x_1, y_1, x_2, y_2, and the coefficients will be functions of z_1, z_2; the terms rejected contain cubes and higher powers of x_1, y_1, x_2, y_2.

It is now seen that $$U = -V_0 + V_2,$$

when expressed in terms of $\alpha_1, \beta_1, z_1, \alpha_2, \beta_2, z_2$; for, by Euler's theorem,

$$x_1 \frac{\partial V}{\partial x_1} + y_1 \frac{\partial V}{\partial y_1} + x_2 \frac{\partial V}{\partial x_2} + y_2 \frac{\partial V}{\partial y_2} = 2V_2,$$

to our degree of approximation. Also the terms of most importance in determining the shape of the pencil are the quadratic terms, and for these

$$U_2 = V_2.$$

Next, we have $$\gamma_1^2 = \mu_1^2 - (\alpha_1^2 + \beta_1^2),$$

and thus $$\gamma_1 = \mu_1 - \tfrac{1}{2} (\alpha_1^2 + \beta_1^2)/\mu_1,$$

approximately. Similarly,

$$\gamma_2 = \mu_2 - \tfrac{1}{2} (\alpha_2^2 + \beta_2^2)/\mu_2.$$

Hence the change in U_2 due to moving the origins is

$$\tfrac{1}{2} \left[(\alpha_2^2 + \beta_2^2)\, \zeta_2/\mu_2 - (\alpha_1^2 + \beta_1^2)\, \zeta_1/\mu_1 \right].$$

Comparing this expression with the complicated forms given by Larmor* for the transformation of V_2 from one pair of origins to a second pair, it will be seen that we shall be able to detect the invariants of U_2 more easily than those of V_2. Further, the invariants of U_2 will be found to present themselves more naturally than those of Prof. Sampson's scheme of coefficients, which expresses $x_1, y_1, \alpha_1, \beta_1$ in terms of $x_2, y_2, \alpha_2, \beta_2$.

On the other hand, it must be said that in my experience it is usually easier to calculate Prof. Sampson's scheme for a given optical system than to find either V_2 or U_2. In fact, I had used this as the best method of solving certain types of optical questions for some time before the publication of Prof. Sampson's paper.

---

* *Loc. cit. supra.*

4. Another argument for the use of U_2 is to be found in the fact that the coefficients of V_2 will become infinite for the values $z_1 = 0$, $z_2 = 0$ if the origins be conjugate points of the optical system; that is, if a pencil proceeding from one origin have one of its focal lines at the other origin after passing through the system. Thus the approximations involved in the expression for V would be untenable; and such a pair of origins cannot be used in that analysis. The truth of this statement may be seen from Larmor's paper.* Now it is in some cases convenient to use origins of this nature; and this was the reason that originally led to my performing the Hamiltonian transformation.

To illustrate the behaviour of U_2, V_2 in this case we may calculate them for direct incidence on a symmetrical instrument. By the ordinary theory we have

$$-\mu_1 x_1 = u_1 a_1 + f_1 a_2, \quad -\mu_1 y_1 = u_1 \beta_1 + f_1 \beta_2,$$

$$+\mu_2 x_2 = f_2 a_1 + u_2 a_2, \quad +\mu_2 y_2 = f_2 \beta_1 + u_2 \beta_2,$$

where f_1, f_2 are the focal lengths of the instrument (connected by the equation $f_1/\mu_1 = f_2/\mu_2$), and u_1, u_2 are the distances of the origins from the corresponding principal focal planes, to be taken positive when the origins are outside these planes. We then have

$$2U_2 = (u_1/\mu_1)\, a_1^2 + (f_1/\mu_1 + f_2/\mu_2)\, a_1 a_2 + (u_2/\mu_2)\, a_2^2$$

$$+ \text{a symmetrical term in } \beta,$$

$$2V_2 = \left[(\mu_1 u_2)\, x_1^2 + (\mu_1 f_2 + \mu_2 f_1)\, x_1 x_2 + (\mu_2 u_1)\, x_2\right] / (u_1 u_2 - f_1 f_2)$$

$$+ \text{a symmetrical term in } y,$$

by using the theorems

$$2U_2 = 2V_2 = (a_2 x_2 + \beta_2 y_2) - (a_1 x_1 + \beta_1 y_1).$$

Examining these expressions, it is clear that V_2 will be useless when $u_1 u_2 = f_1 f_2$, or when the origins are conjugate foci of the system. On the other hand, U_2 may always be employed unless f_1, f_2 be infinite, or u_1, u_2 be infinite; in these cases the instrument will be telescopic (that is, parallel incident rays will be parallel at emergence) and the general theory must be modified to some extent.

* *Cf.* Sections 10 and 14 in his paper (Vol. xx.).

5. We proceed to consider some deductions from the form of U_2. The most general form of a quaternary quadric is

$$2U_2 = a_1\alpha_1^2 + 2c_1\alpha_1\beta_1 + b_1\beta_1^2 + a_2\alpha_2^2 + 2c_2\alpha_2\beta_2 + b_2\beta_2^2$$
$$+ 2p\alpha_1\alpha_2 + 2q\alpha_1\beta_2 + 2r\alpha_2\beta_1 + 2s\beta_1\beta_2.$$

Applying the theorem proved above (Section 2), we have

$$\begin{aligned} -x_1 &= a_1\alpha_1 + c_1\beta_1 + p\alpha_2 + q\beta_2, \\ -y_1 &= c_1\alpha_1 + b_1\beta_1 + r\alpha_2 + s\beta_2, \\ x_2 &= p\alpha_1 + r\beta_1 + a_2\alpha_2 + c_2\beta_2, \\ y_2 &= q\alpha_1 + s\beta_1 + c_2\alpha_2 + b_2\beta_2, \end{aligned}$$

from which we can at once express the coefficients of Prof. Sampson's scheme by solving for x_1, y_1, α_1, β_1 in terms of x_2, y_2, α_2, β_2. We see that there will be sixteen coefficients; but, as these are all expressed by the aid of ten independent constants, there must be six relations amongst the sixteen. Prof. Sampson gives twelve, and shows that the second six can be deduced from the first six. I have verified these relations in the general case, but do not reproduce the work, as the algebra is straightforward but tedious; and, further, it is sufficient to verify them for the simplest canonical form of U_2, since Prof. Sampson has shown that his relations are invariant for all axes.*

6. *Invariants of U_2.*

By the ordinary theory of quadratics, the quantities

$$a_1 + b_1, \quad a_1 b_1 - c_1^2$$

are invariants for all rotations of the first set of axes about the axis of z_1. Further, by what was proved in Section 3, a change of origin will not affect the quantities

$$a_1 - b_1, \quad c_1.$$

Thus $$(a_1 - b_1)^2 + 4c_1^2 = (a_1 + b_1)^2 - 4(a_1 b_1 - c_1^2) \qquad \text{(i.)}$$

is invariant for all axes.

---

* It is perhaps worthy of comment that, if Prof. Sampson's relations be written in the Jacobian form (given in Section 7 *infra*), they follow, from the six conditions of the type $\frac{\partial x_1}{\partial \beta_1} = \frac{\partial y_1}{\partial \alpha_1}$, by direct transformation of the differential coefficients to new independent variables. We infer that, written in this form, they would be true for a beam of finite dimensions, not merely for a thin pencil.

[redacted]

Similarly, we find the invariant

$$(a_2-b_2)^2+4c_2^2=(a_2+b_2)^2-4\,(a_2b_2-c_2^2). \tag{ii.}$$

Next, the terms in p, q, r, s may be written

$$(p\alpha_1+r\beta_1)\,\alpha_2+(q\alpha_1+s\beta_1)\,\beta_2,$$

and thus for all displacements of the second set of axes we have the invariant

$$(p\alpha_1+r\beta_1)^2+(q\alpha_1+s\beta_1)^2=(p^2+q^2)\,\alpha_1^2+2\,(pr+qs)\,\alpha_1\beta_1+(r^2+s^2)\,\beta_1^2.$$

From this we get two absolute invariants

$$p^2+q^2+r^2+s^2 \tag{iii.}$$

and $$(p^2+q^2)(r^2+s^2)-(pr+qs)^2=(ps-qr)^2.$$

Instead of the second of these we use its square root

$$ps-qr. \tag{iv.}$$

Finally, combining the two quadratic forms

$$a_1\alpha_1^2+2c_1\alpha_1\beta_1+b_1\beta_1^2,$$

$$(p^2+q^2)\,\alpha_1^2+2\,(pr+qs)\,\alpha_1\beta_1+(r^2+s^2)\,\beta_1^2,$$

we find that their mutual invariant is reducible to the type

$$(a_1-b_1)(p^2+q^2-r^2-s^2)+4c_1\,(pr+qs) \tag{v.}$$

or to $$(a_1-b_1)(pr+qs)-c_1\,(p^2+q^2-r^2-s^2). \tag{vi.}$$

Neither of the forms (v.), (vi.) is the ordinary harmonic invariant of the two quadratic forms, but their relations to it can be easily put down. Calling the harmonic invariant S, we have

$$S=a_1\,(r^2+s^2)+b_1\,(p^2+q^2)-2c_1\,(pr+qs),$$

and it will be seen that

$$(\text{v.})=(a_1+b_1)(p^2+q^2+r^2+s^2)-2S,$$

and $$(\text{v.})^2+4\,(\text{vi.})^2=(\text{i.})\times\left[(\text{iii.})^2-4\,(\text{iv.})^2\right].$$

The reason for selecting (v.) or (vi.) in preference to S is that S is not invariant for displacements of the origins; only for rotations of the axes about Oz_1, Oz_2.

In the same way as (v.) and (vi.), we get the invariants

$$(a_2-b_2)(p^2+r^2-q^2-s^2)+4c_2(pq+rs), \tag{vii.}$$

$$(a_2-b_2)(pq+rs)-c_2(p^2+r^2-q^2-s^2). \tag{viii.}$$

These satisfy the relation

$$(\text{vii.})^2+4\,(\text{viii.})^2 = (\text{ii.})\times\left[(\text{iii.})^2-4\,(\text{iv.})^2\right].$$

7. We have thus found six independent invariants of U_2, and this is a complete set for the transformations to which we are restricted. Prof. Sampson has shown that there are six, and only six, invariants of his optical scheme; and the same argument may be applied here. We have ten constants in U_2 and four degrees of freedom for the axes, namely, displacements along and rotations about the two axes of z_1, z_2. Hence there must be six, and only six, fundamental constants of an optical system for thin pencils.

It will be now easy to find the relations between the invariants of Prof. Sampson's scheme and those discussed in the last section. To do this, let our axes be chosen so as to make some of the coefficients zero; by rotating the axes we can make $q=0$, $r=0$, and, in addition, it will be convenient to make $Q=0$, $R=0$ if possible; we use the notation Q, R for the minors of q, r in Δ, the discriminant of U_2. After putting $q=0$, $r=0$, we find

$$Q = pb_1c_2+sa_2c_1,$$

$$R = pb_2c_1+sa_1c_2.$$

Now shifting the origins distances $\mu_1\rho_1$, $\mu_2\rho_2$ along the axes of z_1, z_2 away from the instrument will change a_1, b_1, a_2, b_2 to the quantities $a_1+\rho_1$, $b_1+\rho_1$, $a_2+\rho_2$, $b_2+\rho_2$; so that it is usually possible to select the origins in such a way as to give $Q=0$, $R=0$. Exceptional cases will arise if $c_1=0$, $c_2=0$, when the conditions are satisfied for all positions of the origins; and if $p^2=s^2$, when the conditions cannot be satisfied unless $(a_1-b_1)/c_1 = (a_2-b_2)/c_2$. Excluding these cases, we deduce from our expressions for x_1, y_1, x_2, y_2 in terms of α_1, β_1, α_2, β_2 the following scheme, of the type given by Prof. Sampson (p. 66, *loc. cit.*):—

$$\begin{aligned}
x_1 &= -(a_1/p)\,x_2+(a_1a_2/p+c_1c_2/s-p)\,\alpha_2-(c_1/s)\,y_2,\\
\alpha_1 &= (1/p)\,x_2-(a_2/p)\,\alpha_2 \qquad\qquad\qquad\qquad -(c_2/p)\,\beta_2,\\
y_1 &= -(c_1/p)\,x_2 \qquad\qquad\qquad\qquad -(b_1/s)\,y_2+(b_1b_2/s+c_1c_2/p-s)\,\beta_2,\\
\beta_1 &= \qquad\qquad -(c_2/s)\,\alpha_2 \qquad +(1/s)\,y_2-(b_2/s)\,\beta_2.
\end{aligned}$$

The determination of the origins by the conditions $Q = 0$, $R = 0$ will ensure the vanishing of the coefficients of β_2 in x_1, and of α_2 in y_1; the coefficients of y_2 in α_1, and of x_2 in β_1, are zero in consequence of $q = 0$, $r = 0$.

We can immediately verify the twelve relations amongst the coefficients given by Prof. Sampson, contained in equations (1) ... (6), (1′) ... (6′) of his paper. Three typical equations are

$$\frac{\partial (x_1, \alpha_1)}{\partial (x_2, \alpha_2)} + \frac{\partial (y_1, \beta_1)}{\partial (x_2, \alpha_2)} = 1, \tag{1}$$

$$\frac{\partial (x_1, \alpha_1)}{\partial (x_2, y_2)} + \frac{\partial (y_1, \beta_1)}{\partial (x_2, y_2)} = 0, \tag{2}$$

$$\frac{\partial (x_1, \alpha_1)}{\partial (\alpha_2, \beta_2)} + \frac{\partial (y_1, \beta_1)}{\partial (\alpha_2, \beta_2)} = 0. \tag{5}$$

Of these the first two follow at once from the expressions above; and the third is verified on observing that $a_1 a_2/p^2 = b_1 b_2/s^2$ from the conditions $Q = 0$, $R = 0$. As remarked before, the relations written in these forms will be true for a beam of any size, not only for a thin pencil.

8. In the last section we obtained a form of U_2 which contains six independent constants; or, rather, eight, with the two relations $Q = 0$, $R = 0$. This form may be called canonical, and it will be convenient to allude to it by the letter (A) in order to distinguish it from another canonical form to be used later.

I find that, with this scheme, the chief invariants of Prof. Sampson's notation are

$$(a) = \frac{1}{p^2} + \frac{1}{s^2}, \qquad (d) = c_1\left(\frac{1}{s^2} - \frac{1}{p^2}\right),$$

$$(b) = \frac{1}{ps}, \qquad (f) = c_2\left(\frac{1}{s^2} - \frac{1}{p^2}\right),$$

$$(ac') = \frac{a_1 - b_1}{ps}\left(\frac{1}{s^2} - \frac{1}{p^2}\right), \qquad (af') = \frac{a_2 - b_2}{ps}\left(\frac{1}{p^2} - \frac{1}{s^2}\right),$$

$$(D) = \frac{(a_1 - b_1)^2}{p^2 s^2} + c_1^2\left(\frac{1}{p^2} + \frac{1}{s^2}\right)^2, \quad (F) = \frac{(a_2 - b_2)^2}{p^2 s^2} + c_2^2\left(\frac{1}{p^2} + \frac{1}{s^2}\right).$$

Further, in my notation,

$$\text{(i.)} = (a_1-b_1)^2+4c_1^2, \qquad \text{(ii.)} = (a_2-b_2)^2+4c_2^2,$$

$$\text{(iii.)} = p^2+s^2, \qquad \text{(iv.)} = ps,$$

$$\text{(v.)} = (a_1-b_1)(p^2-s^2), \qquad \text{(vi.)} = -c_1(p^2-s^2),$$

$$\text{(vii.)} = (a_2-b_2)(p^2-s^2), \qquad \text{(viii.)} = -c_2(p^2-s^2).$$

Thus we have

$$(a) = \text{(iii.)}/\text{(iv.)}^2, \qquad (b) = 1/\text{(iv.)},$$

$$(d) = -\text{(vi.)}/\text{(iv.)}^2, \qquad (f) = -\text{(viii.)}/\text{(iv.)}^2,$$

$$(ac') = \text{(v.)}/\text{(iv.)}^3, \qquad (af') = -\text{(vii.)}/\text{(iv.)}^3,$$

$$(D) = (d)^2+\text{(i.)}/\text{(iv.)}^2, \qquad (F) = (f^2)+\text{(ii.)}/\text{(iv.)}^2.$$

The other invariants given by Prof. Sampson can be expressed in terms of these; and it is thus unnecessary to write out the complete list. The only one presenting any difficulty is (K), which may be shown to be

$$c_1 c_2 ps\left(\frac{a_1 a_2}{p^2}+\frac{c_1 c_2}{ps}-1\right)\left(\frac{1}{p^2}-\frac{1}{s^2}\right)^2,$$

the apparent want of symmetry of this being accounted for by the relation $a_1 a_2/p^2 = b_1 b_2/s^2$. The expression for (K) in terms of the other invariants is given by Prof. Sampson, but need not be written out here, as it is rather lengthy.

It will be seen that, if $(d) = 0$, and $(f) = 0$ in Prof. Sampson's notation; or if (vi.) $= 0$, (viii.) $= 0$ in the invariants above, we are brought to the exceptional cases of $c_1 = 0$, $c_2 = 0$, or $p = s$. If $p = s$, the reduction to the form (A) is not possible unless

$$(a_1-b_1)/c_1 = (a_2-b_2)/c_2.$$

9. Another canonical form, denoted by (B), is found by taking as the origin on each side of the system one of the points conjugate to infinity. This has some advantages, as it contains only six coefficients, which are thus all invariants. This reduction will be always possible unless the system is *quasi*-telescopic; that is, unless rays incident parallel to the central ray emerge parallel to the emergent central ray.

With these origins we shall have that, if $x_1 = 0$, $y_1 = 0$, then $\alpha_2 = 0$, $\beta_2 = 0$, provided a suitable relation be taken between α_1, β_1. Hence the equations

$$a_1\alpha_1 + c_1\beta_1 = 0,$$

$$a_1\alpha_1 + b_1\beta_1 = 0$$

will hold for the same ratio $\alpha_1 : \beta_1$, that is

$$a_1 b_1 - c_1^2 = 0,$$

But by rotation of axes we can make $c_1 = 0$, and hence one of the two a_1, b_1 will be zero, say b_1.

In the same way we may have

$$b_2 = 0, \quad c_2 = 0.$$

Thus the canonical form is given by

$$2U_2 = a_1\alpha_1^2 + a_2\alpha_2^2 + 2\,(p\alpha_1\alpha_2 + q\alpha_1\beta_2 + r\alpha_2\beta_1 + s\beta_1\beta_2).$$

The second focal line of an incident pencil whose rays will emerge parallel to the central ray may be easily found. For let this be $z_1 = -\mu_1\rho_1$, so that ρ_1 is positive when measured away from the instrument. Then we are to have that $x_1 = \rho_1\alpha_1$, $y_1 = \rho_1\beta_1$ give $\alpha_2 = 0$, $\beta_2 = 0$ provided a suitable relation between α_1, β_1 be chosen. We find at once that $a_1 + \rho_1 = 0$. In the same way the second focal line of an emergent pencil, whose rays are incident parallel to the central ray, is given by $a_2 + \rho_2 = 0$.

10. With the canonical form (B), we have invariants

$$\text{(i.)} = a_1^2, \qquad \text{(ii.)} = a_2^2,$$

$$\text{(v.)} = a_1\,(p^2+q^2-r^2-s^2), \qquad \text{(vi.)} = a_1\,(pr+qs),$$

$$\text{(vii.)} = a_2\,(p^2+r^2-q^2-s^2), \qquad \text{(viii.)} = a_2\,(pq+rs),$$

(iii.) and (iv.) are, of course, unaltered from their values in the general case.

Hence we have a geometrical interpretation of the invariants (i.) and (ii.); in fact (i.)$\times\mu_1^2$ is the square of the distance between the first principal focal lines of the instrument; or between the pair of lines from which a pencil must proceed in order to emerge as a parallel beam. Similarly we interpret (ii.)$\times\mu_2^2$.

With this arrangement of axes the relation between conjugate foci takes a simple form. Suppose that a pencil from the point $z_1 = -\mu_1\rho_1$ proceeds on emergence from a focal line at $z_2 = \mu_2\rho_2$; then we shall have $x_1 = \rho_1\alpha_1$, $y_1 = \rho_1\beta_1$, $x_2 = -\rho_2\alpha_2$, $y_2 = -\rho_2\beta_2$ for the same ratios

$$\alpha_1 : \beta_1 : \alpha_2 : \beta_2,$$

or

$$\begin{vmatrix} a_1+\rho_1 & 0 & p & \\ 0 & \rho_1 & r & s \\ p & r & a_2+\rho_2 & 0 \\ q & s & 0 & \rho_2 \end{vmatrix} = 0.$$

This gives on expansion

$$\rho_1\rho_2(a_1+\rho_1)(a_2+\rho_2)-p^2\rho_1\rho_2-q^2\rho_1(a_2+\rho_2)-r^2\rho_2(a_1+\rho_1)$$
$$-s^2(a_1+\rho_1)(a_2+\rho_2)+(ps-qr)^2=0.$$

Denoting by F, G the points where the first principal focal lines meet the central ray; and by F', G' similar points for the second focal lines, we shall have, if P, Q be the points ρ_1, ρ_2,

$$PF=\mu_1\rho_1,\quad PG=\mu_1(a_1+\rho_1),\quad QF'=\mu_2\rho_2,\quad QG'=\mu_2(a_2+\rho_2).$$

Hence the relation just written becomes

$$PF.PG.QF'.QG'$$
$$-\mu_1\mu_2\left[p^2(PF.QF')+q^2(PF.QG')+r^2(PG'.QF')+s^2(PG.QG')\right]$$
$$+(\mu_1\mu_2)^2(ps-qr)^2=0.$$

Of course all the coefficients involved may be readily put down in terms of invariants if required; but the expressions are not simple, excepting the one $(ps-qr)$ which is the invariant (iv.).*

11. It may be remarked in connexion with the classification of optical systems given by Prof. Sampson at the end of his paper that two conditions are requisite for his class (iv.), namely, $(d)=0$ and $(f)=0$ independently; † both conditions being necessary in order that this class may possess the property stated by Prof. Sampson.

This property is that the projections of a ray on two suitably chosen coordinate planes may be treated as if refracted independently of each other. Investigating this we find the two conditions $c_1=0$, $c_2=0$ in scheme (A) and $q=0$, $r=0$ in scheme (B); and then the two canonical reductions are virtually the same.

It seems possible that Prof. Sampson was led to suppose that (d) could not vanish without (f) by examining his scheme of simplified

* Consider a cone of rays from P; the emergent rays will mark out a certain area on a plane at any point Q, drawn at right angles to the central ray. We may then define d, the *apparent distance* of Q as seen from P, by the equation

$$d^2=(\text{area at }Q)\,/\,(\text{solid angle of cone at }P).$$

It is easy to show that the left-hand side of the last equation, beginning with $PF.PG.QF'.QG'$ is equal to $\mu_2^2 d^2(ps-qr)$.

† Prof. Sampson states, without proof, that the invariant (d) cannot vanish in systems of finite curvatures without making (f) zero as well.

invariants (p. 67 of his paper). Here $(d) = 0$ certainly appears to involve $(f) = 0$; but this is only the case either if $\varpi_1^2 = \varpi_2^2$, or if $k = 0$. Neither of these conditions enables us to satisfy the fundamental property stated above; the true condition is not easily expressed in terms of the coefficients which are there employed; for, if the true condition be satisfied, $k = \infty$, while $k\varpi_1$, $k\varpi_2$ are finite.

The Theorem of Residuation, Noether's Theorem, and the Riemann-Roch Theorem. *By* F. S. Macaulay. Received March 28th, 1899. Read April 13th, 1899.

The following paper is more in the nature of an essay than of a rigorous investigation. Its object is to advance and explain general notions rather than to give incontrovertible proofs of all the statements made. Sections I. and II. contain a discussion of the most general aspect of the Theorem of Residuation, and lead, in Section III., to an analytical and generalized interpretation of results previously deduced geometrically. (*Proc. Lond. Math. Soc.*, Vol. XXIX., pp. 673–695.)

The fundamental idea of the paper is a very simple one, viz., that the whole intersection of two given curves C_l, C_m at a common point A may be resolved into an equivalent aggregate, a, of simple points, no matter how complex the forms of the two curves at A may be. These a points are brought into evidence analytically by the fact that they supply a independent linear equations for the coefficients of a general algebraic curve of sufficiently high order. It is highly probable that the a equations could always be written in such a form, and arranged in such order, that each new equation, interpreted in connexion with those which precede it and apart from those which succeed it, expresses the condition that the curve passes through a new point, or, more strictly, possesses a property equivalent to that of passing through a new point. Such an arrangement is not, however, attempted *per se* in the paper.

The fundamental notion of a complex point being equivalent to an aggregate of simple points is in no sense a novel one; but its very simplicity has been considered as liable to lead to erroneous

deductions. That there is, however, much inherent possibility of usefulness in the idea cannot reasonably be disputed. In particular, the resolution of a complex point into its equivalent simple points affords a means of viewing the theorem of residuation in its most general and extended aspect.

Several lengthy paragraphs of proof or explanation in Section I. have been relegated to footnotes, in order to obscure as little as possible the sequence of ideas.

I. *The General Theorem of Residuation.*

We assume as our starting point the fundamental theorem that if the whole intersection of two given algebraic plane curves C_l, C_m consists of *lm separate* points, then the equation of any other curve C_n which passes through these *lm* points is capable of being written in the form*

$$C_n \equiv C_l S_{n-l} + C_m S_{n-m} = 0.$$

Conversely, any curve whose equation is of this form passes through the *lm* points, as is evident.

If, however, C_l, C_m have a common multiple point at A (with or without contact), then, although they have at A, strictly speaking, but one common point, their whole intersection at A is equivalent to a certain definite number of simple points, which have become absorbed in the single point A. If we imagine an infinitesimal change imposed on the two curves C_l, C_m, by an infinitesimal variation of their coefficients† (including, if desirable, the adding of terms with infinitesimal coefficients beyond those of highest order in C_l, C_m), then the whole intersection of the two curves becomes in general changed to simple and separate points, the infinitesimally displaced curves having nowhere any absolute contact.

If now any curve obtained by a like infinitesimal change of C_n passes through all the separate points, we have at once

$$C_n \equiv C_l S_{n-l} + C_m S_{n-m},$$

since this identity will hold for the infinitesimally displaced curves, by our original theorem.

---

* Noether, *Mathematische Annalen*, Vol. II., p. 314. In order to exclude apparent exceptions to the theorem it should be assumed that all the points of intersection of C_l, C_m are in the finite region. (See footnote, p. 18.)

† The method of infinitesimal variation in the coefficients is employed by Halphen (*l.c.*, footnote, p. 21). The method has been challenged, on insufficient ground, in my estimation, as lacking clearness and rigour.

[redacted]

Hence the two following statements, properly interpreted, are absolutely equivalent, and will be hereafter treated as such:—

(i.) *C_n passes through the whole intersection of C_l, C_m.*

(ii.) *C_n is of the form $C_l S_{n-l} + C_m S_{n-m}$.*

We may also express the equivalence as follows:—

It cannot, in any valid sense, be said that C_n passes through the whole intersection of C_l, C_m unless C_n is of the form $C_l S_{n-l} + C_m S_{n-m}$.

The meaning of statement (i.) is that there exists *one* pair of curves which are infinitesimal displacements of C_l, C_m and intersect wholly in separate points, to which corresponds *a* curve which is an infinitesimal displacement of C_n and passes through all the separate points. This being true of one such displacement of C_l, C_m is true of any, since it at once gives

$$C_n \equiv C_l S_{n-l} + C_m S_{n-m}.$$

The general theorem of residuation is contained, in an undeveloped or embryonic form, in the absolute equivalence of statements (i.) and (ii.) above. Hence an investigation of all the properties involved in the theorem of residuation may be made to rest on an investigation of the properties of the curves which are of the form $C_l S_{n-l} + C_m S_{n-m}$.*

---

* More definitely, it depends on an investigation of the conditions which must be satisfied by three unknown curves S, S', S'' in order that $CS + C'S' + C''S''$ may be identically zero, C', C'' being any two given curves, and C any curve of a given linear system, the conditions for S having to be independent of the arbitrary parameters involved in C (p. 20).

The necessary and sufficient number of independent linear equations that the coefficients of C_n must satisfy in order that C_n may be of the form $C_l S_{n-l} + C_m S_{n-m}$ (*C_l, C_m having no common factor*) is exactly lm if $n \geqslant l+m$, and

$$lm - \tfrac{1}{2}(l+m-n-1)(l+m-n-2)$$

if n is less than $l+m$ but not less than l or m. *This is true no matter how general or specialized the forms of C_l, C_m may be, subject to the condition mentioned*, and is proved in the *Proc. Lond. Math. Soc.*, Vol. XXVI., p. 503. One result of this theorem is that none of the equations among the coefficients of C_n which express the condition that C_n passes through the whole intersection of C_l, C_m are lost by virtue of the fact of any number of the points of intersection of C_l, C_m becoming absorbed in a single point.

In connexion with the above theorem there is, however, a paradox, which, like other geometrical paradoxes, leads to important consequences. If the curve C_n is degenerate, *i.e.*, if $C_n \equiv C_{n'} C_{n''}$, then the total number of independent equations may be very much reduced. The independent linear equations for the coefficients of C_n will not be linear in the coefficients of $C_{n'}$, $C_{n''}$, and will not be independent if C_l, C_m have any common multiple points. The reason is that, if C_l, C_m have a common multiple point at A, the form of their whole intersection at A may be assumed to some extent arbitrarily. If, for example, $C_{n'}$ be subject to the single condition of passing through A, C_l and C_m may be assumed to have i of their simple points of intersection at A on $C_{n'}$, i being the smaller of the two orders of the multiple points of C_l, C_m at A. On the other hand, if another curve pass through these i points, it will be subject to, not 1, but i, conditions, $C_{n'}$ being known. It is to be understood that $C_{n'}$ and $C_{n''}$ are mutually connected.

This method of procedure, simple as it may appear at first sight, has not, so far as I know, been anywhere elaborated. This may be regarded as a justification for attempting the imperfect elaboration which follows.

In the rest of this paper we shall assume that C_l, C_m have no common point at an infinitely great distance from the origin.* If this is not actually the case, we can choose a line, $ax+by=1$, which neither passes through, nor infinitely near to, any common point of C_l, C_m, and then linearly transform C_l, C_m by substituting $x':y':ax'+by'-1$ for $x:y:1$. This ensures that the desired property shall hold for the transformed curves, and we may deal with these in the place of C_l, C_m.

Our next consideration is that of a curve C_n, or rather the general curve $C=\lambda'C'+\lambda''C''+\lambda'''C'''+\ldots$ of a given linear system C', C'', ..., which is not of the form $C_lS_{n-l}+C_mS_{n-m}$. We require an answer to the following question:—What is the number of the lm points of

---

* Two of the many reasons for making this assumption are the following:—

(i.) In varying the coefficients, whereby coincident points are changed to separate points, it is sometimes necessary to add infinitesimal terms to C_l, C_m extending beyond the terms of highest order in C_l, C_m. When this is the case the infinitesimally displaced curves will have a common group of asymptotic points in addition to a common group of lm points corresponding to the whole intersection of C_l, C_m. When none of the lm points are asymptotic, the two groups of points are absolutely distinct, one group being entirely in the finite region, and the other being entirely in the infinite region. If, on the other hand, some of the lm points are asymptotic, the two sets of asymptotic points would have to be dissociated from one another, which might prove a troublesome matter.

(ii.) On p. 677 of Vol. XXIX. of the *Proc. Lond. Math. Soc.* the value found for ρ_p, as the number of arbitrary coefficients in u_p, is only valid if all the N points are in the finite region, a limitation which previously escaped notice. This interpretation of the value of ρ_p constitutes, in our method, the connecting link between the geometrical and analytical formulation of results (pp. 27, 28). Hence, in using this value for ρ_p, without modification, it is essential that none of the N points should escape to an infinite distance.

The disadvantage of the condition that C_l, C_m are to have no common asymptotic points is that the actual labour of any operations connected with them may be thereby materially increased.

The case in which two curves have no common asymptotic points bears a close analogy to the "simple" case of the intersection of two curves at a common multiple point which have no contact of branches. There is, however, this difference, that we know nothing about two given curves in the region of *absolute* infinity (as distinguished from the infinite region), whereas we do know the course of a given curve as it emerges from the infinitely small region surrounding a multiple point.

It is well to note that the n-ic excess of a given point-group N is that number of the N equations supplied by the point-group for a general n-ic which are identically satisfied, owing to the values of the coefficients of the terms of the n-ic beyond the n^{th} order being all zero. In other words, the n-ic excess of a point-group is that number of the points whose effect for an n-ic is lost or nugatory, owing to the fact that the form of any algebraic curve at absolute infinity is indeterminate.

intersection of C_l, C_m which the general curve C of the system may be said to pass through? If C passes through no multiple points common to C_l, C_m, the answer is of course obvious, although the case where C, C_l, C_m have all three contact of a higher order than the first has to be carefully treated. But, if C, C_l, C_m have one or more common multiple points, the answer is not obvious. The answer in this case depends on whether the given linear system C is defined by the geometrical conditions of passing through a given point-base,* or by the analytical conditions of being comprised in a given linear form. For analytical theory an answer may be given as follows:—

The maximum number of (simple) points which the general curve C of a given linear system C', C'', ... may be said to have in common with C_l, C_m is $N = lm - N'$, where N' is the number of independent linear equations which must be satisfied by the coefficients of a general polynomial S, of order not less than $l+m-2$, in order that CS may be of the form $C_l S' + C_m S''$, *i.e.*, in order that $C'S$, $C''S$, ... may one and all be of the form $C_l S' + C_m S''$.†

---

* A point-base of a simple kind is defined in the footnote, *Proc. Lond. Math. Soc.*, Vol. XXIX., p. 676; but in this paper the term is used in its most general meaning. A set of a independent linear equations among the coefficients of a general polynomial S, giving to the curve S a property equivalent to that of passing through a simple points all situated at A, determines a *base-point* at A of *degree* a and *order* equal to the order of the multiple point which the curve S must have at A. *A base-point may thus be defined as a point-group collected at a point, and may have properties just as varied as those of a point-group in respect to order, degree, excess, and defect.* A *point-base* is made up of base-points, its degree being the sum of the degrees of its base-points, and its order being the order of the lowest curve which passes through it. If the base-points are all simple points finitely separated, the point-base is called a point-group.

Examples of point-bases (N, N') occur in this section, and examples of base-points in the next section. Base-points which determine no directions, much less curvatures, beyond those which are inherent in the specification of a higher singularity, may be called *simple* base-points. They are of two kinds:—(i.) the ordinary i-point ($i \geqslant 1$), of degree $\frac{1}{2}i\,(i+1)$ and order i, which gives an ordinary i-fold point to any curve, and (ii.) the k-point of degree $\Sigma\frac{1}{2}i\,(i+1)$ and order k (k being the greatest of the i's), specifying the component i-points of a higher singularity, with the directions and curvatures, &c., which determine the situations of the component i-points relative to the point itself.

† The reasoning by which this conclusion seems to be justified is as follows:—We imagine such infinitesimal changes to be made in C_l, C_m, C', C'', ... that the curves to which C_l, C_m are changed have lm separate points of intersection in the finite region, while the whole set of curves to which C_l, C_m, C', C'', ... are changed have the *greatest possible* number N of these lm separate points in common. It seems probable that the infinitesimal terms to be added to C_l, C_m need not extend beyond the terms of orders l and m; but the truth of this is not evident, and we therefore suppose that C_l, C_m are changed to $C_{l'}$, $C_{m'}$, where $l' \geqslant l$, $m' \geqslant m$. The curves $C_{l'}$, $C_{m'}$ therefore intersect in $l'm' - lm$ points in the infinite region, besides the lm points in the finite region. Taking now C', C'', ... to denote the curves to which the original C', C'', ... are changed, we may suppose, by continuing the

The points common to C_l, C_m through which C does not pass (N' in number) are the points through which S does pass, by virtue of the N' independent linear equations satisfied by the coefficients of S. The theorem of residuation treated analytically thus leads to the extremely difficult problem of determining the most general linear system S, of order not less than $l+m-2$, which satisfies the identity

$$CS \equiv C_l S' + C_m S''$$

for all values of the arbitrary parameters involved in C. Theoretically, however, S is absolutely determinate, and without ambiguity, since the determination depends only on the solution of linear equations. Also, having determined S, we can determine the most general system K, of order not less than $l+m-2$, which satisfies the identity

$$KS \equiv C_l S' + C_m S''$$

for all values of the arbitrary parameters involved in S.* The linear system K constitutes the "complete" system, through N, which contains the given system C. The equations for the coefficients of K will express the fact that K passes through the $N = lm - N'$ points common to C_l, C_m and the system C. If we substitute the coefficients

---

infinitesimal terms of C', C'', ... far enough beyond their original terms of highest order, that C', C'', ... not only all pass through the N points, but also through the *whole* of the $l'm'-lm$ points; for the conditions of their passing through these affect only the coefficients of their terms of higher order, that is, terms with infinitesimal coefficients which may be chosen in any way desirable, these terms extending to an order as high as we please. The conditions that $C'S$, $C''S$, ... should each be of the form $C_{l'} S' + C_{m'} S''$ now only require that S should pass through the remaining $N' = lm - N$ of the lm points. The coefficients of S have then only to satisfy N' conditional equations. These equations are not only independent, but must continue to remain independent when all the infinitesimal parts of the coefficients of $C_{l'}$, $C_{m'}$, C', C'', ... become zero, provided only that S is of sufficiently high order. This number N' is therefore the irreducible minimum of independent equations which must exist for S, *i.e.*, it is the same as the number N' in the text. This proves our theorem.

Also the condition that KS should be of the form $C_{l'} S' + C_{m'} S''$ only requires that K should pass through the N points and the $l'm'-lm$ points. But, assuming K (like C', C'', ...) to have infinitesimal terms proceeding far enough, the conditional equations corresponding to the $l'm'-lm$ points will affect only the infinitesimal coefficients of K, and the only equations among the finite coefficients of K are those supplied by the N points. It is to be noticed that K and S are not mutually connected.

* The number of independent equations for the coefficients of K will be N. This is proved in the last footnote. If K were taken of less order than $l+m-2$, the number of independent equations might be less than N. See, however, the last paragraph of the paper, p. 30.

The fact that only $N = lm - N'$ equations have to be satisfied by the coefficients of K, notwithstanding that S has all but N' of its coefficients arbitrary, is a special property. This property ought to be capable of direct analytical proof; and the same remark applies to properties mentioned later, pp. 24–26.

of any curve of the system C for the coefficients of K in the N equations, we shall obtain a set of N identities among the coefficients of C_l, C_m, C', C'', ... which will not of course be linear in any of these coefficients.

If now we take C_m as base-curve, the two point-bases N, N' (footnote, p. 19) are residual, having C_l for their connecting curve; while N is the point-base of highest degree on C_m through which C_l, C', C'', ... all pass. The general system S through N' cuts C_m for the rest in the whole series of point-bases coresidual to N. So also the general system K through N (which includes the linear system C) cuts C_m for the rest in the whole series of point-bases residual to N. Finally any two curves of the systems K, S through N, N' cut C_m again in two residual point-bases for which S' is the connecting curve. This is the general theorem of residuation on the base-curve C_m.

II. *Noether's Fundamental Theorem.*

Denoting by C_l, C_m, C_n given non-homogeneous polynomials in two variables x, y, of orders l, m, n, and by S, S', &c., unknown polynomials to be chosen as desired, we may enunciate Noether's "fundamental theorem in the theory of Algebraic Functions" as follows:*—

The necessary and sufficient conditions that C_n may be capable of being written in the form

$$C_l S_{n-l} + C_m S_{n-m}$$

are that for each and every point of intersection $x = a$, $y = b$ of the two curves $C_l = 0$, $C_m = 0$ there should exist a curve

$$C_n - C_l S' - C_m S'' = 0$$

which has a t-fold point at $x = a$, $y = b$, the number t having any

---

* The following papers in the *Mathematische Annalen* directly discuss Noether's theorem:—

Vol. VI., 1873, pp. 351–359 (M. Noether); XXVII., 1886, pp. 527–536 (A. Voss); XXX., 1887, pp. 401–409 (L. Stickelberger); pp. 410–417 (Noether); XXXIV., 1889, pp. 447–449 (E. Bertini); pp. 450–453 (Noether); XXXIX., 1891, pp. 129–141 (A. Brill); XL., 1892, pp. 140–144 (Noether); XLII., 1893, pp. 601–604 (H. F. Baker).

One of the most interesting proofs of Noether's theorem is that by M. Halphen in the *Bulletin de la Société Mathématique de France*, Vol. V., 1877, pp. 160–163 (reproduced in Clebsch-Benoist, *Leçons sur la Géométrie*, Vol. II., 1880, pp. 49–51). Halphen, however, assumes a result which appears to require proof. This proof has been supplied by A. Berry in the *Proc. Lond. Math. Soc.*, Vol. XXX., pp. 271–276. (See also second footnote, p. 16.)

integral value not less than a certain minimum, which minimum depends on the character of the whole intersection of the two curves $C_l = 0$, $C_m = 0$ at the point $x = a$, $y = b$. S', S'' may be different for different points.

A short explanation will serve to make the theorem clear. In the first place the conditions of the theorem are obviously *necessary*, no matter how large t may be; for this is at once seen by taking $S' \equiv S_{n-l}$, $S'' \equiv S_{n-m}$. The only question then is as to the *sufficiency* of the conditions.

Consider the simplest example to which the theorem applies; viz., when C_l, C_m are single-branched and do not touch at the common point a, b, so that their whole intersection at a, b consists of one simple point. By taking $t = 1$, it is seen that the conditions of the theorem require that C_n should pass through the point a, b. And the conditions of the theorem require no more than this; for it can be easily proved, by taking the point a, b as origin and the tangents to C_l, C_m as axes of coordinates, that, provided only C_n passes through the origin, S', S'' can be so chosen that the curve

$$C_n - C_l S' - C_m S'' = 0$$

has a multiple point of any desired order (from 1 upwards) at the origin. In this simplest case of all the minimum value of t is therefore unity.

So, in the most general case, however complex the character of the whole intersection of C_l, C_m at the point a, b may be, Noether proves it to be sufficient, in order to know that C_n is of the form $C_l S_{n-l} + C_m S_{n-m}$, that S', S'' can be found such that the curve

$$C_n - C_l S' - C_m S'' = 0$$

has a multiple point of sufficiently high order at a, b, with similar conditions for each point of intersection of the curves C_l, C_m.

The conditions of the theorem require that for every point of intersection of C_l, C_m there should exist two curves S', S'' such that *C_n is the same as $C_l S' + C_m S''$ to any degree of approximation;* and, this being so, the condition is satisfied for *every* point in the plane. Looked at in this light the significance of the theorem is readily comprehended.

The only modifications that have been made in the theorem since Noether first gave it (*l.c.*, Vol. VI.) relate to the determination of the minimum value of t. The knowledge of the minimum is of some interest, but has not yet been proved to be of any essential importance, except in the two cases (ii.) and (iv.) below.

(i.) The number t need not exceed the number, a, of simple points to which the whole intersection of C_l, C_m at a, b is equivalent (Brill, *l.c.*). This is the minimum value of t if, and only if, one at least of the two curves C_l, C_m is single-branched at the point a, b; but in this case simpler conditions can be substituted for those of the theorem, viz., that C_n should have contact of order $a-1$ at a, b with the single-branched curve.

(ii.) If C_l, C_m have respectively i-fold and j-fold points at a, b, and have no contact (so that ij is the number of simple points to which their whole intersection at a, b is equivalent), the minimum value of t is $i+j-1$ (Noether, *l.c.*, Vol. VI.).

This is called the "simple" case.*

(iii.) If C_l, C_m have i-fold and j-fold points at a, b, and their whole intersection at a, b is equivalent to $ij+\beta$ simple points, the minimum value of t does not exceed $i+j+\beta-1$. (Bertini, *l.c.*)

(iv.) If C_l, C_m have multiple points of higher singularity at a, b which can be resolved into ordinary multiple points (including ordinary cusps) common to C_l, C_m, of which any one pair of corresponding components is i-fold for C_l and j-fold for C_m, with the corresponding whole intersection equivalent to ij simple points, then it is sufficient that $C_n - C_l S' - C_m S''$ should have a multiple point of higher singularity at a, b whose corresponding component is of order $i+j-1$. (Noether, *l.c.*, Vol. XXXIV.)

As regards its application to the theorem of residuation Noether's theorem seems open to criticism. Noether possibly did not regard his theorem from this point of view when he first gave it, but subsequently both he and others have so regarded it. It should at least be made clear that the conditions in the theorem supply a *theoretical* rather than a *practical* test; but the direct contrary seems to be implied in much that is written on the subject. Although a great step towards the general theorem of residuation, it does not advance the whole way. The theorem only gives us a test for answering the question whether C_n is of the form $C_l S_{n-l} + C_m S_{n-m}$ or not, whereas, as we have seen in Section I., we want to know the conditions which S must satisfy in order that CS (C being partially or wholly

---

* The "simple" case requires only that the two curves have no contact at the common multiple point. The two multiple points may be of any kind of singularity provided this condition holds.

given) may be of the form $C_l S' + C_m S''$. The latter question includes the former, but the former does not include the latter.

It will be seen that Noether's theorem supplies us with a sufficiency of *local* tests for deciding, in the most general case, whether C_n is of the form $C_l S_{n-l} + C_m S_{n-m}$ or not; or rather, as we should prefer to say, it provides a local test for deciding whether C_n does or does not pass through the whole intersection of C_l, C_m at a common multiple point. The local tests are quite independent of one another; and, if satisfied at every point of intersection of C_l, C_m, the final result follows that C_n is of the form $C_l S_{n-l} + C_m S_{n-m}$. We proceed to explain how much is theoretically required for the satisfying of Noether's tests.

The number of simple points to which the whole intersection of C_l, C_m at a common multiple point is equivalent may be found by taking the multiple point as origin and equating coefficients in the identity

$$C_l S' + C_m S'' \equiv \Sigma,$$

where Σ, S', S'' are general ordinary power series, *i.e.* unterminated series arranged in ascending positive integral powers of x, y. The number of independent linear equations that result for the coefficients of Σ alone is the number of simple points required.*

For the equating of coefficients, write the identity

$$C_l S' + C_m S'' \equiv \Sigma$$

in the form

$$C_l\,(u_0' + u_1' + u_2' + \dots) + C_m\,(u_0'' + u_1'' + u_2'' + \dots) \equiv u_0 + u_1 + u_2 + \dots,$$

where u_p, as usual, stands for a homogeneous polynomial in x, y of order p. Arrange the equations in sets, the p^{th} set coming from the equating of coefficients of terms of order $p-1$. The $(p+1)^{\text{th}}$ set of equations will involve the coefficients of u_p, u'_{p-i}, u''_{p-j} (i, j being the orders of the multiple points of C_l, C_m at the origin) together with the coefficients of Σ, S', S'' which have appeared in the p previous

---

* Assuming that the number of equations, α_1, for the coefficients of Σ does not exceed the number, α, of the simple points of intersection of C_l, C_m which are absorbed at the origin, it can be proved that $\alpha_1 = \alpha$ as follows:—If we substitute for Σ a general polynomial S_n of sufficiently high order $n \geqslant l+m-2$ (S', S'' being still power series), the equations for the coefficients of S_n will be α_1 in number; and, if we add to these all the other equations for S_n corresponding to the other points of intersection of C_l, C_m, we shall obtain a total of $\Sigma\alpha_1$ equations, *i.e.* a number of equations not exceeding $\Sigma\alpha$ or lm. But these equations, in their totality, require S_n to be of the form $C_l S_{n-l} + C_m S_{n-m}$, by Noether's theorem, and are therefore equivalent to lm independent equations (footnote, p. 17). Hence $\Sigma\alpha_1$ is not less than lm, and assuming it is not greater than lm, from above, we have $\Sigma\alpha_1 = lm = \Sigma\alpha$; therefore $\alpha_1 = \alpha$, $\beta_1 = \beta$, &c.

sets of equations. If the $(p+1)^{\text{th}}$ set is the first which does not supply any new equations for the coefficients of Σ alone, that is, if the first $p+1$ sets are such that the coefficients of S', S'' cannot all be eliminated so as to result in an equation which involves the coefficients of u_p, *then, and not till then, the equating of coefficients may stop.* From the first p sets the coefficients of S', S'' may then be eliminated so as to give all the equations which hold for the coefficients of Σ alone. It goes without saying that this method cannot in general be carried out practically.

The number p is the minimum value of t mentioned in the enunciation of Noether's theorem. It seems impracticable, and of no great consequence, to find a simple and general analytical formula for it.

The above is an extension to the general case of a method (or illustration) employed by Noether for the "simple" case in the *Math. Ann.*, Vol. VI. Noether assumes without proof that the equations, taken in sets, actually determine the coefficients of S', S'' in terms of those of Σ. This is true in the "simple," but not in the general, case. For example, if

$$C_l \equiv y^2+ax^3+\dots, \qquad C_m \equiv y^2+bx^3+\dots,$$

the first two sets of the equations only involve u_0 and the coefficients of u_1, which are zero; the third set brings in u_0' and u_0'', but does not determine both; the fourth set determines u_0' and u_0'', but not the newly introduced coefficients of u_1' and u_1''; while the fifth set is the first which does not supply any new equations for the coefficients of Σ alone. The minimum value of t is 4 in this example.

By determining all the independent equations for the coefficients of Σ, and making them all to hold when $C'S$, $C''S$, $C'''S$, ... are substituted for Σ, we obtain all the independent equations which must hold for the coefficients of S; and we thus determine the most general system S such that any curve of the system CS passes through the whole intersection of C_l, C_m at the origin. The number of independent equations for the coefficients of S will always be less than the number for Σ, provided all the curves of the system C pass through the origin. The difference gives the number of simple points common to C_l, C_m at the origin through which the general curve of the system C may be supposed to pass. Also the difference will be the number of independent equations that must hold for the coefficients of a general polynomial K in order that KS may be of the form Σ, or $C_l S'+C_m S''$, to the necessary degree of approximation

at the origin. In other words, the number of independent equations for the coefficients of K together with the number of independent equations for the coefficients of S will be equal to the number of independent equations for the coefficients of Σ, or the number of simple points to which the whole intersection of C_l, C_m at the origin is equivalent. From these systems S and K, which satisfy local conditions at one place only, we can proceed to those systems S and K which satisfy the like conditions at as many places as we please, and, in particular, at all places, as on p. 20.

The theorem of residuation is, however, essentially of a geometrical character; and the problems it suggests are not likely to be completely solved without the free use of geometrical methods. We can solve, by the aid of geometry, the problem of the determination of the general linear systems S and K for (at least) that case in which the linear system C reduces to a single fixed curve. I hope to prove this, and other statements made in this section, in a later paper.

III. *The Generalized Riemann-Roch Theorem.*

By the generalized Riemann-Roch theorem we mean the theorem given on pp. 526, 527 of Vol. XXVI. of the *Proc. Lond. Math. Soc.*, which is restated in a more convenient form at the beginning of the footnote on p. 688 of Vol. XXIX. The proof in Vol. XXVI. holds for the general case if the theorem of residuation is assumed to hold generally. The theorem is applied below to the general case.

We suppose the general curve C of the given linear system $C', C'', \ldots$ to have no fixed constituent. We may then choose two curves C_l, C_m of the system, whose orders are as low as possible, so as to have no common constituent; and we assume their whole intersection to lie in the finite region.

We have explained in Section I. that the least degree N' of the whole point-base common to C_l, C_m which is not common to all the members of the linear system is the number of independent linear equations that must be satisfied by the coefficients of a general polynomial S, of order $l+m-2$, in order that the identity

$$CS \equiv C_l S' + C_m S''$$

may hold for all values of the arbitrary parameters involved in C. It may be observed in passing that the identity of the polynomials CS and $C_l S' + C_m S''$ is a very different thing from the identity of the same expressions when, as in Section II., S, S', S'' are unterminated

power series. The latter case only requires the equating of coefficients from the beginning, arriving at a stage where it may stop; but the former case requires the equating of coefficients to the end bringing no new unknown coefficients into the new equations after a certain time.

We imagine now that all the coefficients of S', S'' have been eliminated and that we have all the resulting equations for the coefficients of S.

We suppose the N' equations for the coefficients of S to be solved in the following way:—Suppose $S \equiv u_0 + u_1 + u_2 + \ldots + u_{l+m-2}$. The coefficient u_0 may be determined in terms of the remainder from one of the equations, and its value substituted in the rest, so that we get a new set of $N'-1$ equations from which u_0 has been eliminated. From this new set of equations the coefficients of u_1 may be determined, and their values substituted in the rest of the equations. From the new set of equations the coefficients of u_2 may be determined, and their values substituted in the rest of the equations; and so on. Suppose that all the coefficients of $u_0 + u_1 + \ldots + u_{p-1}$ are determinable in this manner, but that all the coefficients of u_p cannot be so determined. This last must happen if $N' < \frac{1}{2}(p+1)(p+2)$; and it may also happen if $N' \geqslant \frac{1}{2}(p+1)(p+2)$.

The fact that all the coefficients of $u_0 + u_1 + \ldots + u_{p-1}$ are determinable in terms of the remaining coefficients of S accounts for $\frac{1}{2}p(p+1)$ of the N' equations, so that there are $N' - \frac{1}{2}p(p+1)$ equations among the coefficients of $u_p + \ldots + u_{l+m-2}$. Also, if all the coefficients of u_p are not determinable in terms of the coefficients of $u_{p+1} + \ldots + u_{l+m-2}$, then there are a certain number ρ_p of the coefficients of u_p which are arbitrary;* while there will be $N' + \rho_p - \frac{1}{2}(p+1)(p+2)$ equations among the coefficients of $u_{p+1} + \ldots + u_{l+m-2}$. Solving this new set of equations for the coefficients of u_{p+1}, it will be found that there are a certain number ρ_{p+1} of these which are arbitrary, where $\rho_{p+1} > \rho_p$. *The limits of possibility of the value of* ρ_{p+1} *are* $\rho_p + 1$ *and* $p+2$. Proceeding in this method of solving, the whole N' equations will in time become exhausted (say) when a certain number of the

* The way in which this may happen, even when $N' \geqslant \frac{1}{2}(p+1)(p+2)$, is that some of the coefficients of u_p cannot be determined separately, but only in sets of two or more. A set being determined may be said to determine one in the set, whichever one we like, leaving the rest in the set arbitrary. If the elimination of all the coefficients of u_p should result in the complete disappearance of some of the coefficients of $u_{p+1} + \ldots + u_{l+m-2}$ from the $N' + \rho_p - \frac{1}{2}(p+1)(p+2)$ equations, such coefficients would be arbitrary.

coefficients of u_q have been determined in terms of the arbitrary coefficients of u_q, and the coefficients of $u_{q+1}+\ldots+u_{l+m-2}$, which last are therefore *all* arbitrary. We ought then to have

$$N'+\rho_p+\rho_{p+1}+\ldots+\rho_q=\tfrac{1}{2}(q+1)(q+2),$$

and

$$q+1>\rho_q>\rho_{q-1}>\ldots>\rho_p>0.$$

The N' equations become exhausted as soon as $\rho_{q+1}=q+2$. This would not, however, be the case if C_l, C_m had any asymptotic points in common, through which the general curve C of the linear system did not pass; for then there would be equations in which only the coefficients of the terms of highest order in S would be involved, even if the order of S exceeded $l+m-2$ by any amount, and very possibly also other equations in which only the coefficients of the terms of the two highest orders in S would be involved, and so on.

The lowest curve through the point-base N' is of order p, since in u_p there are one or more arbitrary coefficients; and the order of the point-base is therefore p,* and its degree N'. It must not, however, be supposed that N' is the simplest point-base derivable from N. This last would be found by drawing the two curves of lowest order through N, having no common constituent, to intersect again. The order of the point-base N, which is the order of the lowest curve through N, is given below; but we have no theorem at present which determines with certainty the order of the other lowest curve. The orders, degrees, and forms of the several base-points of N' may be found by the methods of Section II., by transferring the origin to each base-point in succession.

We take now the numbers

$$\ldots\ 0,\ 0,\ \rho_p,\ \rho_{p+1},\ \ldots\ \rho_{q-1},\ \rho_q,\ q+2,\ q+3,\ \ldots,$$

which are the differences of the successive defects of the point-base N'; and write down their successive differences, or second differences of the defects, viz.

$$\ldots\ 0,\ 0,\ \delta_p,\ \delta_{p+1},\ \ldots\ \delta_{q-1},\ \delta_q,\ \delta_{q+1},\ 1,\ 1,\ 1,\ \ldots\ .$$

This series of numbers constitutes the *characterization* of the point-base N', which we express by dropping the zeros at the beginning

---

* The order p of the point-base N' cannot exceed, but may be less than, the smaller of the two numbers l, m, since these are the orders of two curves passing through the point-base.

[redacted]

and units at the end, changing the suffixes to $1, 2, \ldots a$, and writing

$$N' = (\delta_1, \delta_2, \ldots \delta_a),$$

where $$p+a = q+2 = \Sigma\delta, \quad \delta_a > 1.$$

The generalized Riemann-Roch theorem then gives us the characterization of N, which is as follows (assuming $m \geqslant l$) :*—

(i.) $N = (1^{m-l}, 2^{l-p-a}, \delta_a+1, \delta_{a-1}+1, \ldots \delta_1+1)$, if $l \geqslant p+a \; (=\Sigma\delta)$;

(ii.) $N = (1^{m-p-a}, \delta_a, \ldots \delta_{l-p+1}, \delta_{l-p}+1, \ldots \delta_1+1)$, if $p < l < p+a \leqslant m$;

(iii.) $N = (\delta_a-1, \ldots \delta_{m-p+1}- \quad \delta_{m-p}, \ldots \delta_{l-p+1}, \delta_{l-p}+1, \ldots \delta_1+1)$,

if $p < l \leqslant m < p+a$;

(iv.) $N = (1^{m-l-a}, \delta_a, \delta_{a-1}, \ldots \delta_1)$, if $p = l$, $l+a \leqslant m$;

(v.) $N = (\delta_a-1, \ldots \delta_{m-l+1}-1, \delta_{m-l}, \ldots \delta_1)$, if $p = l$, $m < l+a$.

The symbols 1^{m-l}, 2^{l-p-a} in (i.) stand for 1 repeated $m-l$ times, followed by 2 repeated $l-p-a$ times. Similarly for 1^{m-p-a} in (ii.) and 1^{m-l-a} in (iv.). Cases (iv.) and (v.) are the simplified forms of (ii.) and (iii.) when $p = l$, and may be still further simplified (if $\delta_1 = 1$) by the omission of any units at the end.

The order of N' is p, which is the value of $\Sigma(\delta-1)$. The order of N is the value of $\Sigma(\delta-1)$ for N, *i.e.*, it is l in cases (i.), (ii.), (iv.), $l+m-p-a$ in case (iii.), and $m-a$ in case (v.).

The lowest order that can be chosen for S so that the equations among its coefficients may determine the point-base N' without ambiguity is, I think, $q+1 = \Sigma\delta-1$, which would leave the coefficients of the terms of highest order in S entirely arbitrary. It would be of great importance to find methods for determining the value of $q+1$, so that S might be taken of order $q+1$, instead of order $l+m-2$. The order $q+1$ will be sufficiently high if the whole system of curves of order $q+1$, whose coefficients satisfy the N' equations, could not have a point-base in common of higher degree than N'. This property holds in what appears to be the most unlikely case, viz., the linear system of curves determined by x^n, $x^{n-1}y$, $x^{n-2}y^2$, $\ldots$ y^n have no point-base in common of higher degree than

* In this application we regard N', and consequently $\delta_1, \delta_2, \ldots \delta_a$, as known. The numbers p, q, a, l, m are then also known, viz., $p = \Sigma(\delta-1)$, $q = \Sigma\delta-2$, a is the number of the δ's ($\delta_a > 1$), and l, m are the orders of the two curves drawn through N' and intersecting for the rest in N.

$\frac{1}{2}n(n+1)$. Hence, for the complete specification of the point-bases N' and N, it appears that we may choose for the order of S the value $q+1$, or $\Sigma\delta-1$; and for the order of K the value of $\Sigma\delta-1$ for N, *i.e.*, $l+m-p-1$ in cases (i.), (ii.), (iii.), $m-1$ in (iv.) and (v.) if $\delta_1 > 1$, and a lower order in (iv.), and (v.) if $\delta_1 = 1$, since then we should omit all units at the end of (iv.) and (v.), and the value of $\Sigma\delta-1$ for N would diminish accordingly. The general algebraic curve through N' is $PS+P'S'+\ldots=0$, where $S, S', \ldots$ are polynomials of order $q+1$ whose coefficients satisfy the N' equations, and $P, P', \ldots$ are polynomials with arbitrary coefficients. Similarly for the general curve through N.

Concerning the Four Known Simple Linear Groups of Order 25920, *with an Introduction to the Hyper-Abelian Linear Groups.* *By* Dr. L. E. Dickson.

Received March 18th, 1899. Read April 13th, 1899.

Introduction.

In a paper* giving a *résumé* of the known systems of simple groups and a table of the orders of all known simple groups not exceeding one million, I find that, apart from the order 25920, every case in which two or more simple groups of the same order exist has been completely investigated as to their simple isomorphism or non-isomorphism. The greater part of the present paper deals with the four known simple groups of order 25920, viz.,†

(1) The simple group $A(4, 3)$, defined by the decomposition of the Abelian group on four indices taken modulo 3.

---

* "The Known Finite Simple Groups," *Bulletin of the American Mathematical Society*, July, 1899.

† [*Note of August 14th.*—I should have referred to a number of important investigations in which occur substitution-groups and groups of collineations isomorphic with the above groups of order 25920. Jordan (*Traité des Substitutions*, pp. 316–329) shows that the Galois group of the equation for the 27 lines on a general cubic surface has the order 2·25920, and proves (pp. 365–369) that it is isomorphic with the Abelian group for the trisection of hyperelliptic functions of four periods. A proof involving less calculation has been given by the writer (*Comptes Rendus*, Vol. CXXVIII., p. 873, 1899). The present article was written

(2) The second hypo-Abelian group $SH(6, 2)$, a sub-group of index 2 under the general hypo-Abelian group on six indices taken modulo 2.

(3) The orthogonal group $O(5, 3)$, a sub-group of index 4 under the total orthogonal group on five indices taken modulo 3.

(4) The hyper-Abelian group $HA(4, 2^2)$ of quaternary hyper-Abelian substitutions of determinant unity in the Galois field of order 2^2.

The groups (1) and (2) are due to Jordan,* the groups (3) and (4) are due to the writer. Complete references are given in the paper above cited. Group (4) is here studied in a new form (see § 1). Between the groups (1) and (3) a holohedric isomorphism has already been established by the writer.† In the present paper I set up an abstract group and a substitution-group on thirty-six letters, with which each of the groups (1), (2), and (4) is proven holohedrically isomorphic. Incidentally, a simple group-theoretic proof is given in § 4 of the holohedric isomorphism between the Abelian group on four indices modulo 2 and the symmetric group on six letters. The knowledge of the simple isomorphism of the above four groups is of considerable aid in the study of the properties of any one of them; for example, sub-groups which are evident in the case of one group are recognized with difficulty in the case of another. By means of the sub-group of order 60·16 of the group (3), which results from the extension of the group of order 60 of the even permutations of the five indices $\xi_1, \ldots, \xi_5$ by the commutative group of order 16 formed by the substitutions

$$\xi_i' = \epsilon_i \xi_i \quad (i = 1, \ldots, 5),$$

where each $\epsilon_i = \pm 1$ and $\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5 = +1$, we could set up a substitution group on twenty-seven letters simply isomorphic with (3).

The general theorems of §§ 2, 3 are not essential to the later developments, as the particular cases used are established incidentally in the later proofs.

---

before I knew of the papers by H. Burkhardt (*Math. Ann.*, Vol. XXXVIII. and Vol. XLI.), in which quaternary and senary collineation groups isomorphic with the above groups are studied at length, incorporating the work of Klein, Maschke, and Witting. The results of these papers are intimately related to, but quite distinct from, those of the present article.]

* *Traité des Substitutions*, pp. 171 and 207. The order of the second hypo-Abelian group is not, however, given correctly.

† "Concerning the Abelian and Hypo-Abelian Groups," *Bulletin of the American Mathematical Society*, p. 332, 1899; read before the Society, February 25th, 1899.

1. *Definition of the General Hyper-Abelian Group.*

In a paper recently contributed to the *Mathematische Annalen*, I have determined the order, generators, and structure of the linear homogeneous group on M indices in the $GF[p^{2n}]$ which is defined by the invariant

$$\xi_1^{p^n+1}+\xi_2^{p^n+1}+\dots+\xi_M^{p^n+1}.$$

Those of its substitutions which have as coefficients marks of the $GF[p^n]$ form a sub-group,* which for $p>2$ is the general orthogonal group on M indices in the $GF[p^n]$. I therefore propose for the larger group the name *hyper-orthogonal* group, and denote by $HO[M, p^{2n}]$ the simple groups of order

$$\frac{1}{g}[p^{nM}-(-1)^M]\,p^{n(M-1)}\,[p^{n(M-1)}-(-1)^{M-1}]\,p^{n(M-2)}\dots[p^{2n}-1]\,p^n,$$

g being the greatest common divisor of M and p^n+1, the cases $(p^n=2, M=2)$, $(p^n=3, M=2)$, and $(p^n=2, M=3)$ being exceptional.

If we make the transformation of indices

$$\xi_1'=J\xi_1+\xi_2,\quad \xi_2'=\rho\,(J^{p^n}\xi_1+\xi_2),$$

where J and ρ are primitive roots (belonging, in fact, to the $GF[p^{2n}]$) of the respective equations

$$J^{p^n+1}=1,\quad \rho^{p^n+1}=-1,$$

the function $\xi_1^{p^n+1}+\xi_2^{p^n+1}$ is transformed into

$$(J-J^{p^n})\,(\xi_1\xi_2^{p^n}-\xi_2\xi_1^{p^n}).$$

Hence, for $M=2m$, the hyper-orthogonal group is simply isomorphic with the linear group defined by the invariant

$$\psi\equiv\sum_{i=1}^{m}\begin{vmatrix}\xi_{2i-1} & \xi_{2i}\\ \xi_{2i-1}^{p^n} & \xi_{2i}^{p^n}\end{vmatrix}.$$

The totality of linear $2m$-ary substitutions with coefficients belonging to the included field $GF[p^n]$ and leaving ψ absolutely invariant form a sub-group, which is immediately recognized as the Abelian

---

* For $p=2$, it is a solvable group of order $2^{nM(M-1)/2}$ generated by the binary substitutions $\begin{pmatrix} a & a+1\\ a+1 & a\end{pmatrix}$.

group on $2m$ indices in the $GF[p^n]$. I therefore propose for the larger group the name *hyper-Abelian* group, and denote by $HA(2m, p^{2n})$ the simple quotient group corresponding to the above $HO(2m, p^{2n})$.

Consider the general linear $2m$-ary substitution in the $GF[p^{2n}]$.

$$S:\quad \xi_i' = \sum_{j=1}^{2m} a_{ij}\xi_j \quad (i = 1, \dots, 2m).$$

It transforms ψ into

$$\sum_{l=1}^{m}\left\{\sum_{j=1}^{2m}\begin{vmatrix} a_{2l-1\,j} & a_{2l\,j}\\ a^{p^n}_{2l-1\,j} & a^{p^n}_{2l\,j}\end{vmatrix}\xi_j^{p^n+1}+\sum_{j\neq k}^{j,k=1,\dots,2m}\begin{vmatrix} a_{2l-1\,j} & a_{2l\,j}\\ a^{p^n}_{2l-1\,k} & a^{p^n}_{2l\,k}\end{vmatrix}\xi_j\xi_k^{p^n}\right\}.$$

The conditions that S shall leave ψ absolutely invariant are therefore the following:—

$$(1)\qquad \sum_{l=1}^{m}\begin{vmatrix} a_{2l-1\,j} & a_{2l\,j}\\ a^{p^n}_{2l-1\,k} & a^{p^n}_{2l\,k}\end{vmatrix} = \epsilon_{jk} \quad (j, k = 1, \dots, 2m),$$

where $\epsilon_{jk} = 0$, unless j and k differ by unity, when

$$\epsilon_{2i-1\,2i} = 1, \quad \epsilon_{2i\,2i-1} = -1 \quad (i = 1, \dots, m).$$

The relations (1) in which $j > k$ are derived from those in which $j < k$ by raising the latter to the power p^n.

The reciprocal of the hyper-Abelian substitution S is

$$S^{-1}:\quad \left.\begin{cases} \xi'_{2l-1} = \sum_{j=1}^{m}\left(a^{p^n}_{2j\,2l}\,\xi_{2j-1} - a^{p^n}_{2j-1\,2l}\,\xi_{2j}\right) \\ \xi'_{2l} \phantom{{}_{-1}} = \sum_{j=1}^{m}\left(-a^{p^n}_{2j\,2l-1}\,\xi_{2j-1} + a^{p^n}_{2j-1\,2l-1}\,\xi_{2j}\right)\end{cases}\right\}\quad (l = 1, \dots, m).$$

Indeed, the product SS^{-1} replaces ξ_{2l-1} by

$$\sum_{j,k}^{1,\dots,m}\begin{vmatrix} a_{2j-1\,2k-1} & a_{2j\,2k-1}\\ a^{p^n}_{2j-1\,2l} & a^{p^n}_{2j\,2l}\end{vmatrix}\xi_{2k-1}+\sum_{j,k}^{1,\dots,m}\begin{vmatrix} a_{2j-1\,2k} & a_{2j\,2k}\\ a^{p^n}_{2j-1\,2l} & a^{p^n}_{2j\,2l}\end{vmatrix}\xi_{2k}$$

$$\equiv \sum_{k=1}^{m}\left(\epsilon_{2k-1\,2l}\,\xi_{2k-1} + \epsilon_{2k\,2l}\,\xi_{2k}\right) \equiv \xi_{2l-1}.$$

Similarly, it replaces ξ_{2l} by

$$-\sum_{k=1}^{m}\left(\epsilon_{2k-1\,2l-1}\,\xi_{2k-1} + \epsilon_{2k\,2l-1}\,\xi_{2k}\right) \equiv \xi_{2l}.$$

For convenience of reference, we give to the relations (1) the more explicit form

$$(2)\quad \sum_{l=1}^{m} \begin{vmatrix} a_{2l-1\,j} & a_{2l\,j} \\ a_{2l-1\,k}^{p^n} & a_{2l\,k}^{n} \end{vmatrix} = \begin{cases} 1 \text{ (if } k=j+1=\text{even)} \\ 0 \text{ (unless } k=j+1=\text{even, if } j \lesseqqgtr k). \end{cases}$$

The corresponding relations for the reciprocal S^{-1} are found by replacing

$$a_{2l-1\,2j-1},\quad a_{2l-1\,2j},\quad a_{2l\,2j-1},\quad a_{2l\,2j}$$

by respectively $a_{2j\,2l}^{p^n}$, $-a_{2j-1\,2l}^{p^n}$, $-a_{2j\,2l-1}^{p^n}$, $a_{2j-1\,2l-1}^{p^n}$.

Writing out the four sets of relations (2) according to the evenness or oddness of j and k, and making the replacement described, we obtain four sets of relations, which may be combined into the single formula

$$(3)\quad \sum_{l=1}^{m} \begin{vmatrix} a_{j\,2l-1}^{p^n} & a_{j\,2l}^{p^n} \\ a_{k\,2l-1} & a_{k\,2l} \end{vmatrix} = \begin{cases} 1 \text{ (if } k=j+1=\text{even)} \\ 0 \text{ (unless } k=j+1=\text{even, if } j \lesseqqgtr k). \end{cases}$$

The relations (3) are together equivalent to the relations (2).

The determinant Δ of the hyper-Abelian substitution S must satisfy the relation

$$(4)\quad \Delta^{p^n+1} = 1.$$

Indeed, if we reflect on the main diagonal the determinant of S^{-1}, change the signs of the $2l-1^{\text{st}}$ row and column (for $l = 1, \ldots, m$), and interchange the $2l-1^{\text{st}}$ row with the $2l^{\text{th}}$ row (for $l = 1, \ldots, m$) and likewise the corresponding columns, we obtain the determinant

$$|\, a_{ij}^{p^n} \,| \equiv |\, a_{ij} \,|^{p^n} \equiv \Delta^{p^n} \quad (i, j = 1, \ldots, 2m).$$

Hence $\Delta\Delta^{p^n} = 1$, being the determinant of the product SS^{-1}.

2. *The Maximal Sub-group of the Hyper-Abelian Group within which the Abelian Group is Self-conjugate.*

We determine all the hyper-Abelian substitutions

$$S:\quad \xi_i' = \sum_{j=1}^{2m} a_{ij}\,\xi_j \quad (i = 1, \ldots, 2m)$$

which transform the Abelian group into itself. S transforms the Abelian substitution, affecting a single index,

$$\xi_{2r-1}' = \xi_{2r-1} + \xi_{2r}$$

into the substitution

$$\xi_i' = \xi_i + a_{i\,2r-1} \sum_{j=1}^{m} \left(-a^{p^n}_{2j\,2r-1}\,\xi_{2j-1} + a^{p^n}_{2j-1\,2r-1}\xi_{2j}\right)$$
$$(i = 1, \ldots, 2m),$$

whose coefficients must therefore belong to the $GF[p^n]$, viz.,

$$a_{i\,2r-1}\,a^{p^n}_{j\,2r-1} \quad (i, j = 1, \ldots, 2m;\ r = 1, \ldots, m).$$

Likewise, S must transform the Abelian substitution

$$\xi_{2r}' = \xi_{2r} + \xi_{2r-1}$$

into a substitution belonging to the $GF[p^n]$. Hence the products

$$a_{i\,2r}\,a^{p^n}_{j\,2r} \quad (i, j = 1, \ldots, 2m;\ r = 1, \ldots, m)$$

must belong to the $GF[p^n]$. The reciprocal S^{-1} must transform the Abelian group into itself. From the above results, it follows therefore that the products

$$a_{2r\,s}\,a^{p^n}_{2r\,t}, \quad a_{2r-1\,s}\,a^{p^n}_{2r-1\,t} \quad (s, t = 1, \ldots, 2m;\ r = 1, \ldots, m)$$

must belong to the $GF[p^n]$. Combining our results, every product

$$(5) \quad a_{ir}\,a^{p^n}_{jr}, \quad a_{ri}\,a^{p^n}_{rj} \quad (i, j, r = 1, \ldots, 2m)$$

must belong to the $GF[p^n]$.

But, if β, γ be marks of the $GF[p^{2n}]$ such that

$$\beta\gamma^{p^n} = \mu = \text{mark of } GF[p^n],$$

then, if $\gamma \neq 0$, $\beta\gamma^{p^n+1} = \mu\gamma$, $\beta/\gamma = \text{mark of } GF[p^n]$.

Hence, by (5), the ratios of the coefficients in any row or any column of the matrix of S must all belong to the $GF[p^n]$.

S transforms the Abelian substitution

$$\xi_{2r-1}' = \xi_{2r-1} + \xi_{2s}, \quad \xi_{2s-1}' = \xi_{2s-1} + \xi_{2r} \quad (r \neq s)$$

into
$$\xi_i' = \xi_i + a_{i\,2r-1} \sum_{j=1}^{m} \left(-a^{p^n}_{2j\,2s-1}\,\xi_{2j-1} + a^{p^n}_{2j-1\,2s-1}\,\xi_{2j}\right)$$
$$+ a_{i\,2s-1} \sum_{j=1}^{m} \left(-a^{p^n}_{2j\,2r-1}\,\xi_{2j-1} + a^{p^n}_{2j-1\,2r-1}\,\xi_{2j}\right).$$

Hence the sums

$$a_{i\,2r-1}\,a^{p^n}_{j\,2s-1} + a_{i\,2s-1}\,a^{p^n}_{j\,2r-1} \quad (i, j = 1, \ldots, 2m;\ r, s = 1, \ldots, m)$$

must all belong to the $GF[p^n]$. In like manner, if S transform each of the following three Abelian substitutions (in which $r \neq s$),

$$\xi'_{2r-1} = \xi_{2r-1} + \xi_{2s-1}, \quad \xi'_{2s} = \xi_{2s} - \xi_{2r};$$

$$\xi'_{2r} = \xi_{2r} + \xi_{2s}, \quad \xi'_{2s-1} = \xi_{2s-1} - \xi_{2r-1};$$

$$\xi'_{2r} = \xi_{2r} + \xi_{2s-1}, \quad \xi'_{2s} = \xi_{2s} - \xi_{2r-1}$$

into substitutions belonging to the $GF[p^n]$, then must the respective sums

$$\left\{\begin{array}{l} a_{i\,2r-1}a_{j\,2s}^{p^n} + a_{i\,2s}a_{j\,2r-1}^{p^n} \\ a_{i\,2r}a_{j\,2s-1}^{p^n} + a_{i\,2s-1}a_{j\,2r}^{p^n} \\ a_{i\,2r}a_{j\,2s}^{p^n} + a_{i\,2s}a_{j\,2r}^{p^n} \end{array}\right\} \quad \binom{i, j = 1, \ldots, 2m}{r, s = 1, \ldots, m}$$

belong to the $GF[p^n]$. Combining our results, every sum

$$(6) \quad a_{ir}a_{s}^{p^n} + a_{is}a_{jr}^{p^n} \quad (i, j, r, s = 1, \ldots, 2m;\ r \neq s)$$

belongs to the $GF[p^n]$.

Of the coefficients in the i^{th} row of the matrix of S, we may suppose that $a_{ir} \neq 0$, for example. If, then, $a_{jr} \neq 0$, the ratios of the coefficients in the i^{th} and j^{th} rows must all belong to the $GF[p^n]$ [by the result following from (5)]. If, however, $a_{jr} = 0$, we may suppose that, for example, $a_{js} \neq 0$ $(s \neq r)$. Then, by (6), the products $a_{ir}a_{js}^{p^n}$ belong to the $GF[p^n]$. We have in either case the result that the ratios of the coefficients in the i^{th} and j^{th} rows belong to the $GF[p^n]$. Hence the ratios of all the coefficients in S to any one non-vanishing coefficient belong to the $GF[p^n]$, so that S may be written

$$(7) \quad \xi'_i = a \sum_{j=1}^{2m} \lambda_{ij}\xi_j \quad (i = 1, \ldots, 2m),$$

where the λ_{ij} belong to the $GF[p^n]$.

Inversely, we readily verify that every hyper-Abelian substitution of the form (7) transforms into itself the Abelian group defined for the $GF[p^n]$.

3. *Number of Sub-groups of HA $(2m, p^{2n})$ conjugate with the Simple Abelian Group A $(2m, p^n)$.*

The maximal invariant sub-group of the group of hyper-Abelian substitutions of determinant unity is composed of the substitutions* [see (3)],

$$(8) \quad \xi_i' = \kappa \xi_i \quad (i = 1, \ldots, 2m),$$

$$[\kappa^{p^n+1} = 1, \quad \kappa^{2m} = 1],$$

The quotient group has been denoted by $HA\,(2m, p^{2n})$. The writer has shown (*Quarterly Journal of Pure and Applied Mathematics*, pp. 169–178, 1897; *ibid.*, 1899, for the case $2m = 4$, $p^n = 2^n$) that the maximal invariant sub-group of Abelian substitutions of determinant unity on $2m$ indices in the $GF\,[p^n]$ is composed of the substitutions

$$(9) \quad \xi_i' = \epsilon \xi_i \quad (i = 1, \ldots, 2m) \quad [\epsilon = \pm 1],$$

and is consequently of order 1 or 2 according as $p = 2$ or $p > 2$. We denote the quotient group by the symbol $A\,(2m, p^n)$. The cases

$$(2m = 2,\ p^n = 2), \quad (2m = 2,\ p^n = 3), \quad \text{and} \quad (2m = 4,\ p^n = 2)$$

are exceptional. For the last case see § 4.

If the substitution (8) belong to the $GF\,[p^n]$, then must $\kappa^2 = 1$, when (8) becomes identical with (9). It follows that the quotient group $A\,(2m, p^n)$ is a sub-group of the quotient group $HA\,(2m, p^{2n})$.

We now seek the substitutions of the quotient group $HA\,(2m, p^{2n})$ which correspond to the hyper-Abelian substitutions (7). The latter have the determinant

$$a^{2m}\,|\lambda_{ij}| = 1.$$

Raising this to the power $p^n - 1$, we must have

$$a^{2m(p^n-1)} = 1.$$

Further, a being a mark $\neq 0$ of the $GF\,[p^{2n}]$, we have

$$(a^{p^n-1})^{p^n+1} \equiv a^{p^{2n}-1} = 1.$$

Hence, by (8), the hyper-Abelian substitution of determinant unity

$$(10) \quad \xi_i' = (a^{p^n-1})^t\, \xi_i \quad (i = 1, \ldots, 2m)$$

* This result follows immediately from the analogous theorem for the isomorphic hyper-orthogonal group proven in the *Annalen* paper above cited. For $2m = 2$, the cases $p^n = 2$ and $p^n = 3$ are exceptional.

corresponds, for every integer t, to the identity in the quotient group HA. Hence (7) and the product of (7) and (10), viz.,

$$(7_1) \quad \xi_i' = A \sum_{j=1}^{2m} \lambda_{ij} \xi_j \quad (i = 1, \ldots, 2m),$$

where
$$A \equiv a^{1+t(p^n-1)},$$

correspond to the same substitution in the quotient group. Since $\frac{1}{2}(p^n-1)$ and $\frac{1}{2}(p^n+1)$ are relatively prime, we can, if $p>2$, determine an integer t such that $1+t(p^n-1)$ shall be a multiple of $\frac{1}{2}(p^n+1)$. For such a choice of t, we have

$$A^{2(p^n-1)} = 1,$$

so that A^2 will belong to the $GF[p^n]$. For $p=2$, (p^n-1) and (p^n+1) are relatively prime, so that an integer t may be found such that

$$1+t(p^n-1) \equiv 0 \pmod{p^n+1}.$$

Hence, will
$$A^{p^n-1} = 1,$$

so that the substitution (7_1) belongs to the $GF[p^n]$, and is therefore contained in $A(2m, p^n)$.

Consider, for $p>2$, the substitutions of the form (7_1) in which A^2, but not A, belongs to the $GF[p^n]$. Then will

$$(a) \quad A^{p^n-1} = -1.$$

If A_1 be one solution of this equation, the p^n-1 distinct solutions in the $GF[p^{2n}]$ are given by βA_1, if β runs through the marks $\neq 0$ of the $GF[p^n]$. Since the factor β can be absorbed into the coefficients λ_{ij}, we need only consider the substitutions (7_1) in which A_1 is a definite solution of (a). If m be even and -1 be a not-square in the $GF[p^n]$, we can determine a substitution (8) such that its product by (7_1) gives a substitution with coefficients belonging to the $GF[p^n]$. Indeed, the conditions

$$(\kappa A_1)^{p^n-1} = 1, \quad \kappa^{p^n+1} = 1, \quad \kappa^{2m} = 1$$

combined with (a) require $-\kappa^{p^n-1} = 1$, and therefore $-1 = \kappa^2$. Since -1 belongs to the $GF[p^n]$, it is a square in the $GF[p^{2n}]$. Our conditions can thus be satisfied if, and only if, $\dfrac{p^n+1}{2}$ be even and m even. In this case, every substitution of $HA(2m, p^{2n})$ which transforms $A(2m, p^n)$ into itself belongs to the latter group. As

shown above, the same result holds if $p = 2$. If, however, m be even and -1 be a square in the $GF[p^n]$, the number of substitutions of $HA\,(2m, p^{2n})$ which transform $A\,(2m, p^n)$ into itself is double the order of the latter group. Indeed, (7_1) can be expressed as the product of the Abelian substitution

$$\left.\begin{cases} \xi'_{2l-1} = \sum\limits_{j=1}^{2m} \lambda_{2l-1\,j}\,\xi_j \\ \xi'_{2l} \;\;\;= \sum\limits_{j=1}^{2m} A_1^{p^n+1}\lambda_{2l\,j}\,\xi_j \end{cases}\right\} \quad (l = 1, \dots, m),$$

by the hyper-Abelian substitution

$$\xi'_{2l-1} = A_1\xi_{2l-1}, \quad \xi'_{2l} = A^{-p^n}\xi_{2l} \quad (l = 1, \dots, m),$$

of determinant, when m is even,

$$(A_1^{-p^n+1})^m = (-1)^m = 1.$$

I have not completed the investigation for m odd. For $m = 1$, a hyper-Abelian substitution of the form (7_1) has the determinant $+1$ only if A belongs to the $GF[p^n]$. Indeed, the hyper-Abelian condition (2), or (3), gives, for (7_1), when $m = 1$,*

$$A^{p^n+1}(\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21}) = 1.$$

If (7_1) have determinant unity,

$$A^2(\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21}) = 1.$$

Hence must
$$A^{p^n-1} = 1,$$

so that indeed A belongs to the $GF[p^n]$.

For the case $2m = 4$, $p^n = 2$, above excluded, the determinant of (7) is unity only when $a^4 = 1$, and hence $a = 1$. In this case the substitutions (8) and (9) each reduce to the identical substitution. Hence every hyper-Abelian substitution transforming into itself the Abelian group on four indices modulo 2 must belong to the latter group.

Finally, the order of $A\,(2m, p^n)$ is known (*Quarterly Journal*, *l.c.*) to be

$$\frac{1}{a}(p^{2nm}-1)\,p^{n(2m-1)}\,(p^{n(2m-2)}-1)\,p^{n(2m-3)} \dots (p^{2n}-1)\,p^n,$$

if $a = 1$ or 2 according as $p = 2$ or $p > 2$.

---

* We verify similarly that every hyper-Abelian substitution of determinant unity on two indices in the $GF[p^n]$ must be an Abelian substitution belonging to the $GF[p^n]$. Hence $HA\,(2, p^{2n}) \equiv A\,(2, p^n)$.

We derive at once the following theorem :—

The number of sub-groups of $HA\ (2m, 2^{2n})$ conjugate with the Abelian group on $2m$ indices in the $GF\,[2^n]$ is

$$\frac{1}{g}\,(2^{n(2m-1)}+1)\,2^{n(2m-2)}\,(2^{n(2m-3)}+1)\,2^{n(2m-4)}\,\dots\,(2^{3n}+1)\,2^{2n},$$

where g is the greatest common divisor of m and 2^n+1. For $p>2$ and m even, the number of sub-groups of $HA\ (2m, p^{2n})$ conjugate with the sub-group $A\ (2m, p^n)$ is

$$\frac{2}{gq}\,(p^{n(2m-1)}+1)\,p^{n(2m-2)}\,(p^{n(2m-3)}+1)\,p^{n(2m-4)}\,\dots\,(p^{3n}+1)\,p^{2n},$$

where g is the greatest common divisor of $2m$ and p^n+1, and where $q=1$ or 2 according as -1 is a not-square or a square in the $GF\,[p^n]$, viz., according as p^n is of the form $4l-1$ or $4l+1$. For $p>2$ and m odd, the number of conjugates is given by a similar formula, but it is not determined whether the value of q is 1 or 2.

For example, $HA\ (4, 2^2)$ has 36 sub-groups conjugate with the quaternary Abelian group modulo 2 of order 720 [see also § 5]. $HA\ (4, 3^2)$ has $\frac{1}{2}\,(3^3+1)\,3^2 \equiv 126$ sub-groups conjugate with $A\ (4, 3)$.

4. *Abstract Form of the Quaternary Abelian Group Modulo 2.*

THEOREM.—*The Abelian group on four indices modulo 2 is simply isomorphic with the symmetric group on six letters.**

The symmetric group on six letters is simply isomorphic with the abstract group defined by the five generators B_1, B_2, B_3, B_4, B_5 with the generational relations†

(11) $$B_1^2 = B_2^2 = B_3^2 = B_4^2 = B_5^2 = 1,$$

(12) $$(B_1B_2)^3 = (B_2B_3)^3 = (B_3B_4)^3 = (B_4B_5)^3 = 1,$$

(13) $$(B_1B_3)^2 = (B_1B_4)^2 = (B_1B_5)^2 = (B_2B_4)^2 = (B_2B_5)^2 = (B_3B_5)^2 = 1.$$

In seeking a generational correspondence between this abstract group and the Abelian group A_{720} on four indices modulo 2, I started

* This result is due to Jordan, *Traité des Substitutions*, No. 335, who obtained it by means of the groups of Steiner. Not only is my proof immediate, but also it serves as the basis of my work below on the $HA\ (4, 2^2)$.

† Moore, "Concerning the Abstract Groups of Order $k\,!$ and $\frac{1}{2}k\,!$ Holohedrically Isomorphic with the Symmetric and the Alternating Substitution-Groups on k letters," *Proceedings of the London Mathematical Society*, Vol. xxviii., No. 597.

with the simple knowledge that the Abelian substitutions* L_1, L_2, M_1, M_2 were of period 2 and satisfied the relations

$$(L_1L_2)^2 = (M_1M_2)^2 = (L_1M_2)^2 = (L_2M_1)^2 = 1,$$
$$(L_1M_1)^3 = (L_2M_2)^3 = 1.$$

It was then natural to seek an Abelian substitution S such that

$$(a) \quad S^2 = 1, \quad (M_1S)^2 = (SM_2)^2 = 1, \quad (L_1S)^3 = (SL_2)^3 = 1.$$

If such a substitution S exists, then the desired generational correspondence would be given by taking

$$B_1 \sim M_1, \quad B_2 \sim L_1, \quad B_3 \sim S, \quad B_4 \sim L_2, \quad B_5 \sim M_2.$$

Now, the most general quaternary linear substitution commutative with M_1 and M_2 is seen by inspection to be

$$\begin{array}{c} \xi_1 \\ \eta_1 \\ \xi_2 \\ \eta_2 \end{array} \left\{ \begin{array}{cccc} \alpha_{11} & \gamma_{11} & \alpha_{12} & \alpha_{12} \\ \gamma_{11} & \alpha_{11} & \alpha_{12} & \alpha_{12} \\ \alpha_{21} & \alpha_{21} & \alpha_{22} & \gamma_{22} \\ \alpha_{21} & \alpha_{21} & \gamma_{22} & \alpha_{22} \end{array} \right\}.$$

The conditions that it be Abelian are as follows:—

$$\alpha_{11}^2 - \gamma_{11}^2 = 1, \quad \alpha_{22}^2 - \gamma_{22}^2 = 1, \quad \alpha_{21}(\alpha_{11} - \gamma_{11}) + \alpha_{12}(\gamma_{22} - \alpha_{22}) = 0,$$

or, reducing modulo 2,

$$\gamma_{11} = \alpha_{11} + 1, \quad \gamma_{22} = \alpha_{22} + 1, \quad \alpha_{12} = \alpha_{21}.$$

The resulting substitution is of period 2 (follows from $\alpha_{12} = \alpha_{21}$). In order that L_1S and L_2S shall be of period 3, we readily find that α_{11} and α_{22} must be congruent to zero. Hence

$$S \equiv \left\{ \begin{array}{cccc} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{array} \right\}.$$

We readily verify that S satisfies the conditions (a). Since the Abelian group on four indices modulo 2 is of order 720, the proof of its isomorphism with the symmetric group on six letters is complete.

---

* In the notation used by Jordan, *Traité des Substitutions*, p. 174.

Using Moore's Theorem A′, we obtain a second abstract definition of the Abelian group A_{720}, viz., by setting

$$B_1 \sim M_1, \quad C \sim M_2 L_2 S L_1 M_1 \equiv \begin{Bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{Bmatrix}.$$

The generational relations

$$B_1^2 = 1, \quad C^6 = 1, \quad (B_1 C)^5 = 1,$$

$$(B_1 C^{-1} B_1 C)^3 = (B_1 C^2 B_1 C^4)^2 = (B_1 C^3 B_1 C^3)^2 = 1$$

are readily seen to be satisfied. Indeed, we have

$$C^2 \sim \begin{Bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{Bmatrix}, \quad C^3 \sim \begin{Bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{Bmatrix}, \quad C^6 \sim 1.$$

$$B_1 C^4 B_1 C^2 \sim S M_1, \quad B_1 C^3 B_1 C^3 \sim M_1 L_2, \quad B_1 C^{-1} B_1 C \sim M_1 L_1,$$

$$(B_1 C)^5 \sim 1.$$

5. *Abstract Form of the Quaternary Hyper-Abelian Group in the $GF[2^2]$ and of the Senary Second Hypo-Abelian Group.*

THEOREM.—*The simple groups $HA(4, 2^2)$ and $SH(6, 2)$ are simply isomorphic with the abstract groups H of order* 25920 *generated by the operators B_1, B_2, B_3, B_4, B_5, and B, satisfying the generational relations* (11), (12), (13) *together with*

(14) $B^3 = 1, \ BB_1 = B_1 B, \ BB_2 = B_2 B, \ BB_4 = B_4 B, \ BB_5 = B_5 B;$

(15) $$(BB_3)^3 = B_2 B_5;$$

(16) $$B (B_1 B_5 B_3 B_2 B_4)^3 B = (B_1 B_5 B_3 B_2 B_4)^3;$$

(17) $$(B_1 B_2 B_3 B_1 B_2)(B^{-1} B_3 B)(B_1 B_2 B_3 B_1 B_2)(B^{-1} B_3 B) = B_5.$$

Consider the rectangular array of the operators of H, the first line comprising the 720 operators of the abstract sub-group $A \sim A_{720}$. It will be shown that the thirty-six right-hand multipliers necessary to the formation of this array are those exhibited in the following scheme. Giving to the j^{th} line the notation O_j, for $j = 1, 2, \ldots, 36$,

we will prove that the operators of H are given uniquely in the following table:—

$O_1 \equiv A$, $O_2 \equiv AB$, $O_3 \equiv AB^2$, $O_4 \equiv ABB_3$, $O_5 \equiv ABB_3B_2$,

$O_6 \equiv ABB_3B_2B_1$, $O_7 \equiv ABB_3B_4$, $O_8 \equiv ABB_3B_4B_5$,

$O_9 \equiv ABB_3B_2B_4$, $O_{10} \equiv ABB_3B_2B_4B_1$, $O_{11} \equiv ABB_3B_2B_4B_5$,

$O_{12} \equiv ABB_3B_2B_4B_1B_5$, $O_{13} \equiv AB^2B_3$, $O_{14} \equiv AB^2B_3B_2$,

$O_{15} \equiv AB^2B_3B_2B_1$, $O_{16} \equiv AB^2B_3B_4$, $O_{17} \equiv AB^2B_3B_4B_5$,

$O_{18} \equiv AB^2B_3B_2B_4$, $O_{19} \equiv AB^2B_3B_2B_4B_1$, $O_{20} \equiv AB^2B_3B_2B_4B_1B_5$,

$O_{21} \equiv AB^2B_3B_2B_4B_5$, $O_{22} \equiv AB^2B_3B$, $O_{23} \equiv AB^2B_3BB_2$,

$O_{24} \equiv AB^2B_3BB_2B_1$, $O_{25} \equiv AB^2B_3BB_2B_4$, $O_{26} \equiv AB^2B_3BB_2B_4B_1$,

$O_{27} \equiv AB^2B_3BB_4$, $O_{28} \equiv AB^2B_3BB_4B_5$,

$O_{29} \equiv AB^2B_3BB_2B_4B_1B_5$, $O_{30} \equiv AB^2B_3BB_2B_4B_5$,

$O_{31} \equiv AB^2B_3BB_2B_4B_3$, $O_{32} \equiv AB^2B_3BB_2B_4B_3B_1$,

$O_{33} \equiv AB^2B_3BB_2B_4B_3B_5$, $O_{34} \equiv AB^2B_3BB_2B_4B_3B_1B_5$,

$O_{35} \equiv AB^2B_3BB_2B_4B_3B_1B_2$, $O_{36} \equiv AB^2B_3BB_2B_4B_3B_1B_2B_5$.

In order to prove that the lines $O_1, \ldots, O_{36}$ of our table are merely permuted on applying as a right-hand multiplier any operator of the group H, we note several relations following from the generational relations (11)–(17). From (15), we have

$$BB_3B = B_1B_5B_3B^2B_3, \quad B^2B_3B^2 = B_1B_5B_3BB_3.$$

Hence $$ABB_3B = AB^2B_3, \quad AB^2B_3B^2 = ABB_3.$$

Similarly, $$AB^2B_3BB_3 = AB^2B_3B,$$

$$O_{35}B_3 = AB^2B_3BB_2B_4B_1 \,.\, B_2B_3B_2 = AB^2B_3BB_4 \,.\, B_1B_2B_1 \,.\, B_3B_2 = O_{35},$$

since the first B_1 may be carried to the left and merged into A. From (11)–(14) with (17) it follows that

$$(18) \quad AB^2B_3BB_2B_1B_3 = AB^2B_3BB_2.$$

From (16), using (11)–(14), it follows that

$$(19) \quad AB^2B_3B_2B_4B_3 = ABB_3B_2B_4B_1B_5.$$

Indeed, the condition for (19) is that

$$B^2B_3B_2B_4B_3 \,.\, B_5B_1B_4B_2B_3B^2$$

shall belong to A. But, by (16),

$$B^2(B_1B_5B_3B_2B_4)^3B^2 = B_1B_5(B^2B_3B_2B_4B_3B_5B_1B_4B_2B_3B^2)B_1B_5B_2B_4$$

belongs to A.

We next verify that $O_{31}B = O_{31}$. Indeed, applying (15),

$$O_{31}B = AB^2B_3B_2B_4 \,.\, B_1B_5B_3B^2B_3 = ABB_3B_2B_4B^2B_3$$
$$= ABB_3B^2B_2B_4B_3 = AB^2B_3BB_2B_4B_3 = O_{31},$$

where the second equality follows by using (19). It follows that

$$O_jB = O_j \quad (j = 31, 32, 33, 34, 35, 36).$$

We may now verify by inspection of our table that B applied as a right-hand multiplier interchanges the lines $O_1, \ldots, O_{36}$ according to the substitution

$$[B]: \quad (O_1O_2O_3)(O_4O_{13}O_{22})(O_5O_{14}O_{23})(O_6O_{15}O_{24})(O_7O_{16}O_{27})$$
$$(O_8O_{17}O_{28})(O_9O_{18}O_{25})(O_{10}O_{19}O_{26})(O_{11}O_{21}O_{30})(O_{12}O_{20}O_{29}).$$

If we note that the operator, belonging to A,

$$P \equiv (B_1B_5B_3B_2B_4)^3$$

corresponds in the group A_{720} to the Abelian substitution P_{12} which transforms M_1, L_1, S, L_2, M_2 into respectively M_2, L_2, S, L_1, M_1, we find that P transforms [see (16)] B_1, B_2, B_3, B_4, B_5, B into respectively B_5, B_4, B_3, B_2, B_1, B^2. Hence, by transforming relation (18) by P, we find, since

$$ABB_3B^2 = AB^2B_3B,$$

$$(20) \quad AB^2B_3BB_4B_5B_3 = AB^2B_3BB_4.$$

The condition that $O_{35}B_1 = O_{36}$ is that A shall contain

$$B^2B_3BB_2B_4B_3B_5(B_1B_2B_1B_2B_1)B_3B_4B_2B^2B_3B$$
$$\equiv B^2B_3BB_4B_5(B_2B_3B_2B_3B_2)B_4B^2B_3B$$
$$\equiv B^2B_3BB_4B_5B_3 \,.\, B_4B^2B_3B.$$

But, by (20), the latter product belongs to A. Since the second product belongs to A, we have at once

$$O_{31}B_2 = O_{33}.$$

Transforming this result by P, we find

$$O_{31}B_4 = O_{32}.$$

Hence $$O_{33}B_4 = O_{32}B_2 \equiv O_{35}.$$

It follows without further calculation or device that B_1, B_2, B_3, B_4, B_5 give rise to the following substitutions on the O_j, when applied as right-hand multipliers :—

$$[B_1]:\quad (O_5O_6)(O_9O_{10})(O_{11}O_{12})(O_{14}O_{15})(O_{18}O_{19})(O_{20}O_{21})(O_{23}O_{24})(O_{25}O_{26})(O_{29}O_{30})(O_{31}O_{32})(O_{33}O_{34})(O_{35}O_{36}),$$

$$[B_2]:\quad (O_4O_5)(O_7O_9)(O_8O_{11})(O_{13}O_{14})(O_{16}O_{18})(O_{17}O_{21})(O_{22}O_{23})(O_{25}O_{27})(O_{28}O_{30})(O_{31}O_{33})(O_{32}O_{35})(O_{34}O_{36}),$$

$$[B_3]:\quad (O_2O_4)(O_3O_{13})(O_9O_{20})(O_{10}O_{21})(O_{11}O_{19})(O_{12}O_{18})(O_{23}O_{24})(O_{25}O_{31})(O_{27}O_{28})(O_{26}O_{32})(O_{29}O_{34})(O_{30}O_{33}),$$

$$[B_4]:\quad (O_4O_7)(O_5O_9)(O_6O_{10})(O_{13}O_{16})(O_{14}O_{18})(O_{15}O_{19})(O_{22}O_{27})(O_{23}O_{28})(O_{24}O_{26})(O_{31}O_{32})(O_{33}O_{35})(O_{34}O_{36}),$$

$$[B_5]:\quad (O_7O_8)(O_9O_{11})(O_{10}O_{12})(O_{16}O_{17})(O_{18}O_{21})(O_{19}O_{20})(O_{25}O_{30})(O_{26}O_{29})(O_{27}O_{28})(O_{31}O_{33})(O_{32}O_{34})(O_{35}O_{36}).$$

Each of the five substitutions $[B_j]$ is the product of twelve permutations, and thus leaves twelve letters fixed.

It follows that the substitution-group $[H]$ on thirty-six letters, generated by the substitutions $[B]$ and $[B_j]$ ($j = 1, \ldots, 5$), is isomorphic with the abstract group H. As a check upon the calculations, we readily verify that

$$[B_2][B][B_2] = [B],\quad [B_1][B][B_1] = [B],\quad [B_1][B_3][B_1] = [B_3],$$
$$[B_2][B_3][B_2] = [B_3][B_2][B_3],\ \&c.$$

It may be shown in several ways that the order of H is

$$36 \,.\, 720 = 25920.$$

It follows at once from the above investigation that its order does not exceed 36 times the order of A, which is 720. We proceed to prove that the order of H is at least $36 \,.\, 720$. We obtain a set of substitutions belonging to $HA\,(4, 2^2)$ which satisfy the generating relations (11)–(17) by setting B_1, B_2, B_3, B_4, B_5, B into correspondence with M_1, L_1, S, L_2, M_2, I respectively, where I denotes the substitution

$$I:\quad \xi_1' = I\xi_1,\quad \eta_1' = I\eta_1,\quad \xi_2' = I^2\xi_2,\quad \eta_2' = I^2\eta_2,$$

I being a root of the congruence irreducible modulo 2,

$$I^2 \equiv I+1 \pmod{2}.$$

Then I is of period 3 and satisfies the relations corresponding to (14)–(17). Indeed, I is commutative with M_1, L_1, L_2, M_2, while

$$(15_1) \quad (IS)^3 = M_1 M_2.$$

Since
$$P \equiv (B_1 B_5 B_3 B_2 B_4)^3$$
corresponds to $\quad (M_1 M_2 S L_1 L_2)^3 \equiv P_{12} \equiv (\xi_1 \xi_2)(\eta_1 \eta_2),$

we verify, by inspection, that

$$(16_1) \quad I P_{12} I = P_{12}.$$

By actual calculation, we find

$$(17_1) \quad (M_1 L_1 S M_1 L_1)(I^2 S I)(M_1 L_1 S M_1 L_1)(I^2 S I) = M_2.$$

It would remain to prove that M_1, L_1, S, L_2, M_2, I generate $HA\,(4, 2^2)$ of order 25920. This follows from the fact that the substitutions in the rectangular array corresponding to the above $\{O_1, O_2, \ldots, O_{36}\}$ are all distinct.* We give here another method, which will prove the order of H to be equal that of $SH\,(6, 2)$ and therefore 25920.

We next determine a set of *generators* of $SH\,(6, 2)$ which satisfy the generational relations (11)–(17). By means of the writer's theory† of the second compound of a given linear group, we may set up a simple isomorphism between the Abelian group A_{720} and the sub-group L_{720} of $SH\,(6, 2)$, which sub-group is defined by the invariants

$$x_1 + y_1, \quad x_1 y_1 + x_2 y_2 + x_3 y_3.$$

To the general substitution of A_{720},

$$(21) \quad \begin{array}{c} \xi_1 \\ \eta_1 \\ \xi_2 \\ \eta_2 \end{array} \left\{ \begin{array}{cccc} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{array} \right\},$$

---

* In my earlier work, I verified this result directly, using two (unpublished) general lemmas and making use of the separation of the thirty-six O_j's into the five sets given in the table of § 6.

† *Bulletin of the American Mathematical Society*, December, 1898; *Proceedings of the London Mathematical Society*, Vol. xxx., pp. 70–98, 1899.

there corresponds the following substitution of L_{720},

$$\begin{array}{c} x_1 \\ \\ x_2 \\ x_3 \\ y_3 \\ y_2 \\ \\ y_1 \end{array}\left\{\begin{array}{cccc} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, & \ldots, & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}, & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, & \ldots, & \begin{vmatrix} a_{13} & a_{14} \\ a_{33} & a_{34} \end{vmatrix} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix}, & \begin{vmatrix} a_{31} & a_{33} \\ a_{41} & a_{43} \end{vmatrix}, & \ldots, & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{array}\right\},$$

built by the rule for forming the second compound of the determinant of the fourth order $| a_{ij} |$. In particular, to

$$M_1,\ L_1,\ S,\ L_2,\ M_2,\ P_{12},$$

correspond in the second compound the following substitutions :—

$$(b) \quad P_{23}M_2M_3,\ N_{23},\ \Sigma,\ Q_{23},\ P_{23},\ M_1M_3,$$

where
$$\Sigma \equiv \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ y_3 \\ y_2 \\ y_1 \end{array}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Letting the first five substitutions (b) correspond to B_1, B_2, B_3, B_4, B_5, respectively, it follows that the relations corresponding to (11), (12), (13) are satisfied by the substitutions (b). Now, the hypo-Abelian substitution L_1M_1 is commutative with $P_{23}M_2M_3$, N_{23}, Q_{23}, and P_{23} (since it affects only x_1, y_1, which are not altered by the latter four substitutions). Further,

$$(L_1M_1)^2 = 1, \quad (L_1M_1\Sigma)^3 = M_2M_3, \quad L_1M_1 \,.\, M_1M_3 \,.\, L_1M_1 = M_1M_3.$$

Hence, if we let $L_1M_1 \sim B$, the relations corresponding to (14), (15), (16) will be satisfied. The relation corresponding to (17) may be written

$$(17_2) \quad (M_1L_1\Sigma L_1M_1)\,V = W\,(M_1L_1\Sigma L_1M_1),$$

where V and W are the second compounds of the substitutions

$$L_1 M_1 S L_1, \quad M_1 L_1 M_1 S L_1 M_1 M_2,$$

respectively. It is easily shown that

$$V \equiv \begin{Bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{Bmatrix}, \quad W \equiv \begin{Bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{Bmatrix}.$$

The relation (17_2) is then readily verified. It follows that $SH(6, 2)$ is isomorphic with the abstract group H. We readily verify, indeed, that the substitutions (b) together with $L_1 M_1$ generate the group $SH(6, 2)$. To the substitution N_{12} of A_{720} corresponds the substitution $N_{12} Q_{21}$ of L_{720}. Hence $N_{12} Q_{21}$ can be derived from the substitutions (b). It is transformed into $Q_{12} R_{12}$ by $M_2 M_3$. We therefore reach the substitution

$$L_1 M_1 \,.\, N_{12} Q_{21} \,.\, L_1 M_1 \,.\, R_{12} Q_{12} \,.\, L_1 M_1 = U.$$

But* the substitutions $M_2 M_3$, N_{23}, Q_{23}, $L_1 M_1$, $M_1 M_2$, U generate the second hypo-Abelian group $SH(6, 2)$.

Note.—By § 4, the abstract group A is generated by the two operators B_1 and $C \equiv B_5 B_4 B_3 B_2 B_1$. We may verify that the corresponding substitution $[C] \equiv [B_5][B_4][B_3][B_2][B_1]$ has the form

$$[C]: \quad (O_2 O_6 O_{17} O_3 O_{15} O_8)(O_4 O_{10} O_{16} O_{20} O_{14} O_{11})(O_5 O_{21} O_{13} O_{19} O_7 O_{12})$$
$$(O_{23} O_{34} O_{27} O_{26} O_{35} O_{30})(O_9 O_{18})(O_{22} O_{29})(O_{24} O_{36} O_{28})(O_{25} O_{32} O_{33}).$$

A further check is obtained by forming the product $[B_1][C]$:

$$(O_2 O_6 O_{21} O_{14} O_8)(O_3 O_{15} O_{11} O_5 O_{17})(O_4 O_{10} O_{18} O_7 O_{12})(O_9 O_{16} O_{20} O_{13} O_{19})$$
$$(O_{22} O_{29} O_{23} O_{36} O_{30})(O_{24} O_{34} O_{25} O_{35} O_{28})(O_{26} O_{32} O_{31} O_{33} O_{27}).$$

The latter is of period 5 while $[C]$ is of period 6.

---

* "The Structure of the Hypo-Abelian Groups," *Bulletin of the American Mathematical Society*, July, 1898. See, in particular, §§ 10, 11.

6. *The Operators of* $HA\,(4,\,2^2)$ *of Period* 2.

The quaternary Abelian group modulo 2 may (by § 4) be put into simple isomorphism with the symmetric group on six letters by the following correspondence of generators:—

$$\begin{array}{cccccc} & M_1, & L_1, & S, & L_2, & M_2 \\ \sim & (12), & (23), & (34), & (45), & (56). \end{array}$$

Within the symmetric group, every substitution of period 2 is the conjugate of either (12), (12)(34), or (12)(34)(56). Hence, within the Abelian group, every substitution of period 2 is conjugate with either M_1, M_1S, or M_1M_2S. The latter three substitutions have each the characteristic determinant $(\rho+1)^4$, but are not conjugate within the Abelian group. The latter two are, however, conjugate within the hyper-Abelian group $HA\,(4,\,2^2)$. Indeed,

$$\left\{\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & \lambda & \lambda \\ \lambda^2 & 0 & 1 & 0 \\ \lambda^2 & 0 & 0 & 1 \end{array}\right\} \quad [\lambda^2 \equiv \lambda+1 \pmod 2]$$

belongs to $HA\,(4,\,2^2)$ and transforms M_1S into M_1M_2S. But there exists no quaternary linear substitution belonging to the $GF[2^n]$ which transforms M_1 into M_1S. Indeed, if such a substitution be

$$(22)\qquad T = \left\{\begin{array}{cccc} \alpha_{11} & \gamma_{11} & \alpha_{12} & \gamma_{12} \\ \beta_{11} & \delta_{11} & \beta_{12} & \delta_{12} \\ \alpha_{21} & \gamma_{21} & \alpha_{22} & \gamma_{22} \\ \beta_{21} & \delta_{21} & \beta_{22} & \delta_{22} \end{array}\right\},$$

the conditions for the identity $M_1T = TM_1S$ are seen to be as follows:—

$$\alpha_{22} = \beta_{22}, \quad \gamma_{22} = \delta_{22}, \quad \alpha_{12} = \beta_{12}, \quad \gamma_{12} = \delta_{12},$$

$$\alpha_{11}+\gamma_{11} = \beta_{11}+\delta_{11} = \alpha_{21}+\beta_{21} = \gamma_{21}+\delta_{21},$$

$$\alpha_{11}+\beta_{11} = \gamma_{11}+\delta_{11} = \alpha_{21}+\delta_{21} = \gamma_{21}+\beta_{21}.$$

The determinant of T would then be zero, as seen immediately upon adding the first to the second row and the third to the fourth row.

THEOREM. — *Within the simple group* $HA\,(4,\,2^2)$ *the operators of period* 2 *fall into two distinct sets of conjugates,* 45 *operators being conjugate with* M_1, *and* 270 *with* M_1S.

From the form (see § 1) of the reciprocal of the general substitution S of $HA\ (4, 2^2)$ it follows by inspection that every such substitution of period 2 has the form

$$(23)\qquad \begin{Bmatrix} \alpha_{11} & \gamma_{11} & \alpha_{12} & \gamma_{12} \\ \beta_{11} & \alpha_{11}^2 & \beta_{12} & \delta_{12} \\ \delta_{12}^2 & \gamma_{12}^2 & \alpha_{22} & \gamma_{22} \\ \beta_{12}^2 & \alpha_{12}^2 & \beta_{22} & \alpha_{22}^2 \end{Bmatrix},$$

where $\gamma_{11}, \beta_{11}, \gamma_{22}, \beta_{22}$ are integers modulo 2. The coefficients of (23) are also subject to the hyper-Abelian conditions (2) or (3).

By § 5, the 36 . 720 substitutions of $HA\ (4, 2^2)$ are given uniquely by the following table, where A denotes the quaternary Abelian group modulo 2 having the order 720 :—

$$\begin{array}{lll} (3.720) & AI^t & (t = 0, 1, 2), \\ (18.720) & AI^\tau SL_1^a M_1^{a\alpha} L_2^b M_2^{\beta b} & (\tau = 1, 2;\ a, b, \alpha, \beta = 0, 1), \\ (9.720) & AI^2SIL_1^c M_1^{\gamma c} L_2^d M_2^{\delta d} & (c, d, \gamma, \delta = 0, 1), \\ (4.720) & AI^2SIL_1L_2SM_1^r M_2^s & (r, s = 0, 1), \\ (2.720) & AI^2SIL_1L_2SM_1L_1M_2^\sigma & (\sigma = 0, 1). \end{array}$$

Since S, L_1, M_1, L_2, M_2 belong to A, and since P_{12} transforms AI^2 into AI, it follows that every substitution of $HA\ (4, 2^2)$ is conjugate within the latter with some substitution of the form $A_1 I$ or $A'I^2SI$. If a substitution $A_1 I$ have the period 2, it is readily shown that

$$IA_1I = A_1^{-1} = A_1 = \begin{Bmatrix} 0 & 0 & \alpha & \gamma \\ 0 & 0 & \beta & \delta \\ \delta & \gamma & 0 & 0 \\ \beta & \alpha & 0 & 0 \end{Bmatrix},$$

where $$\alpha\delta - \beta\gamma \equiv 1 \pmod 2.$$

There are six substitutions of the type A_1, each conjugate with I'_{12}. Indeed, by transforming A_1 by $1, M_1, M_2$, or M_1M_2, we can take $\alpha \equiv 1$.

If $\alpha \equiv 1$, then $$\beta \equiv \gamma \equiv 1 \pmod 2.$$

Transforming by $M_1 L'_{2,1}$, we reach

$$\begin{Bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{Bmatrix} \equiv P_{12}.$$

If $\delta \equiv 1$, then $\beta\gamma \equiv 0$. Transforming by $M_1 M_2$ if necessary, we can suppose $\gamma = 0$. Transforming by $L'_{2,1}$, if $\beta = 1$, we can make $\beta = 0$. The resulting substitution is P_{12}. But P_{12} corresponds* to (16)(25)(34). Hence every substitution A is conjugate with $M_1 M_2 S$, and therefore with $M_1 S$.

It follows that there are 20 . 6 substitutions of period 2 falling under the types

$$A\Gamma, \quad A\Gamma S L_1^a M_1^{\alpha a} L_2^b M_2^{\beta b} \quad (r = 1, 2;\ a, b, \alpha, \beta = 0, 1),$$

all being conjugate with $M_1 S$.

Of the substitutions of period 2 of type A, 15 are conjugate with M_1 and 60 with $M_1 S$. Indeed, there are 15 distinct substitutions on six letters of the type (ab), 3 . 15 of the type $(ab)(cd)$, and 15 of the type $(ab)(cd)(ef)$.

Taking as A' the general substitution (22), with coefficients integers modulo 2, we find for the product $A'\Gamma^2 S\Gamma$ the expression

$$\begin{Bmatrix} \beta_{11}+\Gamma^2(\alpha_{21}+\beta_{21}), & \delta_{11}+\Gamma^2(\gamma_{21}+\delta_{21}), & \beta_{12}+\Gamma^2(\alpha_{22}+\beta_{22}), & \delta_{12}+\Gamma^2(\gamma_{22}+\delta_{22}) \\ \alpha_{11}+\Gamma^2(\alpha_{21}+\beta_{21}), & \gamma_{11}+\Gamma^2(\gamma_{21}+\delta_{21}), & \alpha_{12}+\Gamma^2(\alpha_{22}+\beta_{22}), & \gamma_{12}+\Gamma^2(\gamma_{22}+\delta_{22}) \\ \beta_{21}+\Gamma(\alpha_{11}+\beta_{11}), & \delta_{21}+\Gamma(\gamma_{11}+\delta_{11}), & \beta_{22}+\Gamma(\alpha_{12}+\beta_{12}), & \delta_{22}+\Gamma(\gamma_{12}+\delta_{12}) \\ \alpha_{21}+\Gamma(\alpha_{11}+\beta_{11}), & \gamma_{21}+\Gamma(\gamma_{11}+\delta_{11}), & \alpha_{22}+\Gamma(\alpha_{12}+\beta_{12}), & \gamma_{22}+\Gamma(\gamma_{12}+\delta_{12}) \end{Bmatrix}.$$

If it be of period 2, it must have the form (23). Hence, modulo 2,

$$\gamma_{21} \equiv \delta_{21}, \quad \alpha_{21} \equiv \beta_{21}, \quad \alpha_{12} \equiv \beta_{12}, \quad \gamma_{12} \equiv \delta_{12}, \quad \gamma_{11} \equiv \beta_{11}, \quad \gamma_{22} \equiv \beta_{22},$$

$$\alpha_{11}+\beta_{11} \equiv \gamma_{22}+\delta_{22} \equiv \gamma_{11}+\delta_{11} \equiv \alpha_{22}+\beta_{22},$$

$$\alpha_{21} \equiv \alpha_{12}, \quad \beta_{21} \equiv \gamma_{12}, \quad \gamma_{21} \equiv \beta_{12}, \quad \delta_{21} \equiv \delta_{12}.$$

---

* This follows from the identity

$$P_{12} \equiv (M_1 M_2 S L_1 L_2)^3,$$

or directly from the fact that $P_{12} \equiv (\xi_1\xi_2)(\eta_1\eta_2)$ transforms M_1, L_1, S into M_2, L_2, S respectively. Indeed, the letter common to (12) and (23) must be transformed into the letter common to (45) and (56). Hence $P_{12} \sim (25)(16)(34)$.

Expressing the fact that the resulting substitution A' shall be Abelian, we find a single new condition,

$$\alpha_{11}^2+\gamma_{11}^2 \equiv 1 \quad \text{or} \quad \gamma_{11} \equiv \alpha_{11}+1 \quad (\text{mod } 2).$$

Hence A' takes the form

$$(24) \qquad \begin{Bmatrix} a_{11} & a_{11}+1 & a_{12} & a_{12} \\ a_{11}+1 & a_{11} & a_{12} & a_{12} \\ a_{12} & a_{12} & a_{22} & a_{22}+1 \\ a_{12} & a_{12} & a_{22}+1 & a_{22} \end{Bmatrix},$$

which has determinant unity and satisfies all Abelian conditions. It follows that $A'I^2SI$ has the form

$$\begin{Bmatrix} a_{11}+1 & a_{11} & a_{12}+I^2 & a_{12}+I^2 \\ a_{11} & a_{11}+1 & a_{12}+I^2 & a_{12}+I^2 \\ a_{12}+I & a_{12}+I & a_{22}+1 & a_{22} \\ a_{12}+I & a_{12}+I & a_{22} & a_{22}+1 \end{Bmatrix}.$$

Transforming by $L_{2,1}\, L_{1,1}$ we obtain the substitution

$$\begin{Bmatrix} 1 & 0 & 0 & 0 \\ a_{11} & 1 & a_{12}+I^2 & 0 \\ 0 & 0 & 1 & 0 \\ a_{12}+I & 0 & a_{22} & 1 \end{Bmatrix}.$$

If $a_{12} \equiv 1$, we transform by the substitution I; if $a_{12} \equiv 0$, we transform by I^2. In either case the resulting substitution

$$B = \begin{Bmatrix} 1 & 0 & 0 & 0 \\ a_{11} & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & a_{22} & 1 \end{Bmatrix}$$

belongs to the Abelian group A, and is therefore conjugate with M_1 or with M_1S.

If $a_{11} \equiv a_{22} \equiv 0$, we find that

$$B = L_1L_2SM_2L_2M_1L_1,$$

and is therefore conjugate with SM_2M_1 and hence conjugate with M_1S.

By transforming B by P_{12}, the coefficients a_{22} and a_{11} are interchanged. It remains therefore to consider the case $a_{11} \equiv 1$. The transformed of B by the Abelian substitution

$$Q_{12}: \quad \xi_1' = \xi_1 + \xi_2, \quad \eta_2' = \eta_2 + \eta_1$$

gives the substitution
$$\left\{\begin{matrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_{22}+1 & 1 \end{matrix}\right\}.$$

For $a_{22} \equiv 1$, this equals $M_1 L_1 M_1$, which is conjugate with L_1, and hence with M_1. If $a_{22} \equiv 0$, it becomes $M_1 L_1 M_1 M_2 L_2 M_2$, which is conjugate with $L_1 L_2$, and therefore with $M_1 S$. Combining our results, B is conjugate with M_1 or with $M_1 S$ according as the product $a_{11} . a_{22}$ is congruent to 1 or 0 (mod 2) respectively. Hence the substitutions of period 2 in the last three types in our table are of two classes, the number of them conjugate with M_1 being 15 . 2, and the number conjugate with $M_1 S$ being 15 . 6.

7. *Further Theorems on the* $HA\,(4, 2^2)$ *and* $SH\,(6, 2)$.

THEOREM.—*The number of sub-groups of* $SH\,(6, 2)$ *conjugate with the sub-group* $SH\,(4, 2)$ *leaving invariant* $x_1 + y_1 + x_1 y_1 + x_2 y_2$ *is* 216.

I reach this theorem by proving that every substitution of $SH\,(6, 2)$ which transforms the group $SH\,(4, 2)$ into itself is of the form S or $S M_1 M_3$, where S belongs to $SH\,(4, 2)$. Hence the number of conjugates with the latter within $SH\,(6, 2)$ is

$$25920/120 \equiv 216.$$

Indeed, $SH\,(4, 2)$ is simply isomorphic with the simple group of order 60, the statement of Jordan (*Traité des Substitutions*, § 291) being incorrect.*

THEOREM.—*There are exactly* 216 *sub-groups of* $HA\,(4, 2^2)$ *conjugate with the simple sub-group of order* 60 *composed of all the substitutions of* $HA\,(4, 2^2)$ *of the form*

$$\left\{\begin{matrix} \xi_2' = \alpha\xi_2 + \beta\xi_4, & \xi_1' = \delta^2\xi_1 - \gamma^2\xi_3 \\ \xi_4' = \gamma\xi_2 + \delta\xi_4, & \xi_3' = -\beta^2\xi_1 + \alpha^2\xi_3 \end{matrix}\right\} \quad (\alpha\delta - \beta\gamma = 1).$$

---

* Dickson, "The Structure of the Hypo-Abelian Groups," *Bulletin of the American Mathematical Society*, pp. 495–510, 1898.

[redacted]

I obtain this result by proving that every hyper-Abelian substitution of determinant unity on four indices in the $GF[p^n]$ which transforms the G_{60} into itself is of the form g or $g\ (\xi_1\xi_2)(\xi_3\xi_4)$, where g belongs to G_{60}.

I have established an analogous theorem for the group $HA\ (4, p^n)$.

We next consider certain properties of the Abelian group $A\ (4, 3)$, which will enable us to prove it isomorphic with $HA\ (4, 2^2)$.

8. *The Operators of $A\ (4, p^n)$ of Period 2.*

THEOREM.—*Within the simple Abelian group A (4, 3) the operators of period 2 fall into two distinct sets of conjugate operators, the one represented by P_{12} and the other by $M_1 M_2$.*

The Abelian group on four indices in the $GF[p^n]$, $p>2$, has as maximal invariant sub-group the group of order 2 generated by the substitution

$$C:\quad \xi_1' = -\xi_1,\quad \eta_1' = -\eta_1,\quad \xi_2' = -\xi_2,\quad \eta_2' = -\eta_2.$$

Hence, to obtain the simple quotient group $A\ (4, p^n)$, we have merely to consider the products $SC \equiv CS$ to be identical with S.

In order that the substitution of $A\ (4, p^n)$

$$S:\quad \left\{ \begin{aligned} \xi_i' &= \pm \sum_j^{1,2} (\alpha_{ij}\xi_j + \gamma_{ij}\eta_j) \\ \eta_i' &= \pm \sum_j^{1,2} (\beta_{ij}\xi_j + \delta_{ij}\eta_j) \end{aligned} \right\} \quad (i = 1, 2)$$

shall be of period 2, it is necessary and sufficient that

$$\left\{ \begin{matrix} \alpha_{11} & \gamma_{11} & \alpha_{12} & \gamma_{12} \\ \beta_{11} & \delta_{11} & \beta_{12} & \delta_{12} \\ \alpha_{21} & \gamma_{21} & \alpha_{22} & \gamma_{22} \\ \beta_{21} & \delta_{21} & \beta_{22} & \delta_{22} \end{matrix} \right\} \equiv \pm \left\{ \begin{matrix} \delta_{11} & -\gamma_{11} & \delta_{21} & -\gamma_{21} \\ -\beta_{11} & \alpha_{11} & -\beta_{21} & \alpha_{21} \\ \delta_{12} & -\gamma_{12} & \delta_{22} & -\gamma_{22} \\ -\beta_{12} & \alpha_{12} & -\beta_{22} & \alpha_{22} \end{matrix} \right\}.$$

(1) Taking first the plus sign, we have

$$S = \pm \left\{ \begin{matrix} \alpha_{11} & 0 & \alpha_{12} & \gamma_{12} \\ 0 & \alpha_{11} & \beta_{12} & \delta_{12} \\ \delta_{12} & -\gamma_{12} & \alpha_{22} & 0 \\ -\beta_{12} & \alpha_{12} & 0 & \alpha_{22} \end{matrix} \right\}.$$

The conditions that S shall be Abelian are the following:—

$$a_{11}^2+a_{12}\delta_{12}-\beta_{12}\gamma_{12}=1, \quad a_{22}^2+a_{12}\delta_{12}-\beta_{12}\gamma_{12}=1,$$

$$a_{12}(a_{11}+a_{22}) = \gamma_{12}(a_{11}+a_{12}) = \beta_{12}(a_{11}+a_{22}) = \delta_{12}(a_{11}+a_{22}) = 0.$$

Hence must $a_{11}+a_{22}=0$; for, otherwise,

$$a_{12}=\gamma_{12}=\beta_{12}=\delta_{12}=0, \quad a_{11}=a_{22}=\pm 1,$$

when S would be the identity in the group $A(4, p^n)$. Hence, in the case under consideration, S has the form

$$(25) \qquad \pm\left\{\begin{array}{cccc} a_{11} & 0 & a_{12} & \gamma_{12} \\ 0 & a_{11} & \beta_{12} & \delta_{12} \\ \delta_{12} & -\gamma_{12} & -a_{11} & 0 \\ -\beta_{12} & a_{12} & 0 & -a_{11} \end{array}\right\} \quad [a_{11}^2+a_{12}\delta_{12}-\beta_{12}\gamma_{12}=1].$$

If the coefficients a_{12}, γ_{12}, β_{12}, δ_{12} are all zero, S becomes

$$T_{1,-1}: \quad \xi_1'=-\xi_1, \quad \eta_1'=-\eta_1, \quad \xi_2'=\xi_2, \quad \eta_2'=\eta_2.$$

In the contrary case, we may suppose $a_{12}\neq 0$. Indeed, if $a_{12}=0$, $\gamma_{12}\neq 0$, we transform S by M_2; if $\beta_{12}\neq 0$, we transform S by M_1; if $\delta_{12}\neq 0$, we transform by M_1M_2. If we transform S (in which now $a_{12}\neq 0$) by $L_{2,\lambda}$, we obtain a substitution S' of period 2 which replaces ξ_1 by

$$a_{11}\xi_1+a_{12}\xi_2+(\gamma_{12}-\lambda a_{12})\,\eta_2.$$

Hence, if $\gamma_{12}\neq 0$, we can choose λ to make the coefficient of η_2 vanish. Transforming S' (in which now $\gamma_{12}=0$, $a_{12}\neq 0$) by $L'_{1,\rho}$, we reach a substitution S'' which replaces ξ_1 and η by respectively

$$a_{11}\xi_1+a_{12}\xi_2, \quad a_{11}\eta_1+(\beta_{12}+\rho a_{12})\,\xi_2+\delta_{12}\eta_2.$$

By choice of ρ we can make the coefficient of ξ_2 vanish. Transforming S'' (in which $\gamma_{12}=\beta_{12}=0$, $a_{12}\neq 0$) by $Q_{1,2,\sigma}$, we obtain a substitution S''' replacing ξ_1 and η_1 by respectively

$$(a_{11}+\sigma\delta_{12})\,\xi_1+(a_{12}-2\sigma a_{11}-\sigma^2\delta_{12})\,\xi_2, \quad (a_{11}+\sigma\delta_{12})\,\eta_1+\delta_{12}\eta_2.$$

If $\delta_{12}\neq 0$, we can make the coefficients of ξ_1 and η_1 vanish, when

$$S'''=\pm\left\{\begin{array}{cccc} 0 & 0 & a_{12} & 0 \\ 0 & 0 & 0 & \delta_{12} \\ \delta_{12} & 0 & 0 & 0 \\ 0 & a_{12} & 0 & 0 \end{array}\right\} \quad [a_{12}\delta_{12}=1].$$

Transforming S''' by T_2, a_{12}, we obtain

$$P_{12} \equiv (\xi_1 \xi_2)(\eta_1 \eta_2).$$

If, however, $\delta_{12} = 0$ in S'', we transform it by $M_1 M_2$ and reach a substitution of the same form as S'', but having $\delta_{12} \neq 0$. From it we obtain P_{12} as before.

We readily determine an Abelian substitution which transforms P_{12} into $T_{1,-1}$. In fact, the most general one is

$$\left\{\begin{matrix} a_{11} & \gamma_{11} & -a_{11} & -\gamma_{11} \\ \beta_{11} & \delta_{11} & -\beta_{11} & -\delta_{11} \\ a_{21} & \gamma_{21} & a_{21} & \gamma_{21} \\ \beta_{21} & \delta_{21} & \beta_{21} & \delta_{21} \end{matrix}\right\},$$

subject only to the two conditions

$$2\,(a_{11}\delta_{11} - \beta_{11}\gamma_{11}) = 1, \quad 2\,(a_{21}\delta_{21} - \beta_{21}\gamma_{21}) = 1.$$

(2) Taking next the minus sign, we have

$$S = \pm \left\{\begin{matrix} a_{11} & \gamma_{11} & a_{12} & \gamma_{12} \\ \beta_{11} & -a_{11} & \beta_{12} & \delta_{12} \\ -\delta_{12} & \gamma_{12} & a_{22} & \gamma_{22} \\ \beta_{12} & -a_{12} & \beta_{22} & -a_{22} \end{matrix}\right\},$$

subject to the Abelian conditions.

(2_1) Suppose first that $a_{12}, \gamma_{12}, \beta_{12}, \delta_{12}$ are all zero. Then

$$-a_{11}^2 - \beta_{11}\gamma_{11} = 1.$$

Restricting our investigation henceforth to the case in which -1 is a square in the $GF[p^n]$, it follows that $\beta_{11}\gamma_{11} \neq 0$. Transforming S by $L_{1,\lambda}$, we reach a substitution S_1 replacing η_1 by

$$\beta_{11}\xi_1 - (a_{11} + \lambda\beta_{11})\,\eta_1.$$

Since $\beta_{22} \neq 0$, we can make the coefficient of η_1 vanish. Transforming S_1 by $L_{2,\rho}$, we can make a_{22} vanish in the resulting substitution S_2. Hence

$$S_2 = \pm \left\{\begin{matrix} 0 & -\beta_{11}^{-1} & 0 & 0 \\ \beta_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta_{22}^{-1} \\ 0 & 0 & \beta_{22} & 0 \end{matrix}\right\}.$$

[redacted]

For the case $p^n = 3$, we have, taking $\beta_{11} = -1$,

$$S_2 = M_1 M_2 \quad \text{or} \quad M_1 M_2 T_{2,-1}.$$

But M_2 transforms $M_1 M_2$ into $M_1 M_2 T_{2,-1}$.

(2_2) Suppose next that $\alpha_{12}, \gamma_{12}, \beta_{12}, \delta_{12}$ are not all zero. As in case (1) we may suppose that $\alpha_{12} \neq 0$, $\beta_{12} = \gamma_{12} = 0$. The Abelian conditions then give $\alpha_{12}(\alpha_{11} + \alpha_{22}) = 0$. Hence S has the form

$$(26) \qquad \pm \begin{Bmatrix} \alpha_{11} & \gamma_{11} & \alpha_{12} & 0 \\ \beta_{11} & -\alpha_{11} & 0 & \delta_{12} \\ -\delta_{12} & 0 & -\alpha_{11} & \gamma_{22} \\ 0 & -\alpha_{12} & \beta_{22} & \alpha_{11} \end{Bmatrix}.$$

Suppose first that $\beta_{11}, \gamma_{11}, \beta_{22}, \gamma_{22}$ are all zero. If also $\alpha_{11} = 0$, (26) becomes S''' of case (1). If $\alpha_{11} \neq 0$, we have $\alpha_{12}\delta_{12} = -1$. Taking $\alpha_{11} \equiv +1$, and transforming by $T_{2,\alpha_{12}}$, we obtain the substitution

$$X \equiv \pm \begin{Bmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{Bmatrix}.$$

But the following substitution of $A(4, 3)$,

$$\begin{Bmatrix} 0 & -1 & 1 & 1 \\ 1 & 0 & -1 & -1 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{Bmatrix},$$

transforms X into $M_1 M_2$.

If, however, $\beta_{11}, \gamma_{11}, \beta_{22}, \gamma_{22}$ do not all vanish, we can take $\beta_{11} \neq 0$. Indeed, if $\beta_{11} = 0$, $\gamma_{11} \neq 0$, we transform (26) by M_1; if $\beta_{22} \neq 0$, we transform (26) by P_{12}; if $\gamma_{22} \neq 0$, we transform by $P_{12} M_1$. Transforming (26), in which now $\beta_{11} \neq 0$, by $L_{1,\lambda}$, we obtain a substitution which replaces η_1 by

$$\beta_{11}\xi_1 - (\alpha_{11} + \lambda\beta_{11})\,\eta_1 + \delta_{12}\eta_2.$$

If $\alpha_{11} \neq 0$, we can make the coefficient of η_1 vanish. Transforming the resulting substitution by $L_{2,\lambda}$, we can make $\alpha_{12} \neq 0$, $\gamma_{12} = 0$. We now have a substitution of the form (26) in which $\alpha_{11} = 0$, $\beta_{11} \neq 0$, $\alpha_{12} \neq 0$. By one of the Abelian conditions,

$$-\beta_{11}\alpha_{12} - \delta_{12}\beta_{22} = 0.$$

Hence $\delta_{12} \neq 0$. Taking $\alpha_{12} \equiv 1$, $\delta_{12} \equiv \pm 1 \pmod 3$, the Abelian conditions give

$$\gamma_{22} \pm \gamma_{11} \equiv 0, \quad \beta_{11} \pm \beta_{22} \equiv 0, \quad -\beta_{11}\gamma_{11} \pm 1 \equiv -\beta_{22}\gamma_{22} \pm 1 \equiv 1,$$

where the upper signs hold simultaneously and likewise the lower.

Suppose first that $\delta_{12} \equiv -1$, so that

$$\beta_{11} \equiv \gamma_{11} \equiv \beta_{22} = \gamma_{22} \equiv \pm 1 \pmod 3.$$

Transforming by $T_{2,\,\pm 1}$, we have also $\alpha_{12} = \pm 1$, $\delta_{12} = \mp 1$. Hence (26) takes the form

$$\pm \begin{Bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{Bmatrix},$$

which is transformed into $M_1 M_2$ by the Abelian substitution $R_{1,2,-1}$.

Suppose, however, that $\delta_{12} = +1$, so that $\beta_{11}\gamma_{11} \equiv 0$. Then $\gamma_{11} \equiv 0$, $\beta_{11} \equiv \pm 1$. Transforming by $M_1 M_2 T_{2,\,\pm 1}$, we obtain the substitution

$$\pm \begin{Bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{Bmatrix},$$

which is transformed into $M_1 M_2$ by the Abelian substitution

$$\begin{Bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{Bmatrix}.$$

Finally, P_{12} and $M_1 M_2$ are not conjugate within A (4, 3). Indeed, the characteristic determinant of $M_1 M_2$ is

$$\begin{vmatrix} \rho & \pm 1 & 0 & 0 \\ \mp 1 & \rho & 0 & 0 \\ 0 & 0 & \rho & \pm 1 \\ 0 & 0 & \mp 1 & \rho \end{vmatrix} \equiv (\rho^2+1)^2,$$

while that of P_{12} is $(\rho^2-1)^2$.

We have therefore proven that an operator of A (4, 3) is conjugate with P_{12} if, and only if, it be of the form (25). The number of these operators is 45. Indeed, by setting

$$a_{12} = A+D, \quad \delta_{12} = A-D, \quad -\beta_{12} = B+C, \quad \gamma_{12} = B-C,$$

the single condition upon the coefficients of (25), viz.,

$$a_{11}^2 + a_{12}\delta_{12} - \beta_{12}\gamma_{12} \equiv 1 \pmod 3,$$

takes the form $$a_{11}^2 + A^2 + B^2 - D^2 - C^2 \equiv 1 \pmod 3.$$

The latter congruence has $3^4+3^2 \equiv 90$ sets of solutions.* Since, however, the coefficients in (25) are only determined up to a common factor ± 1, the number of corresponding operators in the quotient group A (4, 3) is one-half of 90.

It must therefore follow from §§ 6 and 11 that the number of operators of A (4, 3) conjugate with $M_1 M_2$ is 270, a result not used in our further work.

By the demonstration given in case (1), we have the theorem :—

The quaternary Abelian group in the $GF[p^n]$ *contains, in addition to the self-conjugate substitution* C, *exactly* $p^{4n}+p^{2n}$ *substitutions of period* 2, *all of the latter being conjugate within the Abelian group.*

9. Theorem.—*The simple Abelian group* A (4, 3) *contains sub-groups simply isomorphic with the symmetric group on six letters.*

We seek sub-groups isomorphic with the abstract group G_{720} generated by the operators B_1, B_2, B_3, B_4, B_5 subject to the generational relations (11), (12), (13). From the isomorphism of G_{720} with the substitution-group $G_{6!}^{(6)}$, it follows that the operators $B_1, \ldots, B_5$ are all conjugate within G_{720}. By § 8, there are two sets of conjugate

* Dickson, "Orthogonal Group in a Galois Field," *Bulletin of the America Mathematical Society*, February, 1898.

Abelian substitutions of period 2. From the set (25) conjugate with $T_{1,-1}$, I proceed to determine five substitutions, say $B_1, \ldots, B_5$, which satisfy the generational relations (11)–(13) of the group G_{720}. Applying an Abelian transformation, we may take $B_2 = T_{1,-1}$. Giving initially to B_1 the general form (25), we may simplify its form by applying any transformation as M_1, M_2, $L_{1,\rho}$, $L_{2,\rho}$ which does not alter $T_{1,-1}$. Proceeding therefore as in § 8, case (1), we may suppose that $\beta_{12} = \gamma_{12} = 0$, $a_{12} \neq 0$ in B_1, noting that the exceptional case $B_1 \equiv T_{1,-1}$ is evidently to be excluded. We have therefore

$$B_2 = T_{1,-1}, \quad B_1 = \pm \begin{Bmatrix} -a & 0 & a_1 & 0 \\ 0 & -a & 0 & a_2 \\ a_2 & 0 & a & 0 \\ 0 & a_1 & 0 & a \end{Bmatrix}.$$

The condition $\quad (B_1 B_2)^3 = 1 \quad$ or $\quad (B_1 B_2)^2 = B_2 B_1$

takes the form

$$\begin{Bmatrix} a^2 - a_1 a_2 & 0 & -2aa_1 & 0 \\ 0 & a^2 - a_1 a_2 & 0 & -2aa_2 \\ 2aa_2 & 0 & a^2 - a_1 a_2 & 0 \\ 0 & 2aa_1 & 0 & a^2 - a_1 a_2 \end{Bmatrix}$$

$$= \pm \begin{Bmatrix} a & 0 & a_1 & 0 \\ 0 & a & 0 & a_2 \\ -a_2 & 0 & a & 0 \\ 0 & -a_1 & 0 & a \end{Bmatrix}.$$

Since $a_1 \neq 0$, we must have

$$a \equiv \pm 1, \quad a_1 a_2 \equiv 0, \quad a_2 \equiv 0 \pmod 3.$$

Transforming by $T_{2,-1}$, if necessary, we may give to a_1 the same value as a. Hence

$$(27) \quad B_2 = \pm \begin{Bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Bmatrix}, \quad B_1 = \pm \begin{Bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{Bmatrix}.$$

Taking for B_3 the general substitution (25), the condition

$$(B_3B_2)^2 = (B_3B_2)^{-1}:$$

$$\cdot \left\{\begin{matrix} \tau & 0 & 2\alpha_{11}\alpha_{12} & 2\alpha_{11}\gamma_{12} \\ 0 & \tau & 2\alpha_{11}\beta_{12} & 2\alpha_{11}\delta_{12} \\ -2\alpha_{11}\delta_{12} & 2\alpha_{11}\gamma_{12} & \tau & 0 \\ 2\alpha_{11}\beta_{12} & -2\alpha_{11}\alpha_{12} & 0 & \tau \end{matrix}\right\}$$

$$= \pm \left\{\begin{matrix} -\alpha_{11} & 0 & \alpha_{12} & \gamma_{12} \\ 0 & -\alpha_{11} & \beta_{12} & \delta_{12} \\ -\delta_{12} & \gamma_{12} & -\alpha_{11} & 0 \\ \beta_{12} & -\alpha_{12} & 0 & -\alpha_{11} \end{matrix}\right\},$$

where $\qquad \tau \equiv \alpha_{11}^2 - \alpha_{12}\delta_{12} + \beta_{12}\gamma_{12},$

gives, since $\alpha_{12}, \gamma_{12}, \beta_{12}, \delta_{12}$ are not all zero $[B_3 \neq B_2]$,

$$2\alpha_{11} = \pm 1, \quad \tau = \mp \alpha_{11}.$$

Hence $\qquad \alpha_{12}\delta_{12} \equiv \beta_{12}\gamma_{12} \pmod 3.$

Taking $\alpha_{11} = +1$, the condition $B_3B_1 = B_1B_3$ becomes

$$\left\{\begin{matrix} \delta_{12}-1 & -\gamma_{12} & -\alpha_{12}-1 & -\gamma_{12} \\ 0 & -1 & -\beta_{12} & -\delta_{12} \\ \delta_{12} & -\gamma_{12} & -1 & \\ -\beta_{12} & \alpha_{12}+1 & \beta_{12} & \delta_{12}-1 \end{matrix}\right\}$$

$$= \pm \left\{\begin{matrix} -1 & \gamma_{12} & \alpha_{12}+1 & \gamma_{12} \\ 0 & \delta_{12}-1 & \beta_{12} & \delta_{12} \\ -\delta_{12} & \gamma_{12} & \delta_{12}-1 & 0 \\ \beta_{12} & -\alpha_{12}-1 & -\beta_{12} & -1 \end{matrix}\right\}.$$

The upper sign cannot be chosen; for then

$$\beta_{12} = \gamma_{12} = \delta_{12} = 0, \quad \alpha_{12} = -1,$$

when $B_3 = B_1$. Taking therefore the lower sign, the only condition is seen to be $\delta_{12} = -1$. Dropping all subscripts, B_3 takes the form

$$(28) \quad B_3 = \pm \left\{\begin{matrix} 1 & 0 & -\beta\gamma & \gamma \\ 0 & 1 & \beta & -1 \\ -1 & -\gamma & -1 & 0 \\ -\beta & -\beta\gamma & 0 & -1 \end{matrix}\right\}.$$

Taking for B_4 the general substitution (25), the condition

$$(B_2 B_4)^2 = 1$$

gives

$$B_2 B_4 B_2 = B_4,$$

or

$$\left\{\begin{matrix} \alpha_{11} & 0 & -\alpha_{12} & -\gamma_{12} \\ 0 & \alpha_{11} & -\beta_{12} & -\delta_{12} \\ -\delta_{12} & \gamma_{12} & -\alpha_{11} & 0 \\ \beta_{12} & -\alpha_{12} & 0 & -\alpha_{11} \end{matrix}\right\} = \pm \left\{\begin{matrix} \alpha_{11} & 0 & \alpha_{12} & \gamma_{12} \\ 0 & \alpha_{11} & \beta_{12} & \delta_{12} \\ \delta_{12} & -\gamma_{12} & -\alpha_{11} & 0 \\ -\beta_{12} & \alpha_{12} & 0 & -\alpha_{11} \end{matrix}\right\}.$$

The upper sign would require that $B_4 \equiv B_2$. Taking therefore the lower sign, we have $\alpha_{11} = 0$ as the only condition. The condition

$$(B_1 B_4)^2 = 1$$

gives

$$B_4 B_1 = B_1 B_4,$$

viz.,

$$\left\{\begin{matrix} \delta_{12} & -\gamma_{12} & -\alpha_{12} & -\gamma_{12} \\ 0 & 0 & -\beta_{12} & -\delta_{12} \\ \delta_{12} & -\gamma_{12} & 0 & 0 \\ -\beta_{12} & \alpha_{12} & \beta_{12} & \delta_{12} \end{matrix}\right\} = \pm \left\{\begin{matrix} 0 & \gamma_{12} & \alpha_{12} & \gamma_{12} \\ 0 & \delta_{12} & \beta_{12} & \delta_{12} \\ -\delta_{12} & \gamma_{12} & \delta_{12} & 0 \\ \beta_{12} & -\alpha_{12} & -\beta_{12} & 0 \end{matrix}\right\}.$$

The upper sign requires $\alpha_{12} = \gamma_{12} = \beta_{12} = \delta_{12} = 0$. We must therefore take the lower sign, when the only condition is $\delta_{12} = 0$. The Abelian condition then gives $-\beta_{12}\gamma_{12} \equiv 1 \pmod 3$. Taking $\gamma_{12} = +1$ and writing $\alpha \equiv \alpha_{12}$, we have

$$(29) \quad B_4 = \pm \left\{\begin{matrix} 0 & 0 & \alpha & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & \alpha & 0 & 0 \end{matrix}\right\}.$$

We consider next the condition

$$(B_3 B_4)^3 = 1.$$

We find that $(B_3 B_4)^2$ becomes

$$\left\{\begin{matrix} (\alpha+\beta)^2-\alpha\gamma-\beta\gamma-1 & (\alpha+\beta-\gamma)(\alpha\gamma+\beta\gamma) & \alpha(\alpha+\beta-\gamma) & \alpha+\beta-\gamma \\ -\alpha-\beta+\gamma & \gamma(-\alpha-\beta+\gamma)-1 & -\alpha-\beta+\gamma & 0 \\ 0 & \alpha+\beta-\gamma & \beta(\alpha+\beta-\gamma)-1 & -\alpha-\beta+\gamma \\ -\alpha-\beta+\gamma & \alpha(-\alpha-\beta+\gamma) & (\alpha+\beta-\gamma)(\beta\gamma-\alpha\beta) & (\gamma-\alpha)^2+\alpha\beta-\beta\gamma-1 \end{matrix}\right\}.$$

This must equal the substitution $B_4 B_3$, viz.,

$$\pm\begin{Bmatrix} \gamma & \alpha\gamma+\beta\gamma & \alpha & 1 \\ -1 & -\alpha-\beta & -1 & 0 \\ 0 & 1 & -\alpha+\gamma & -1 \\ -1 & -\alpha & -\alpha\beta+\beta\gamma & -\beta \end{Bmatrix}.$$

The resulting conditions all reduce to identities in virtue of the single condition

$$(30)\quad \alpha+\beta-\gamma\equiv\pm 1.$$

Assuming B_5 in the general form (25), the conditions

$$(B_1 B_5)^2=1,\quad (B_2 B_5)^2=1$$

give

$$\alpha_{11}=0,\quad \delta_{12}=0,$$

as found above in considering B_4. Hence

$$(31)\quad B_5=\pm\begin{Bmatrix} 0 & 0 & \alpha_{12} & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & \alpha_{12} & 0 & 0 \end{Bmatrix}.$$

Then

$$(B_4 B_5)^3=1$$

is satisfied identically. The condition

$$(B_3 B_5)^2=1$$

gives

$$B_3 B_5=B_5 B_3:$$

$$\begin{Bmatrix} -\beta-\alpha_{12} & -\beta\gamma-\alpha_{12}\gamma & -\alpha_{12} & -1 \\ 1 & \gamma & 1 & 0 \\ 0 & -1 & -\beta & 1 \\ 1 & \alpha_{12} & -\beta\gamma+\beta\alpha_{12} & \gamma-\alpha_{12} \end{Bmatrix}$$

$$=\pm\begin{Bmatrix} \gamma & \beta\gamma+\alpha_{12}\gamma & \alpha_{12} & 1 \\ -1 & -\beta-\alpha_{12} & -1 & 0 \\ 0 & 1 & \gamma-\alpha_{12} & -1 \\ -1 & -\alpha_{12} & \beta\gamma-\beta\alpha_{12} & -\beta \end{Bmatrix}.$$

The lower sign must be chosen, when we have the single condition

$$(32)\quad \alpha_{12}+\beta-\gamma\equiv 0.$$

For every set of values α, β, γ, a_{12} satisfying the conditions (30) and (32), we therefore obtain a sub-group of $A(4, 3)$ simply isomorphic with the group G_{720}. We next prove that all of these sub-groups are conjugate within $A(4, 3)$. By the above results, the following substitution belonging to $A(4, 3)$,

$$(33)\qquad \begin{Bmatrix} 0 & 0 & -a_{12} & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & -a_{12} & 0 & 0 \end{Bmatrix},$$

transforms B_1 and B_2 into themselves. Further, it transforms B_5, B_3, and B_4 into, respectively,

$$\pm\begin{Bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{Bmatrix}, \quad \mp\begin{Bmatrix} 1 & 0 & \tau & -a_{12}+\beta \\ 0 & 1 & a_{12}+\gamma & -1 \\ -1 & a_{12}-\beta & -1 & 0 \\ -a_{12}-\gamma & \tau & 0 & -1 \end{Bmatrix},$$

$$\pm\begin{Bmatrix} 0 & 0 & a_{12}-\alpha & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & a_{12}-\alpha & 0 & 0 \end{Bmatrix},$$

where $$\tau \equiv -(a_{12}+\gamma)(-a_{12}+\beta).$$

It follows that every group with generators $B_1, \ldots, B_5$ defined by the formulæ (27), (28), (29), and (31) is conjugate within $A(4, 3)$ with a similar group having

$$a_{12} = 0, \quad \beta-\gamma = 0, \quad \alpha = \pm 1.$$

Transforming the latter group by the substitution (33), in which $a_{12} = 0$, we obtain a similar group with the same generators B_1, B_2, B_5, but having the coefficients β and γ interchanged in B_3 and the sign of α changed in B_4. Hence every group obtained as above is conjugate with one of the groups given by

$$a_{12} = 0, \quad \alpha = 1, \quad \beta = \gamma.$$

The latter groups are, in turn, conjugate with the group given by

$$a_{12} = 0, \quad \alpha = 1, \quad \beta = \gamma = 0.$$

Indeed, by the lemma of § 10, the substitution (35) is commutative with B_1, B_2, B_4, and B_5 (in which $\alpha_{12}=0$). We verify that it transforms B_3 (in which $\beta=\gamma$) into the substitution

$$\pm\left\{\begin{matrix} 1 & 0 & -(\gamma+\gamma_1)^2 & \gamma+\gamma_1 \\ 0 & 1 & \gamma+\gamma_1 & -1 \\ -1 & -(\gamma+\gamma_1) & -1 & 0 \\ -(\gamma+\gamma_1) & -(\gamma+\gamma_1)^2 & 0 & -1 \end{matrix}\right\}.$$

Taking $\gamma_1=-\gamma$, we obtain a substitution of the form B having $\beta=\gamma=0$. We combine our results into the following

THEOREM.—*If a sub-group of A (4, 3) be simply isomorphic with the symmetric group, and if it contain a substitution conjugate with $T_{1,-1}$, that sub-group is conjugate within A (4, 3) with the group generated by the following substitutions:—*

$$(34)\quad B_1=\pm\left\{\begin{matrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{matrix}\right\},\quad B_2=\pm\left\{\begin{matrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}\right\},$$

$$B_3=\pm\left\{\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{matrix}\right\},\quad B_4=\pm\left\{\begin{matrix} 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{matrix}\right\},$$

$$B_5=\pm\left\{\begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix}\right\}.$$

10. LEMMA.—*The most general substitution of A (4, 3) which is commutative with B_1, B_2, B_4, and B_5, defined by* (34), *has the form*

$$(35)\quad \pm\left\{\begin{matrix} 1 & \gamma_1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\gamma_1 & 1 \end{matrix}\right\}.$$

By inspection, the most general substitution of A (4, 3) which is transformed into itself by B_2 is of one of the two forms

$$A_1 = \pm \begin{Bmatrix} \alpha_{11} & \gamma_{11} & 0 & 0 \\ \beta_{11} & \delta_{11} & 0 & 0 \\ 0 & 0 & \alpha_{22} & \gamma_{22} \\ 0 & 0 & \beta_{22} & \delta_{22} \end{Bmatrix}, \quad A_2 = \pm \begin{Bmatrix} 0 & 0 & \alpha_{12} & \gamma_{12} \\ 0 & 0 & \beta_{12} & \delta_{12} \\ \alpha_{21} & \gamma_{21} & 0 & 0 \\ \beta_{21} & \delta_{21} & 0 & 0 \end{Bmatrix}.$$

The condition $A_1 B_1 = B_1 A_1$ gives

$$\begin{Bmatrix} -\alpha_{11} & -\gamma_{11} & \alpha_{22} & \gamma_{22} \\ -\beta_{11} & -\delta_{11} & 0 & 0 \\ 0 & 0 & \alpha_{22} & \gamma_{22} \\ \beta_{11} & \delta_{11} & \beta_{22} & \delta_{22} \end{Bmatrix} = \pm \begin{Bmatrix} -\alpha_{11} & -\gamma_{11} & \alpha_{11} & 0 \\ -\beta_{11} & -\delta_{11} & \beta_{11} & 0 \\ 0 & \gamma_{22} & \alpha_{22} & \gamma_{22} \\ 0 & \delta_{22} & \beta_{22} & \delta_{22} \end{Bmatrix}.$$

The upper sign must be chosen, whence we have

$$\beta_{11} = \gamma_{22} = 0, \quad \alpha_{11} = \alpha_{22}, \quad \delta_{11} = \delta_{22}.$$

The condition $A_1 B_5 = B_5 A_1$ then gives

$$\begin{Bmatrix} 0 & 0 & \beta_{22} & \delta_{11} \\ 0 & 0 & -\alpha_{11} & 0 \\ 0 & -\delta_{11} & 0 & 0 \\ \alpha_{11} & \gamma_{11} & 0 & 0 \end{Bmatrix} = \pm \begin{Bmatrix} 0 & 0 & -\gamma_{11} & \alpha_{11} \\ 0 & 0 & -\delta_{11} & 0 \\ 0 & -\alpha_{11} & 0 & 0 \\ \delta_{11} & -\beta_{22} & 0 & 0 \end{Bmatrix}.$$

Hence $$\beta_{22} = \mp \gamma_{11}, \quad \delta_{11} = \pm \alpha_{11}.$$

By one of the Abelian conditions

$$\alpha_{11}\delta_{11} \equiv \pm \alpha_{11}^2 \equiv 1.$$

Hence the upper signs must be chosen. Thus A_1 has the form (35). We verify that $A_1 B_4 = B_4 A_1$ identically.

The condition $A_2 B_1 = B_1 A_2$ gives

$$\begin{Bmatrix} \alpha_{21} & \gamma_{21} & -\alpha_{12} & -\gamma_{12} \\ 0 & 0 & -\beta_{12} & -\delta_{12} \\ \alpha_{21} & \gamma_{21} & 0 & 0 \\ \beta_{21} & \delta_{21} & \beta_{12} & \delta_{12} \end{Bmatrix} = \pm \begin{Bmatrix} 0 & \gamma_{12} & \alpha_{12} & \gamma_{12} \\ 0 & \delta_{12} & \beta_{12} & \delta_{12} \\ -\alpha_{21} & -\gamma_{21} & \alpha_{21} & 0 \\ -\beta_{21} & -\delta_{21} & \beta_{21} & 0 \end{Bmatrix}.$$

On the plus sign, $\beta_{12} = \delta_{12} = 0$, which is impossible. Taking the minus sign, $\alpha_{12} = \delta_{12} = 0$, $\beta_{12} = -\beta_{21}$, $\gamma_{12} = -\gamma_{21}$. By one of the Abelian conditions,

$$\alpha_{12}\delta_{12} - \beta_{12}\gamma_{12} \equiv -\beta_{12}\gamma_{12} \equiv 1.$$

Hence A_2 has the form

$$\pm \begin{Bmatrix} 0 & 0 & \alpha_{12} & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & \delta_{21} & 0 & 0 \end{Bmatrix}.$$

The conditions $A_2B_4 = B_4A_2$ and $A_2B_5 = B_5A_2$ give respectively

$$\delta_{21}-1 \equiv 1-\alpha_{12}, \quad \delta_{21} \equiv -\alpha_{12} \pmod 3,$$

which are incompatible. Hence no Abelian substitution of the form A_2 is commutative with B_1, B_2, B_4, and B_5.

11. Theorem.—*The simple group* $A(4, 3)$ *is simply isomorphic with the abstract group of order* 25920 *with the generators* $B_1, \ldots, B_5$, *and* B, *subject to the generational relations* (11)–(17).

By § 9 the substitutions $B_1, \ldots, B_5$ defined by formulæ (34) satisfy the relations (11)–(13). By § 10, a substitution of $A(4, 3)$ is commutative with B_1, B_2, B_4, B_5 and has the period three [relations (14)], if, and only if, it be of the form

$$B = \pm \begin{Bmatrix} 1 & \gamma_1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\gamma_1 & 1 \end{Bmatrix} \quad [\gamma_1 \equiv \pm 1 \pmod 3].$$

We then verify that relation (15) is satisfied identically. To verify the relation (16), we note that

$$B_1B_5B_3B_2B_4 = \pm \begin{Bmatrix} 1 & 1 & -1 & -1 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 \end{Bmatrix},$$

$$(B_1B_5B_3B_2B_4)^2 = \pm \begin{Bmatrix} -1 & -1 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ -1 & 0 & 1 & 1 \end{Bmatrix}.$$

To verify the relation (17), we note that

$$B_1B_2B_3B_1B_2 = \pm \begin{Bmatrix} -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{Bmatrix},$$

$$B^{-1}B_3B = \pm \begin{Bmatrix} 1 & 0 & -\gamma_1^2 & -\gamma_1 \\ 0 & 1 & -\gamma_1 & -1 \\ -1 & \gamma_1 & -1 & 0 \\ \gamma_1 & -\gamma_1^2 & 0 & -1 \end{Bmatrix}.$$

The left side of (17) thus becomes, remembering that $\gamma_1^2 \equiv 1$,

$$\pm \begin{Bmatrix} 0 & 0 & 0 & -\gamma_1 \\ 0 & 0 & \gamma_1 & 0 \\ 0 & \gamma_1 & 0 & 0 \\ -\gamma_1 & 0 & 0 & 0 \end{Bmatrix} = B_5.$$

Thursday, May 11*th*, 1899.

Prof. H. LAMB, F.R.S., Vice-President, in the Chair.

Sixteen members present.

The following were elected members:—George A. Miller, Ph.D., Instructor in Mathematics, Cornell University, Ithaca, New York; James Pierpont, Ph.D., Professor of Mathematics, Yale University, New Haven, Connecticut, U.S.A.

Major MacMahon communicated a few results he had arrived at in the Theory of Partitions. Mr. Heppel asked a question, in connexion with the subject, to which Major MacMahon replied.

Mr. Macdonald then gave some results from his paper on "The Zeroes of a Spherical Harmonic $P_n^m(\mu)$ considered as a Function of n." The Chairman and Dr. Hobson spoke on the subject of the communication, which they considered to be one of great interest.

Mr. W. F. Sheppard read a paper "On the Statistical Rejection of Extreme Variations, Single or Correlated (Normal Variation and Normal Correlation)." Prof. Hudson and the Chairman asked a few questions, which Mr. Sheppard answered.

The following presents were made to the Library :—

Bordeaux.—"Mémoires de la Société des Sciences," Tome iv. ; Paris, 1898.

Bordeaux.—"Procès Verbaux des Seances de la Société des Sciences," Année 1897–8 ; Paris, 1898.

Rayet, G.—"Observations Pluviométriques faites de Juin 1897 à Mai 1898, 8vo ; Bordeaux, 1898.

Fischer, O.—"Der Gang des Menschen," Roy. 8vo ; Leipzig, 1899. [Offprint, "Sächsischen Gesell.," xxv. Band, No. 1.]

Scheibner, W.—"Ueber die Differentialgleichungen der Mondbewegung," roy. 8vo ; Leipzig, 1899. [Offprint "Sächsischen Gesell.," xxv. Band, No. 2.]

Biddle, D.—"Mathematical Questions and Solutions from the 'Educational Times,'" Vol. lxx., 8vo ; London, 1899.

"Proceedings of the American Philosophical Society," Vol. xxxvii., No. 158 ; Philadelphia.

H. Poincaré.—"La Théorie de Maxwell et les Oscillations Hertziennes." ["Scientia" Series, from the Editors, Messrs. Carré and Naud.]

"Educational Times," May, 1899.

"Indian Engineering," Vol. xxv., Nos. 12–15 ; March 25–April 15, 1899.

The following exchanges were received :—

"Journal de l'Ecole Polytechnique," Série ii., Cahier 4 ; Paris, 1898.

"Proceedings of the Royal Society," Vol. lxiv., Nos. 411–412, 1899.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. xxiii., St. 4 ; Leipzig, 1899.

"Bulletin de la Société Mathématique de France," Tome xxvii., Fasc. 1 ; Paris, 1899.

"Bulletin of the American Mathematical Society," Series 2, Vol. v., No. 7, April, 1899 ; New York.

"Reale Istituto Lombardo—Rendiconti," Serie 2, Vol. xxxi. ; Milano, 1898.

"Bulletin des Sciences Mathématiques," Tome xxiii., Mars, 1899 ; Paris.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 1, Vol. viii., Fasc. 6, 7 ; Roma, 1899.

"Berichte über die Verhandlungen der Königl. Sächs. Gesellschaft der Wissenschaften zu Leipzig," ii., 1899.

"Nyt Tidsskrift for Matematik," A, Aarg. 10, Nr. 3, 4 ; Copenhagen, 1899.

"Journal of the Institute of Actuaries," Vol. xxxiv., Pt. 5 ; April, 1899.

On the Statistical Rejection of Extreme Variations, Single or Correlated. (Normal Variation and Normal Correlation.) *By* W. F. Sheppard, M.A., LL.M. Received May 5th, 1899. Read May 11th, 1899.

1. *Single Organ.*—For finding the law of variation of any particular organ in a homogeneous community, we require first of all to take the measurements of this organ in a large number of individuals, supposed to represent the result of a random selection from the total number constituting the community. Amongst the individuals so observed it is always possible that some may be included which should have been rejected, either as being stray members of some other community, or as being sports possessing some physiological differentiation from their fellows. The number of these, however, may be supposed to be relatively small; and, if their measurements are measurements which are not infrequent amongst normal members of the community, no appreciable error will result from their inclusion. But it may sometimes happen that they present extreme variations, which only occur very rarely, if at all, in normal individuals; and the inclusion of these extreme values may make an appreciable difference in the "frequency-constants" which determine the law of variation. If, therefore, the statistical data show a fairly continuous variation within certain limits, with irregularities of variation beyond these limits, we have to consider whether the individuals presenting these irregularities should be rejected, as being sports or members of another community, or whether they should be retained, as being normal individuals possessing a development which is merely rare.

When, as is usually the case, the doubtful members are not accessible for further observation, the question cannot be decided absolutely. Nor can it be decided relatively, by balancing the probabilities; for we have no information as to the *a priori* probability of sports or of heterogeneity. All we can do, therefore, is to reject the doubtful members, and at the same time take account of the possibility that in doing this we are rejecting measurements which ought to have been retained.

In rejecting observations which appear from the data as a whole to be possible, but to possess a certain degree of *a priori* improbability,

we are really excluding from consideration a certain portion of the figure of frequency of the measurements under observation. How much we should exclude is a question less of principle than of convenience. The considerations which influence us in deciding the question are somewhat similar to those which fix the "criteria" for the retention or rejection of extreme astronomical observations; but there are important differences. In astronomical investigations the observations are few; and we therefore take care not to exclude so many genuine observations as materially to increase the probable error of the ultimate determination. This difficulty does not apply to cases of biological statistics, where twenty or thirty individuals could often be spared without making any material difference in the probable error. On the other hand, the quantities sought are not exactly the same in the two cases. The astronomical deviations are regarded as errors, the mean value about which they are symmetrically distributed being the true value of the quantity under observation; and it is this true value, or mean value, that we wish to find. The mean square of the deviation from the mean is also required, but only for the purpose of finding the probable error in the mean itself; and a small error in its calculation may be disregarded. Hence any rule is admissible which excludes even appreciable portions of the figure of frequency, provided that these portions are symmetrically situated with regard to the central ordinate of the figure. But in biological statistics the mean is often of less importance than the mean square of the deviation from the mean. Now, if we assume that the distribution, whether symmetrical or unsymmetrical, is known to be of a particular type, the exclusion of portions of the figure of frequency can be arranged so as not to affect the value of the mean. But the mean square of the deviation from the mean is necessarily affected; and, if large portions of the figure are excluded, the alteration in value may be comparable with the probable error. This difficulty, however, may be obviated by finding the mean square of deviation for the portion of the figure retained, and then multiplying by such a factor as will give the mean square of deviation for the whole figure. We are thus left free to exclude a relatively large number of observations; and the only practical limitation is one of convenience of calculation. In the cases which most commonly present themselves, the measurements are clustered about their mean value, and the frequency diminishes continuously as we recede in either direction from the mean. In such cases the values of the mean and mean square are calculated by means of formulæ which

[redacted]

involve the bounding ordinates of the figure of frequency. In order, therefore, to make the calculations as simple as possible, the portions of the figure which we exclude should be so small that these bounding ordinates are practically negligible.

The rule which most naturally suggests itself is that we should find the range corresponding to a *representative* selection, and reject all values outside this range. The range is determined as follows. Let L denote the measure of the organ in question, and let $f(X)$ denote the proportion of individuals for which L exceeds X. Then, if we make a representative selection of n individuals, and arrange them in classes corresponding to values $X_{-p}, X_{-p+1}, \ldots, X_{q-1}, X_q$ of L (these values usually proceeding by a common difference which is the unit of measurement), the extreme values X_{-p} and X_q are determined by the condition that

$$\left.\begin{aligned} nf(X_{-p}) &> n-\tfrac{1}{2} \\ nf(X_{-p+1}) &< n-\tfrac{1}{2} \end{aligned}\right\},$$

$$\left.\begin{aligned} nf(X_{q-1}) &> \tfrac{1}{2} \\ nf(X_q) &< \tfrac{1}{2} \end{aligned}\right\}.$$

Hence, if X' and X are the values of L given by the conditions

$$\left.\begin{aligned} nf(X') &= n-\tfrac{1}{2} \\ nf(X) &= \tfrac{1}{2} \end{aligned}\right\},$$

the extreme classes in the representative selection will be those which include the values X' and X. The range from X' to X may therefore be considered to be the actual range of the selected values of L. This may conveniently be called the *theoretical range* for n individuals.

The proportion of individuals, in the complete community, for which L lies between X' and X is

$$(n-1)/n = 1-1/n.$$

Hence, when n individuals are obtained by random selection, the à priori probability that one at least of the n values of L will lie outside these limits is $1-(1-1/n)^n$. The value of this probability is different for different values of n; but when n is great it is approximately $1-1/e = {\cdot}63212\ldots$. If, therefore, we make it a rule to exclude all values of L lying outside the theoretical range, the result is that, in about two-thirds of the cases we consider, it will be

necessary to exclude one individual at least, even though there is no abnormality.

Chauvenet's criterion is really a modification of the above rule, for cases in which the figure of frequency is symmetrical. A representative selection is made of n values of $L \sim L_1$, where L_1 is the mean value; the sign of the deviation from the mean being disregarded. The proportion of individuals for which $L \sim L_1$ exceeds $X \sim L_1$ is $2f(X)$, so that the value of L which determines the theoretical range is given by the condition

$$2nf(X) = \tfrac{1}{2}.$$

The *a priori* probability that one at least of n values of L obtained by random selection will lie outside the range given by Chauvenet's criterion is $1-(1-1/2n)^n$, which, when n is great, is approximately $1-1/\sqrt{e} = {\cdot}39347\ldots$. Hence, by adopting this criterion, we reduce the number of cases of rejection of non-abnormal variations to about two-fifths of the total number of cases considered.

The effect which the adoption of either of these methods will have on the values of the mean and the mean square of deviation will of course depend on the particular law of variation. We shall confine our attention to cases in which the variation is *normal*, *i.e.*, in which the measurements are distributed about their mean value according to the law of error. The distribution being symmetrical, and the exclusion also being symmetrical, the determination of the mean is not affected. Let the true mean and mean square of deviation be L_1 and a^2; and let θ be the proportion of values of L lying outside the prescribed limits, so that $\theta = 1/n$ or $1/2n$ according as we take the theoretical range or Chauvenet's criterion. Let z and $\pm x$ be the corresponding ordinate and abscissa of the standard normal curve, whose area is unity and mean square of deviation unity. Then

$$\left.\begin{aligned} z &= \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2} \\ \theta &= \sqrt{\frac{2}{\pi}} \int_x^\infty e^{-\frac{1}{2}x^2}\, dx \end{aligned}\right\},$$

and our rule leads us to exclude all values of L which do not lie between $L_1 - ax$ and $L_1 + ax$. The effect of this is that we are studying a distribution whose mean square of deviation is

$$a^2 \left(1 - \sqrt{\frac{2}{\pi}} \int_x^\infty e^{-\frac{1}{2}x^2} x^2\, dx\right) \Big/ \left(1 - \sqrt{\frac{2}{\pi}} \int_x^\infty e^{-\frac{1}{2}x^2}\, dx\right)$$
$$= a^2 (1 - 2xz - \theta) / (1 - \theta).$$

Hence, when we have found the most probable value of the mean square of deviation of this distribution, we must multiply this most probable value by $(1-\theta)/(1-2xz-\theta)$ in order to get the most probable value of a^2. Now, since θ is small, we have

$$\theta = \sqrt{\frac{2}{\pi}} \int_x^\infty e^{-\frac{1}{2}x^2} dx \doteq \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{1}{2}x^2}}{x}, \text{ nearly;}$$

so that

$$z = \tfrac{1}{2}x\theta;$$

and therefore the factor by which we must multiply is approximately

$$(1-\theta)/(1-\theta-x^2\theta) = 1+x^2\theta.$$

When n is very great, x^2 is only moderately great; and the correction in a^2, due to multiplying by $1+x^2\theta$, will be small in comparison with the probable error in a^2. Suppose, for instance, that $n = 1{,}000{,}000$, and that we use the theoretical range. Then $x^2 = 24$, nearly; and the correction in a^2 is $\cdot000024a^2$. But the probable error in a^2 is $\cdot00095a^2$, so that the correction is only about one-fortieth of the probable error. In most cases, however, n is a good deal less than this, and the correction must then be taken into account. If, for instance, $n = 1{,}000$, the correction amounts to about one-third of the probable error, even if we adopt Chauvenet's criterion.

Thus we see that neither of these methods does away with the necessity of correcting the calculated value of the mean square of deviation; and that, whichever method is adopted, the individuals rejected will in a large proportion of cases be ordinary members of the community. Moreover, the proportion of cases in which ordinary members are rejected is different for different values of n. It would therefore appear to be simpler to fix on a definite value for this proportion, and calculate the corresponding range. All measurements outside this range should then be rejected. The range which I would suggest as the most convenient is that which I have elsewhere called the *probable* range. It extends from a value X' to a value X, where $L_1 - X' = X - L_1$, and where $X - X'$ is such that it is an even chance that, in making a random selection of n values of L, one at least will lie outside the range. If therefore we write

$$\left.\begin{aligned} X' &= L_1 - ax \\ X &= L_1 + ax \end{aligned}\right\},$$

the values of x are given by the condition that

$$(1-\theta)^n = \tfrac{1}{2},$$

where $$\theta = \sqrt{\frac{2}{\pi}} \int_x^\infty e^{-\frac{1}{2}x^2}\,dx.$$

Since $(1-1/n)^n < \frac{1}{2} < (1-1/2n)^n$, $1-\theta$ lies between $1-1/n$ and $1-1/2n$; *i.e.*, the probable range lies between the theoretical range and the range given by Chauvenet's criterion.

Table I. gives the value of x for the probable range corresponding to any value of n from 10 to 1,000,000. The values are given to five places of decimals, but, except when n is very great, this degree of accuracy is hardly necessary. The table also gives the values of P and of $\log_{10}(P-1)$, where P is the factor by which the most probable value of the mean square of deviation, for the figure of frequency represented by the individuals retained, is to be multiplied in order to get the most probable value of the mean square of deviation for the whole distribution.

The table is arranged by equal increments of $\log_{10} n$, beginning with $\log_{10} n = 1$. But caution must be exercised in using the earlier part of the table. When n is comparatively small, the most probable value of the mean square of deviation depends partly on the *a priori* probabilities of different values; and we have no means of knowing these *a priori* probabilities. It is only when n becomes great that the different possible values tend to be distributed symmetrically about a mean value coinciding with the mean square of deviation of the actual observations.*

If, instead of the probable range, it is preferred to use either the theoretical range or that given by Chauvenet's criterion, the corresponding value of x can be deduced from the table. Thus for the theoretical range we have

$$\theta = 1/n.$$

This will give the value of x corresponding to $\log_{10} n'$ in Table I., where n' is given by the equation

$$(1-1/n)^{n'} = \tfrac{1}{2},$$

so that $$n' = \frac{\log 2}{-\log(1-1/n)}.$$

* This difficulty does not arise when the values of the mean and mean square of deviation are suggested by *a priori* reasoning, and the object before us is to test whether the observations can be regarded as obtained by random selection from a distribution having this mean and mean square of deviation. For use in such cases I have given elsewhere [*Phil. Trans.*, Vol. 192 (1898), A, p. 123] the values of x/Q for values of n from 1 to 20, where $Q = \cdot 67448975 \ldots$. For any larger value of n the value of x/Q may be found from Table I. of the present paper, by multiplying the corresponding value of x by $1/Q = 1\cdot 4826022$.

Similarly for Chauvenet's criterion we should take the value corresponding to $\log_{10} n''$, where

$$n'' = \frac{\log 2}{-\log(1-1/2n)}.$$

The following comparative table shows the values of x, for each of the three ranges, corresponding to selected values of n :—

n	$\log_{10} n$	Values of x for		
		Theoretical Range.	Probable Range.	Chauvenet's Criterion.
10	1·0	1·64485	1·83190	1·95996
100	2·0	2·57583	2·70127	2·80703
1,000	3·0	3·29053	3·39237	3·48076
10,000	4·0	3·89059	3·97863	4·05563
100,000	5·0	4·41717	4·49578	4·56479
1,000,000	6·0	4·89164	4·96327	5·02631

It remains to consider the effect of the limitation of range on the calculation of the actual average square of deviation from the average, when the number of individuals is large, and the measurements are only given to the nearest multiple of a particular unit. In these cases the rule as to exclusion cannot always be applied exactly, as the limiting values X' and X will not generally coincide with intermediate divisions of the scale. Let the unit of measurement be h, and suppose that the range taken is from X_{-p} to X_q. Let λ_2 denote the average square of deviation, calculated on the assumption that each value of L is equal to the nearest multiple of h. Then the most probable value of the average square differs from $\lambda_2 - \frac{1}{12}h^2$ by a series of small terms, the leading term being approximately $\frac{1}{6}h^2(\xi z + \xi' z')$, where $X_{-p} = L_1 - a\xi'$, $X_q = L_1 + a\xi$, and z and z' are the ordinates of the standard normal curve corresponding to abscissæ ξ and ξ'.* If $\frac{1}{2}\theta$ and $\frac{1}{2}\theta'$ are the excluded areas, this is approximately equal to $\frac{1}{12}h^2(\xi^2\theta + \xi'^2\theta')$, which is negligible if h^2 is appreciably less than a^2. Hence we may take $\lambda_2 - \frac{1}{12}h^2$ as the average square of deviation, without further correction.

* "On the Calculation of the most Probable Values of Frequency-Constants," *Proc. Lond. Math. Soc.*, Vol. xxix., p. 359, formula (19). The exact value of the leading term is $\frac{1}{6}h^2(\xi z + \xi' z')/(1 - \frac{1}{2}\theta - \frac{1}{2}\theta')$.

2. *Examples.*—To illustrate the method, let us take Prof. Weldon's measurements of the ratio of carapace to body-length in Naples crabs.* The ratios are taken by increments of ·004 as the unit, commencing with ·7155 as zero, so that the range of the observations is from ·7155 to ·7955.

Ratio.	Number.	Ratio.	Number.
0 to 1	1	10 to 11	126
1 to 2	3	11 to 12	82
2 to 3	5	12 to 13	72
3 to 4	11	13 to 14	41
4 to 5	40	14 to 15	28
5 to 6	55	15 to 16	8
6 to 7	98	16 to 17	7
7 to 8	121	17 to 18	0
8 to 9	152	18 to 19	0
9 to 10	147	19 to 20	2
		Total ...	999

For $n = 999$, $\log n = 2{\cdot}9995655$, we find, from Table I.,

$$x = 3{\cdot}39210,$$

$$P = 1{\cdot}008668.$$

For the average and the average square of deviation of the 999 individuals, we have

$$\left.\begin{aligned} L_1 &= 9{\cdot}185 \\ a^2 &= 7{\cdot}42585 \end{aligned}\right\},$$

whence $$a = 2{\cdot}7250.$$

The limits of the range are

$$9{\cdot}185 \pm (2{\cdot}7250 \times 3{\cdot}39210) = 9{\cdot}185 \pm 9{\cdot}243 = -{\cdot}06 \text{ and } +18{\cdot}43.$$

We ought therefore to exclude the two values between 19 and 20.

---

* *Roy. Soc. Proc.*, Nov. 16th, 1893, p. 322. See also Karl Pearson, in *Phil. Trans.*, Vols. 185 (1894), A, p. 96, and 186 (1895), A, p. 384.

For the revised values of L_1, a^2, and a, we have

$$L_1 = 9{\cdot}164,$$
$$a^2 = 7{\cdot}22704 \times 1{\cdot}008668 = 7{\cdot}28968,$$
$$a = 2{\cdot}6999.$$

Since the probable error in a is ·0411, the correction made by the omission of the extreme values is about $\frac{3}{5}$ of the probable error.

It will be seen that different rules of exclusion might lead to the retention of exactly the same individuals, and yet give us different factors by which their mean square of deviation should be multiplied. If, for instance, the range were from $9{\cdot}185 - 9{\cdot}185$ to $9{\cdot}185 + 9{\cdot}185$, it would correspond to $\log n = 2{\cdot}9654$, which gives

$$P = 1{\cdot}00927,$$

and

$$a^2 = 7{\cdot}22704 \times 1{\cdot}00927 = 7{\cdot}29403,$$
$$a = 2{\cdot}7007.$$

Strictly speaking, it is not legitimate to limit the range by the observations, and then treat the observations as representing a random selection from this range. But the method is sometimes useful for finding limits to the most probable value of the mean square of deviation. It will be seen that the two different ranges give values differing only by $\frac{1}{50}$ of the probable error.

For another example, take the chest-measurements of Scotch soldiers.*

Chest-measurement to nearest inch.	Number.	Chest-measurement to nearest inch.	Number.
33	3	41	935
34	19	42	646
35	81	43	313
36	189	44	168
37	409	45	50
38	753	46	18
39	1062	47	3
40	1082	48	1
		Total ...	5,732

* *Edinburgh Medical Journal*, Vol. XIII., pp. 260–2. See *Phil. Trans.*, Vol. 192 (1898), A, p. 136.

For $n = 5{,}732, \quad \log n = 3 \cdot 7583062,$

we find $x = 3 \cdot 84426,$

$$P = 1 \cdot 0018992.$$

The observations give $L_1 = 39 \cdot 8489,$

$$a = 2 \cdot 05301,$$

showing a range between limits $39 \cdot 8489 \pm 7 \cdot 8923$; *i.e.*, between 31·9566 and 47·7412. It is therefore doubtful whether the one value between $47\frac{1}{2}$ and $48\frac{1}{2}$ should be excluded or not; and the exact measurement is not available. If we assume that it is greater than 47·7412, and exclude it accordingly, we get

$$L_1 = 39 \cdot 8475,$$

$$a^2 = 4 \cdot 204001 \times 1 \cdot 0018992 = 4 \cdot 211985,$$

$$a = 2 \cdot 05231.$$

If, however, we alter the rule so as to make the upper limit of the range $47\frac{1}{2}$ inches, we find

$$x = 7 \cdot 65108 / 2 \cdot 05301 = 3 \cdot 72676,$$

$$\log n = 3 \cdot 55308,$$

$$P = 1 \cdot 0028754,$$

which gives for the corrected value

$$a = 2 \cdot 05331.$$

The probable error in a is ·00409, so that the variation from the original value 2·05301 is practically negligible. It will be found that, on the whole, this original value gives a better fit than the value 2·05331 obtained by adapting the range to the observations.

3. *Two Correlated Organs.*—A more accurate test of abnormality can be obtained when the joint distribution of two correlated organs A and B is given. Suppose, for instance, that the correlation is positive, and that, for each organ separately, the frequency diminishes as we recede in either direction from the mean. Then large positive (or negative) variations of A will generally be accompanied by large positive (or negative) variations of B; but the combination of a moderately large positive variation of A with a moderately large negative variation of B may be very rare. We may therefore in some cases retain an individual presenting an extreme variation of one organ, if the variation of the other organ is such as

might reasonably be expected to accompany it; while, on the other hand, we may be led to reject individuals presenting improbable combinations of variations, even though either variation, if considered by itself, might not suggest any abnormality.

In the class of cases mentioned above, the solid of frequency of the measures of A and B will in all directions slope downwards to its outer boundary. The rule will therefore be to exclude all the individuals represented by the portion of this solid which lies outside a certain cylinder, not necessarily circular. It might be possible to lay down a general rule as to the relation which should hold between this cylinder and the solid; but we shall confine ourselves to the cases in which the correlation is *normal.*

Let L and M be the measures of A and B; and let their means, mean squares of deviation, and mean product of deviation be L_1, M_1, a^2, b^2, and $ab\cos D$. Then the *correlation-solid* of L and M, *i.e.*, the solid of frequency of values of

$$(L-L_1)/(a\sin D) \quad \text{and} \quad (M-M_1)/(b\sin D),$$

referred to coordinate planes including an angle $\pi-D$, is, in this class of cases, a solid of revolution. The natural rule therefore appears to be* that we should exclude all individuals represented by elements of the correlation-solid lying outside a circular cylinder coaxal with the solid; the radius of this cylinder being determined by the total proportion of individuals which we decide to exclude.

This rule is justified by a certain property of the correlation-solid. Let L' and M' be the measures of two organs A' and B', whose development depends solely on the development of A and B. It will usually be found that the distributions of L' and of M' will also be normal, and normally correlated; and that L' and M' will be of the forms

$$\left.\begin{aligned} L' &= L_1'+p\,(L-L_1)+q\,(M-M_1) \\ M' &= M_1'+p'(L-L_1)+q'(M-M_1) \end{aligned}\right\},$$

where L_1' and M_1' are the mean values of L' and M', and p, p', q, q' are constants. But, when this relation holds, the correlation-solid of L' and M' is identical with the correlation-solid of L and M, the relative positions of the elements composing the solid being unaltered.† Hence we shall be led to exclude the same individuals, whether we consider L and M, or L' and M'.

* *Cf.* Yule, "On the Theory of Correlation," *Journal of the Royal Statistical Society*, Vol. LX., Pt. IV. (December, 1897), p. 35.

† *Phil. Trans.*, Vol. 192 (1898), A, p. 139.

[redacted]

The correlation-solid, in the present case, is the solid of revolution of a normal figure whose central ordinate is $1/(2\pi)$, and whose mean square of deviation is unity. Hence, if the radius of the cylinder is r, and if in the total community the proportion of individuals to be excluded is θ, we have

$$e^{-\frac{1}{2}r^2} = \theta.$$

If, as in the case of a single organ, we choose θ so that it may be an even chance that, after making a random selection of n individuals, one at least will have to be rejected, the value of θ is given by the equation

$$(1-\theta)^n = \tfrac{1}{2};$$

so that r is given by $\quad (1-e^{-\frac{1}{2}r^2})^n = \frac{1}{2}.$

Table II. gives the values of r and of r^2 deduced from this equation, for values of n from 10 to 1,000,000.

Let T denote the total number of individuals comprised in the original community; this number being regarded as practically infinite. Then, out of these, $Te^{-\frac{1}{2}r^2}$ are liable to be rejected. The sums of the values of $(L-L_1)^2$, $(M-M_1)^2$, and $(L-L_1)(M-M_1)$ for the remaining $T(1-e^{-\frac{1}{2}r^2})$ are respectively

$$T.a^2 \sin^2 D.\frac{1}{2\pi}\iint e^{-\frac{1}{2}(x^2-2xy\cos D+y^2)}\,x^2\,d\omega,$$

$$T.b^2 \sin^2 D.\frac{1}{2\pi}\iint e^{-\frac{1}{2}(x^2-2xy\cos D+y^2)}\,y^2\,d\omega,$$

$$T.ab \sin^2 D.\frac{1}{2\pi}\iint e^{-\frac{1}{2}(x^2-2xy\cos D+y^2)}\,xy\,d\omega,$$

where x, y are coordinates referred to axes including an angle $\pi-D$, and $d\omega$ denotes an element of area; the integration being over the section of the cylinder of radius r. Taking new axes of ξ and η along and at right angles to the axis of x, we have

$$\left.\begin{aligned} x &= \xi+\eta\cot D \\ y &= \eta\,\mathrm{cosec}\, D \end{aligned}\right\}.$$

Substituting, and then transforming to polar coordinates, it will be found that the sums are respectively

$$T.a^2.\left\{1-(1+\tfrac{1}{2}r^2)\,e^{-\frac{1}{2}r^2}\right\},$$

$$T.b^2.\left\{1-(1+\tfrac{1}{2}r^2)\,e^{-\frac{1}{2}r^2}\right\},$$

$$T.ab\cos D.\left\{1-(1+\tfrac{1}{2}r^2)\,e^{-\frac{1}{2}r^2}\right\}.$$

Hence, when we have found the most probable values of the mean squares and mean product of deviation for the $T(1-e^{-\frac{1}{2}r^2})$ individuals, we must multiply these by

$$P \equiv (1-e^{-\frac{1}{2}r^2}) / \{1-(1+\tfrac{1}{2}r^2)\,e^{-\frac{1}{2}r^2}\},$$

in order to get the most probable values of the mean squares and mean product of deviation for the whole community. The values of P and of $10+\log_{10}(P-1)$ corresponding to the different values of n are given by Table II.

It will be noticed that, if we only require the value of $\cos D$, the factor P need not be introduced at all.

Let X and Y be the values of L and M for a particular individual. Then, if we write

$$(X-L_1)/a = x, \quad (Y-M_1)/b = y,$$

and if ρ denotes the distance of the element, representing this individual, from the axis of the correlation-solid, we have

$$\rho^2 = (x^2-2xy\cos D+y^2)\operatorname{cosec}^2 D\,;$$

and the individual is to be rejected if this is greater than r^2. Now, if we take two straight lines OA, OB, of lengths x and y, and including an angle D, and if we draw AR and BR at right angles to OA and OB respectively, meeting in R, then $OR = \rho$. Hence we get the following rule :—

From Table II. find the values of r (or r^2) and P corresponding to the particular value of n. Calculate the values of L_1, M_1, a^2, b^2, and $ab\cos D$ from the data ; and thence find the values of $(L-L_1)/a$ and $(M-M_1)/b$ for the different values of L and M, according to which the classification has been made. Take two straight lines $X'OX$, $Y'OY$, including an angle D ; and on $X'OX$ and $Y'OY$ mark off the values of $(L-L_1)/a$ and of $(M-M_1)/b$, O being taken as origin. Through these points draw straight lines at right angles to $X'OX$ and $Y'OY$ respectively; these will form a system of parallelograms corresponding to the different compartments in the table of double classification. With centre O and radius r, describe a circle. Then all the individuals represented by parallelograms lying wholly inside the circle must be retained, while those represented by parallelograms wholly outside the circle must be excluded. The cases in which a parallelogram is crossed by the circle must be specially considered; the individuals must be retained or excluded according as $x^2-2xy\cos D+y^2$ is less or greater than $r^2\sin^2 D$, where

$$x = (L-L_1)/a, \quad y = (M-M_1)/b.$$

The values of L_1, M_1, a^2, b^2, and $ab \cos D$ must then be recalculated; and the values of a^2, b^2, and $ab \cos D$ must be multiplied by P.

In drawing the diagram it will be found convenient to make the unit of measurement of $(L-L_1)/a$ and $(M-M_1)/b$ proportional to the numerical value of a (or b); thus the values of $(L-L_1)/a$ will proceed by equidistant divisions of the scale that is used, and the values of $(a/b)(M-M_1)$ and ar will be used instead of those of $(M-M_1)/b$ and r.

4. *Example.*—The table on the next page gives the correlation of strength of squeeze (in lbs.) and breathing capacity (in cubic inches) for 522 men measured at Mr. Galton's laboratory.* The distributions and the correlation are not exactly normal, but they are sufficiently so for purposes of illustration.†

Denoting the measures of squeeze and of breathing capacity by L and M, we find

$$\begin{aligned} L_1 &= 85{\cdot}374, & M_1 &= 226{\cdot}57, \\ a^2 &= 116{\cdot}86, & b^2 &= 1588{\cdot}0, \\ a &= 10{\cdot}810, & b &= 39{\cdot}85, \\ ab \cos D &= 171{\cdot}7, & D &= 66^\circ\, 30'. \end{aligned}$$

First, consider each distribution separately. From Table I., for $n = 522$, $\log n = 2{\cdot}7176705$, we have

$$\left.\begin{aligned} x &= 3{\cdot}21008 \\ P &= 1{\cdot}015063 \end{aligned}\right\},$$

so that the range is from 50·67 to 120·08 in the one case, and from 98·65 to 354·49 in the other. Thus for the corrected values, by the method of §§ 1 and 2, we find

$$\begin{aligned} L_1 &= 85{\cdot}374, & M_1 &= 226{\cdot}86, \\ a^2 &= 118{\cdot}62, & b^2 &= 1570{\cdot}2, \\ a &= 10{\cdot}891, & b &= 39{\cdot}63. \end{aligned}$$

---

* *Journal of the Anthropological Institute*, Vol. xiv. (1885), p. 285.

† It seems clear that a larger number of trials has been made when the measure first obtained has been nearly up to a round number; and this has caused discontinuities in the distribution. For instance, the squeezes 95–100 and 100–105 have been increased at the expense of 90–95; and the same is true for breathing capacities of about 200 and 300.

Strength of Squeeze, and Breathing Capacity, of 522 Englishmen aged 23–26.

Breathing Capacity in cubic inches.	Squeeze in lbs. 55–60	60–65	65–70	70–75	75–80	80–85	85–90	90–95	95–100	100–105	105–110	110–115	Total.
70– 80	...	...	1	...	...	...	...	...	...	...	...	...	1
80– 90	...	...	...	...	...	...	...	...	...	...	...	...	0
90–100	...	...	...	...	...	...	...	...	...	...	...	...	0
100–110	...	...	...	...	...	...	...	...	...	...	...	...	0
110–120	...	...	..	...	...	1	...	...	...	...	...	...	1
120–130	...	...	1	...	1	...	...	...	...	...	...	...	2
130–140	...	...	1	...	1	...	...	...	...	...	...	...	2
140–150	1	...	1	...	1	...	...	...	...	...	...	...	3
150–160	1	...	1	3	1	1	...	1	...	1	...	...	9
160–170	...	1	2	3	4	2	1	2	...	...	...	...	15
170–180	...	...	1	3	7	6	3	1	1	...	...	...	22
180–190	1	3	3	9	10	6	6	3	1	1	1	...	44
190–200	1	...	...	4	6	10	3	2	3	2	...	1	32
200–210	...	1	1	10	7	9	8	9	3	...	1	...	49
210–220	...	1	4	2	9	11	7	6	5	3	3	...	51
220–230	...	1	4	6	11	12	11	6	10	4	...	...	65
230–240	1	1	3	1	6	6	13	5	9	1	2	1	49
240–250	...	...	1	5	4	10	10	8	7	2	1	...	48
250–260	...	1	...	1	2	4	8	4	4	3	2	...	29
260–270	...	...	1	2	3	1	7	4	4	1	...	...	23
270–280	...	...	...	...	2	3	5	1	2	1	...	1	15
280–290	...	...	...	...	2	2	1	2	4	3	2	1	17
290–300	...	...	...	...	...	2	3	7	4	2	1	1	20
300–310	...	...	1	...	3	2	3	6	2	1	...	...	18
310–320	...	...	...	...	...	...	1	...	1	1	...	...	3
320–330	...	...	...	...	...	...	1	...	...	1	1	...	3
330–340	...	...	...	...	...	...	...	...	...	...	...	...	0
340–350	...	...	...	...	...	...	...	...	...	1	...	...	1
Total ...	5	9	26	49	80	88	91	67	60	28	14	5	522

Next, take the two together. From Table II., for $n = 522$, we have

$$\left.\begin{array}{l} r^2 = 13\cdot24969, \quad r = 3\cdot64001 \\ P = 1\cdot008881 \end{array}\right\}.$$

Using the values obtained directly from the data, we find

$$L_1 \pm ar = 46\cdot02 \text{ and } 124\cdot72,$$
$$M_1 \pm br = 81\cdot52 \text{ and } 371\cdot63.$$

We must therefore exclude the case for which L lies between 65 and 70 and M between 70 and 80. To see whether any other cases should

be excluded, we draw the diagram shown in Fig. 1, in the manner

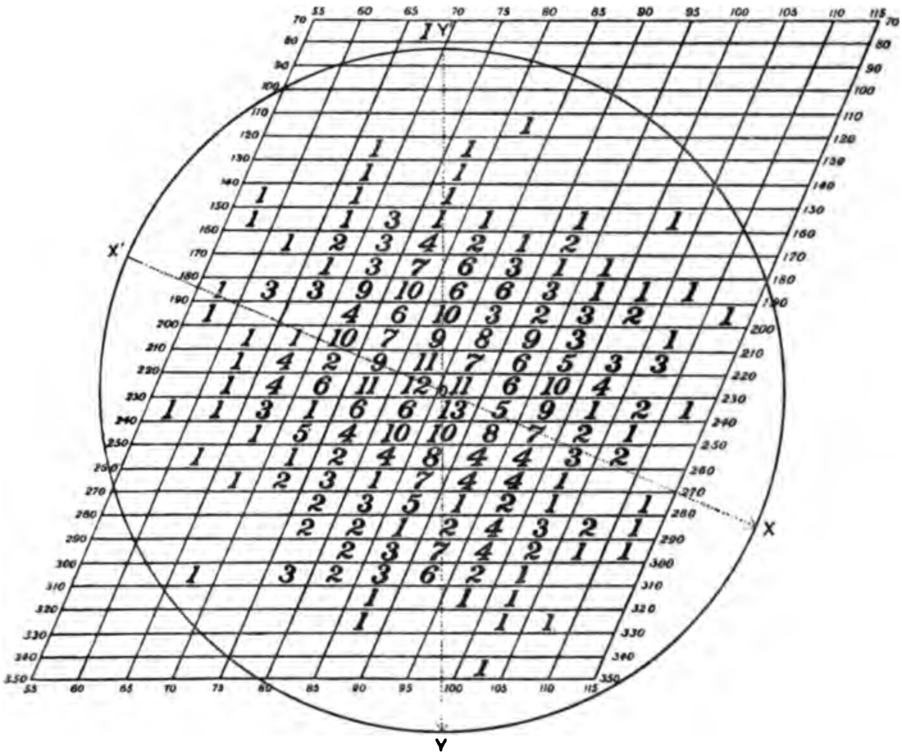


Fig. 1.

described in the last section. The angle between the straight lines $X'OX$ and $Y'OY$ is 66°30′; the different values of $(L-85{\cdot}374)/10{\cdot}810$ and of $(M-226{\cdot}57)/39{\cdot}85$ are marked off on $X'OX$ and $Y'OY$; and the circle is drawn, of radius 3·64001, having its centre at O. The accuracy of the construction of the figure may be tested by observing the points where the circle cuts the set of lines perpendicular to $X'OX$ or $Y'OY$. If, for instance, we take the lines perpendicular to $X'OX$, representing the different values of squeeze, we must solve the equation

$$y^2-2yx\cos D+(x^2-r^2\sin^2 D)=0$$

for different values of x. This equation gives

$$y=x\cos D\pm\sqrt{r^2-x^2}\,.\sin D;$$

whence $M=M_1+by=M_1+\dfrac{b\cos D}{a}ax\pm\dfrac{b\sin D}{a}\sqrt{a^2r^2-a^2x^2}.$

For values of L from 55 to 115, by intervals of 5, we have

$$ax = 5p - {\cdot}37356,$$

where p has values from -6 to $+6$; and thus we find

$$M = 226{\cdot}02193 + 7{\cdot}34758p \pm 3{\cdot}38077 \times \sqrt{1548{\cdot}26063 + 3{\cdot}7356p - 25p^2}.$$

Taking the different values of p, this gives for the range of M for different values of L:—

Squeeze in lbs.	Range of Breathing Capacity in cubic inches.
55	97·36 to 266·51
60	87·60 to 290·96
65	82·82 to 310·44
70	81·52 to 326·44
75	82·97 to 339·68
80	86·89 to 350·46
85	93·00 to 359·05
90	101·26 to 365·48
95	111·73 to 369·71
100	124·56 to 371·57
105	140·11 to 370·72
110	159·00 to 366·52
115	182·55 to 357·66

In practice, it is sufficient to test for one or two values of L. Having done this, we insert in each parallelogram (as shown in Fig. 1) the number of cases observed. It will then be seen that, except in the one case mentioned above, the observations all lie within the limits given by Table II. Recalculating, and multiplying a^2, b^2, and $ab \cos D$ by $P = 1{\cdot}008881$, collecting our results, and calculating the probable errors,* we get the following tables:—

	L_1	M_1	a^2	b^2	a	b
From data	85·374	226·57	116·86	1588·0	10·810	39·85
Corrected from Table I.	85·374	226·86	118·62	1570·2	10·891	39·63
Corrected from Table II.	85·408	226·86	114·48	1560·7	10·747	39·51
Probable error..	·317	1·17	4·82	65·2	·214	·82

* The probable errors here given have been calculated on the assumption of normal distribution and normal correlation. They might have been calculated directly from the observations, without making any assumption; but the additional labour seems hardly necessary.

	$ab \cos D$	D
From data	171·7	66° 30′
Corrected from Table II....	168·3	66° 38′
Probable error	13·5	1° 33′

These results are not very conclusive. But an inspection of Fig. 1 will show that the parallelogram representing the one excluded case is only just outside the bounding circle. Moreover, if we had made this circle a little smaller, it would have been doubtful whether we should not also exclude the case of squeeze between 65 and 70 and breath between 300 and 310. If we exclude this case, as well as the former, it will be found that we get $D = 66° 31'$, which is practically the same as the value originally obtained. There is therefore good ground for supposing that the excluded case is not really abnormal, and that we might retain it, and content ourselves with the values originally obtained from the data, without making any correction in the values of a^2, b^2, and $ab \cos D$.

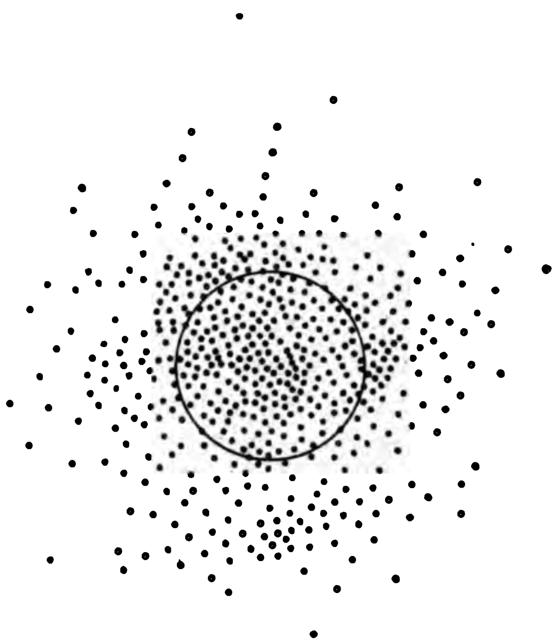


Fig. 2.

If we place dots in each parallelogram in Fig. 1, equal to the number of the corresponding observations, we obtain a rough representation of the correlation-solid, the ordinate of the solid at any point being proportional to the frequency of the dots at that point. The result is shown by Fig. 2, the lines forming the parallelograms having been removed for clearness. The actual position, in each parallelogram, of the dots placed in it, is of course arbitrary; but otherwise the figure gives a representation of the correlation, without any assumption as to its nature. The circle of radius r has been replaced by a circle of radius unity, to show the scale. The square of the radius of this circle is the mean square of the distance of the dots in the figure from *any* straight line through O.

5. *Three Correlated Organs.*—The method may be extended to the case of three organs whose distributions are normal and normally correlated. To treat the case geometrically, the ordinate representing the frequency of any particular value or combination of values must be replaced by a density. Thus, instead of the standard solid, we get a "standard lamina" whose density at a point at distance r from the centre of the lamina is proportional to $e^{-\frac{1}{2}r^2}$. Let L, M, N be the measures of three mutually correlated organs, their means, mean squares of deviation, and mean products of deviation being L_1, M_1, N_1; a^2, b^2, c^2; and $bc\cos D_1$, $ca\cos D_2$, $ab\cos D_3$. Describe a spherical triangle xyz whose sides are D_1, D_2, D_3; take planes at right angles to Ox, Oy, Oz (O being the centre of the sphere); and suppose a material solid to be formed such that the frequency of joint occurrence of values L, M, N is represented by a corresponding density at the point whose coordinates, referred to these planes, are

$$(L-L_1)/(\lambda a \operatorname{cosec} D_1), \quad (M-M_1)/(\lambda b \operatorname{cosec} D_2),$$

$$(N-N_1)/(\lambda c \operatorname{cosec} D_3),$$

where $\lambda^2 = 1-\cos^2 D_1 - \cos^2 D_2 - \cos^2 D_3 + 2\cos D_1 \cos D_2 \cos D_3$.

Then every section of this solid will be a "standard lamina," and the surfaces of equal density will be concentric spheres. If therefore we take the mass of the whole solid to be unity, and describe a sphere of radius R, the portion lying outside this sphere will be

$$\sqrt{\frac{2}{\pi}}\int_R^\infty e^{-\frac{1}{2}R^2} R^2\, dR.$$

Denote this by θ; then, if we choose R so that

$$(1-\theta)^n = \tfrac{1}{2},$$

the adoption of the same rule as before will lead to the exclusion of all individuals represented by points outside this sphere.

Hence, if

$$(L-L_1)/a = x, \quad (M-M_1)/b = y, \quad (N-N_1)/c = z,$$

the rule would be that we should exclude all individuals for which

$$x^2 \sin^2 D_1 + y^2 \sin^2 D_2 + z^2 \sin^2 D_3 - 2yz \sin D_2 \sin D_3 \cos \delta_1$$
$$- 2zx \sin D_3 \sin D_1 \cos \delta_2 - 2xy \sin D_1 \sin D_2 \cos \delta_3$$

is greater than

$$R^2 (1 - \cos^2 D_1 - \cos^2 D_2 - \cos^2 D_3 + 2 \cos D_1 \cos D_2 \cos D_3);$$

where δ_1, δ_2, δ_3 are the angles of the spherical triangle whose sides are D_1, D_2, D_3.*

Supposing this to be done; then we are limiting our data to the part of the solid which lies inside the sphere of radius R. If T denote the whole number of individuals in the original community, then the number of individuals liable to be rejected is

$$T.\sqrt{\frac{2}{\pi}} \int_R^\infty e^{-\frac{1}{2}R^2} R^2 dR.$$

It may be shown that the sums of the squares and products (in pairs) of $L-L_1$, $M-M_1$, and $N-N_1$ for the remaining

$$T.\sqrt{\frac{2}{\pi}} \int_0^R e^{-\frac{1}{2}R^2} R^2 dR$$

are respectively
$$T.a^2.\tfrac{1}{3}\sqrt{\frac{2}{\pi}} \int_0^R e^{-\frac{1}{2}R^2} R^4 dR, \ \ldots,$$
$$T.bc \cos D_1.\tfrac{1}{3}\sqrt{\frac{2}{\pi}} \int_0^R e^{-\frac{1}{2}R^2} R^4 dR, \ \ldots.$$

Hence, when we have determined the most probable values of a^2, b^2, c^2, $bc \cos D_1$, $ca \cos D_2$, and $ab \cos D_3$, for the individuals retained, we must multiply each of these by

$$P \equiv 3 \int_0^R e^{-\frac{1}{2}R^2} R^2 dr \Big/ \int_0^R e^{-\frac{1}{2}R^2} R^4 dR,$$

in order to get the most probable values for the complete community.

The values of R, R^2, $10 + \log_{10}(P-1)$, and P, for values of n from 10 to 1,000,000, are given by Table III.

---

* The angles δ_1, δ_2, δ_3 are the "divergences" of M from N, of N from L, and of L from M, in classes determined by particular values of L, M, and N respectively.

6. *Arbitrary Modification of Rule.*—In fixing θ, the proportion of the figure (or solid) of frequency to be disregarded, at such a value that it may be an even chance that at least one genuine individual may be excluded, we are adopting a quite arbitrary criterion. If we like to modify the rule by taking θ to be of such a value that in one case out of every p the result will be to exclude at least one individual, the corresponding values of x, r, &c., are easily found from the tables. The value of θ is given by

$$1-(1-\theta)^n = 1/p$$

or

$$(1-\theta)^n = (p-1)/p.$$

Now suppose that the corresponding value of (*e.g.*) x is the same as would be found from Table I. by substituting n' for n. Then we have

$$(1-\theta)^{n'} = \tfrac{1}{2}.$$

Comparing these equations, we find

$$\frac{n'}{n} = \frac{\log 2}{\log p - \log(p-1)}.$$

Hence we have only to add $\log_{10}(\log_{10} 2) - \log_{10}\{\log_{10} p - \log_{10}(p-1)\}$ to $\log_{10} n$, and then use the tables without further correction.

The following table gives the values of this amount to be added for different values of p:—

p	$\log_{10} n' - \log_{10} n$
$\frac{100}{99}$	$-(\cdot 8224202)$
$\frac{10}{9}$	$-(\cdot 5213902)$
$\frac{5}{4}$	$-(\cdot 3658488)$
$\frac{4}{3}$	$-(\cdot 3010300)$
$\frac{3}{2}$	$-(\cdot 2000190)$
2	·0000000
3	·2328720
4	·3819127
5	·4922411
10	·8181476
100	1·8386449

Take, for example, the first case considered in § 2, where $n = 999$, $\log_{10} n = 2{\cdot}9995655$. If we fix the range by the condition that in 99 cases out of 100 one genuine individual at least will be excluded, we must take the values of x and P corresponding to

$$\log n' = 2{\cdot}9995655 - {\cdot}8224202 = 2{\cdot}1771453.$$

If, on the other hand, we decide that this ought only to happen in 1 case out of 100, we should have

$$\log n' = 2{\cdot}9995655 + 1{\cdot}8386449 = 4{\cdot}8382104.$$

7. *Construction of the Tables.*—For each of the three tables given at the end of this paper, we have

$$a^n = \tfrac{1}{2}, \tag{1}$$

where

$$a = \sqrt{\frac{2}{\pi}} \int_0^x e^{-\frac{1}{2}x^2}\, dx \qquad \text{(Table I.)},$$

$$a = \int_0^r e^{-\frac{1}{2}r^2}\, r\, dr \qquad \text{(Table II.)},$$

$$a = \sqrt{\frac{2}{\pi}} \int_0^R e^{-\frac{1}{2}R^2}\, R^2\, dR \quad \text{(Table III.)}.$$

From (1) we have

$$\log_{10} a = \frac{-\log_{10} 2}{n}. \tag{2}$$

For convenience, write

$$\omega = \log_e 2$$
$$= {\cdot}69314\ 71805\ 60,$$
$$\mu = \log_{10} e = {\cdot}43429\ 44819\ 03.$$

Then

$$a = e^{-\omega/n}$$
$$= 1 - \omega . n^{-1} + \frac{\omega^2}{2!} n^{-2} - \dots, \tag{3}$$

and

$$\log_e(1-a) = -\log_e 1/(1-e^{-\omega/n})$$

$$= \log_e(\omega/n) - \log_e \frac{\omega/n}{1-e^{-\omega/n}}$$

$$= \log_e \omega - \log_e n - \tfrac{1}{2}\left(\frac{\omega}{n} - \tfrac{1}{1}\frac{B_1}{2!}\frac{\omega^2}{n^2} + \tfrac{1}{2}\frac{B_2}{4!}\frac{\omega^4}{n^4} - \tfrac{1}{3}\frac{B_3}{6!}\frac{\omega^6}{n^6} - \dots\right)$$

$$= \log_e \omega - \log_e n - \tfrac{1}{2}\omega . n^{-1} + \tfrac{1}{2}\left(\tfrac{1}{12}\omega^2\right) n^{-2} - \tfrac{1}{20}\left(\tfrac{1}{12}\omega^2\right)^2 n^{-4}$$
$$+ \tfrac{1}{105}\left(\tfrac{1}{12}\omega^2\right)^3 n^{-6} - \tfrac{3}{1400}\left(\tfrac{1}{12}\omega^2\right)^4 n^{-8} + \dots; \tag{4}$$

whence
$$10+\log_{10}(1-a)$$
$$= (10+\log_{10}\omega)-\log_{10}n-\tfrac{1}{2}\log_{10}2\,.\,n^{-1}$$
$$+\mu\left\{\tfrac{1}{2}\left(\tfrac{1}{12}\omega^2\right)n^{-2}-\tfrac{1}{20}\left(\tfrac{1}{12}\omega^2\right)^2n^{-4}+\tfrac{1}{105}\left(\tfrac{1}{12}\omega^2\right)^3n^{-6}\right.$$
$$\left.-\tfrac{3}{1400}\left(\tfrac{1}{12}\omega^2\right)^4n^{-8}+\ldots\right\}. \quad (5)$$

From these formulæ the values of $\log_{10}a$, a, $\log_e(1-a)$, and $10+\log_{10}(1-a)$ can be calculated for successive values of $\log_{10}n$. As these latter proceed by equal intervals of $\cdot 1$, the significant figures in each constituent term of any one of the above expressions will recur after ten values of $\log_{10}n$; so that the calculation is very simple. For most of the calculations the important formula is (5), which gives

$$\begin{aligned}10+\log_{10}(1-a) = {} & 9{\cdot}84082\ 54610\ 45 \\ & -\log_{10}n \\ & -{\cdot}15051\ 49978\ n^{-1} \\ & +{\cdot}00869\ 40872\ n^{-2} \\ & -{\cdot}00003\ 48092\ n^{-4} \\ & +{\cdot}00000\ 02655\ n^{-6} \\ & -{\cdot}00000\ 00024\ n^{-8} \\ & +\&c.\end{aligned}$$

For Table I., we have

$$\left.\begin{aligned} a &= \sqrt{\frac{2}{\pi}}\int_0^x e^{-\frac{1}{2}x^2}dx \\ P &= \frac{a}{a-2xz} \\ \log_{10}(P-1) &= \log_{10}\frac{2xz}{a-2xz}\end{aligned}\right\},$$

where
$$z = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}.$$

The values of x, up to $\log_{10}n = 4{\cdot}5$, were in the first instance calculated directly from a table based (with corrections) on Kramp's table of values of

$$\log_{10}\int_t^\infty e^{-t^2}dt;$$

but they were afterwards verified and extended, and the values of

$\log_{10}(P-1)$ calculated, by means of Dr. Burgess's recent tables* of

$$\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}\,dt \quad \text{and} \quad \frac{2}{\sqrt{\pi}}e^{-t^2}.$$

We have

$$\left.\begin{aligned} a &= \frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}\,dt \\ P-1 &= \frac{t\,.\,2\zeta}{a-t\,.\,2\zeta} \end{aligned}\right\},$$

where

$$\left.\begin{aligned} t &= x/\sqrt{2} \\ 2\zeta &= 2z\sqrt{2} = \frac{2}{\sqrt{\pi}}e^{-t^2} \end{aligned}\right\}.$$

As a table giving t in terms of $a-t\,.\,2\zeta$ was required for determining the values of R in Table III., I augmented Burgess's table by constructing tables of

$$10+\log_{10}(1-a+t\,.\,2\zeta) \quad \text{and} \quad 10+\log_{10}\left\{(t\,.\,2\zeta)/(1-a+t\,.\,2\zeta)\right\},$$

in addition to a table of $10+\log_{10}(1-a)$ from $t = 2{\cdot}7$ to $t = 4{\cdot}3$. The larger values of x in Table I., and the values of $10+\log_{10}(P-1)$, were calculated from these auxiliary tables.

For Table II., we have $\quad a = 1-e^{-\frac{1}{2}r^2},$

so that $$r^2 = -2\log_e(1-a).$$

The values can be calculated directly from (4), which gives

$$\begin{aligned} r^2 = \ & 4{\cdot}60517\ 01859\ 88\ \log_{10} n \\ & +{\cdot}73302\ 58412 \\ & +{\cdot}69314\ 71806\ n^{-1} \\ & -{\cdot}04003\ 77512\ n^{-2} \\ & +{\cdot}00016\ 03022\ n^{-4} \\ & -{\cdot}00000\ 12225\ n^{-6} \\ & +{\cdot}00000\ 00110\ n^{-8} \\ & -\&c., \end{aligned}$$

or they can be obtained from the values already found for $\log_{10}(1-a)$,

---

* *Transactions of the Royal Society of Edinburgh*, Vol. xxxix., Pt. 2, No. 9 (March, 1898). There are a few errata, which can usually be detected by taking differences. Thus, on p. 321, in value of H for $t = 3{\cdot}5$, "258" should apparently be "256," and, in value for $t = 4{\cdot}1$, "932 999 724" should be "993 299 972." (I have not checked these by direct calculation.) There are also some mistakes in the table on p. 279.

by multiplying by $-2/\mu = -4{\cdot}60517\ 01859\ 88$. Also

$$P = \frac{1-e^{-\frac{1}{2}r^2}}{1-(1+\frac{1}{2}r^2)\,e^{-\frac{1}{2}r^2}},$$

so that
$$P-1 = \frac{r^2}{C-r^2},$$

where
$$\begin{aligned} C &= 2\,(e^{\frac{1}{2}r^2}-1) \\ &= \frac{2}{1-e^{-\omega/n}}-2 \\ &= \frac{2n}{\omega}\left\{1-\tfrac{1}{2}\,\frac{\omega}{n}+\frac{B_1}{2!}\,\frac{\omega^2}{n^2}-\frac{B_2}{4!}\,\frac{\omega^4}{n^4}+\dots\right\}. \end{aligned}$$

The coefficients of powers of n^{-1} in the expression in curled brackets are small multiples of the coefficients, already calculated, in (4), so that they can be written down at once: multiplying by

$$2/\omega = 2{\cdot}88539\ 00817\ 78,$$

we have C in powers of n^{-1}. Calculating the values of C, those of $P-1 = r^2/(C-r^2)$ are obtained by direct division.

For Table III., $\alpha = \sqrt{\dfrac{2}{\pi}} \displaystyle\int_0^R e^{-\frac{1}{2}R^2} R^2\,dR$

$$= \sqrt{\frac{2}{\pi}} \int_0^R e^{-\frac{1}{2}R^2}\,dR - 2R\,\frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}R^2}.$$

Or, writing $t = R/\sqrt{2}$,

$$\alpha = \frac{2}{\sqrt{\pi}} \int_0^t e^{-t^2}\,dt - t\,\frac{2}{\sqrt{\pi}}\,e^{-t^2}.$$

The values of t for the different values of $10+\log_{10}(1-\alpha)$ were found from the table of values of

$$10+\log_{10}\left(1-\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}\,dt + t\,\frac{2}{\sqrt{\pi}}\,e^{-t^2}\right)$$

mentioned above; and these give the values of R and R^2. Denoting $\dfrac{2}{\sqrt{\pi}}e^{-t^2}$ by ζ, we have then

$$P-1 = \frac{\frac{2}{3}t^2\,.\,2t\zeta}{\alpha-\frac{2}{3}t^2\,.\,2t\zeta}.$$

The values of $\log_{10}(\frac{2}{3}t^2\,.\,2t\zeta)$ were found by direct calculation from the values of t, and thence those of $\log_{10}(\alpha-\frac{2}{3}t^2\,.\,2t\zeta)$; the difference giving the values of $\log_{10}(P-1)$.

All the values given are correct to five places of decimals. They were taken to 7, 8, or 9 places, and tested by successive differences; and cases in which the sixth and seventh figures were near ... 50 were specially tested.

A change of sign of Δ^2, in any table, is marked by an asterisk (*).

TABLE I. ONE ORGAN.

Values of x, P, and $10 + log_{10}(P-1)$.

$\log_{10} n$	x	Δ	Δ^2	P	$10 + \log_{10}(P-1)$	Δ	Δ^2
		+	−			−	(±)
1·0	1·83190	9839	313	1·41357	9·61655	8775	207
1·1	1·93029	9544	295	1·33791	9·52880	8614	161
1·2	2·02573	9265	279	1·27712	9·44266	8492	122
1·3	2·11838	9001	264	1·22790	9·35774	8401	91
1·4	2·20839	8752	249	1·18781	9·27373	8336	65
1·5	2·29591	8520	232	1·15501	9·19037	8291	45
1·6	2·38111	8301	219	1·12807	9·10746	8264	27
1·7	2·46412	8095	206	1·10588	9·02482	8250	14
1·8	2·54507	7901	194	1·08756	8·94232	8248	2
1·9	2·62408	7719	182	1·07242	8·85984	8256	8
2·0	2·70127	7547	172	1·05988	8·77728	8270	14
2·1	2·77674	7384	163	1·04950	8·69458	8290	20
2·2	2·85058	7231	153	1·04090	8·61168	8315	25
2·3	2·92289	7087	144	1·03377	8·52853	8344	29
2·4	2·99376	6948	139	1·02787	8·44509	8374	30
2·5	3·06324	6819	129	1·02298	8·36135	8407	33
2·6	3·13143	6694	125	1·01894	8·27728	8442	35
2·7	3·19837	6577	117	1·01559	8·19286	8476	34
2·8	3·26414	6465	112	1·01283	8·10810	8511	35
2·9	3·32879	6358	107	1·01054	8·02299	8547	36
3·0	3·39237	6256	102	1·00866	7·93752	8581	34
3·1	3·45493	6159	97	1·00711	7·85171	8616	35
3·2	3·51652	6065	94	1·00583	7·76555	8650	34
3·3	3·57717	5976	89	1·00478	7·67905	8682	32
3·4	3·63693	5890	86	1·00391	7·59223	8715	33
3·5	3·69583	5808	82	1·00320	7·50508	8745	30
3·6	3·75391	5729	79	1·00262	7·41763	8776	31
3·7	3·81120	5654	75	1·00214	7·32987	8806	30
3·8	3·86774	5580	74	1·00175	7·24181	8833	27
3·9	3·92354	5509	71	1·00142	7·15348	8861	28
4·0	3·97863	5442	67	1·00116	7·06487	8887	26
4·1	4·03305	5376	66	1·00095	6·97600	8913	26
4·2	4·08681	5313	63	1·00077	6·88687	8936	23
4·3	4·13994	5252	61	1·00063	6·79751	8961	25
4·4	4·19246	5193	59	1·00051	6·70790	8983	22
4·5	4·24439	5135	58	1·00042	6·61807	9004	21
4·6	4·29574	5080	55	1·00034	6·52803	9026	22
4·7	4·34654	5027	53	1·00027	6·43777	9046	20
4·8	4·39681	4974	53	1·00022	6·34731	9065	19
4·9	4·44655	4923	51	1·00018	6·25666	9084	19
5·0	4·49578	4875	48	1·00015	6·16582	9102	18
5·1	4·54453	4827	48	1·00012	6·07480	9120	18
5·2	4·59280	4780	47	1·00010	5·9836[illegible]	9136	16
5·3	4·64060	4736	44	1·00008	5·89224	9152	16
5·4	4·68796	4691	45	1·00006	5·80072	9168	16
5·5	4·73487	4649	42	1·00005	5·70904	9183	15
5·6	4·78136	4608	41	1·00004	5·61721	9197	14
5·7	4·82744	4567	41	1·00003	5·52524	9212	15
5·8	4·87311	4527	40	1·00003	5·43312	9225	13
5·9	4·91838	4489	38	1·00002	5·34087	9238	13
6·0	4·96327		37	1·00002	5·24849		13

TABLE II. TWO ORGANS.

Values of r and r^2.

$\log_{10} n$	r	Δ	Δ^2	r^2	Δ	Δ^2
		+	−		+	+
1·0	2·32532	9408	317	5·40711	44641	361
1·1	2·41940	9114	294	5·85352	44929	288
1·2	2·51054	8838	276	6·30281	45158	229
1·3	2·59892	8581	257	6·75439	45341	183
1·4	2·68473	8342	239	7·20780	45486	145
1·5	2·76815	8118	224	7·66266	45602	116
1·6	2·84933	7909	209	8·11868	45695	93
1·7	2·92842	7713	196	8·57563	45768	73
1·8	3·00555	7529	184	9·03331	45826	58
1·9	3·08084	7357	172	9·49157	45872	46
2·0	3·15441	7195	162	9·95029	45910	38
2·1	3·22636	7042	153	10·40939	45938	28
2·2	3·29678	6899	143	10·86877	45962	24
2·3	3·36577	6762	137	11·32839	45980	18
2·4	3·43339	6634	128	11·78819	45995	15
2·5	3·49973	6513	121	12·24814	46007	12
2·6	3·56486	6396	117	12·70821	46016	9
2·7	3·62882	6287	109	13·16837	46023	7
2·8	3·69169	6183	104	13·62860	46029	6
2·9	3·75352	6083	100	14·08889	46034	5
3·0	3·81435	5987	96	14·54923	46037	3
3·1	3·87422	5897	90	15·00960	46041	4
3·2	3·93319	5810	87	15·47001	46042	1
3·3	3·99129	5728	82	15·93043	46045	3
3·4	4·04857	5647	81	16·39088	46046	1
3·5	4·10504	5571	76	16·85134	46047	1
3·6	4·16075	5497	74	17·31181	46048	1
3·7	4·21572	5427	70	17·77229	46049	1
3·8	4·26999	5358	69	18·23278	46050	1
3·9	4·32357	5293	65	18·69328	46050	
4·0	4·37650	5230	63	19·15378	46050	
4·1	4·42880	5169	61	19·61428	46050	
4·2	4·48049	5110	59	20·07478	46051	
4·3	4·53159	5053	57	20·53529	46051	
4·4	4·58212	4998	55	20·99580	46051	
4·5	4·63210	4944	54	21·45631	46052	
4·6	4·68154	4893	51	21·91683	46051	
4·7	4·73047	4843	50	22·37734	46051	
4·8	4·77890	4794	49	22·83785	46052	
4·9	4·82684	4747	47	23·29837	46051	
5·0	4·87431	4701	46	23·75888	46052	
5·1	4·92132	4657	44	24·21940	46052	
5·2	4·96789	4613	44	24·67992	46051	
5·3	5·01402	4572	41	25·14043	46052	
5·4	5·05974	4530	42	25·60095	46051	
5·5	5·10504	4491	39	26·06146	46052	
5·6	5·14995	4452	39	26·52198	46052	
5·7	5·19447	4414	38	26·98250	46051	
5·8	5·23861	4377	37	27·44301	46052	
5·9	5·28238	4341	36	27·90353	46052	
6·0	5·32579		35	28·36405		

TABLE II. TWO ORGANS—*continued.*

Values of P and $10 + log_{10}(P-1)$.

$\log_{10} n$	P	$10 + \log_{10}(P-1)$	Δ	Δ^2
			—	(±)
1·0	1·24076	9·38159	8371	88
1·1	1·19856	9·29788	8307	64
1·2	1·16399	9·21481	8265	42
1·3	1·13557	9·13216	8240	25
1·4	1·11214	9·04976	8228	12*
1·5	1·09279	8·96748	8229	1
1·6	1·07677	8·88519	8238	9
1·7	1·06350	8·80281	8255	17
1·8	1·05251	8·72026	8278	23
1·9	1·04340	8·63748	8305	27
2·0	1·03585	8·55443	8335	30
2·1	1·02959	8·47108	8368	33
2·2	1·02440	8·38740	8404	36
2·3	1·02011	8·30336	8439	35
2·4	1·01656	8·21897	8475	36
2·5	1·01362	8·13422	8513	38
2·6	1·01120	8·04909	8549	36
2·7	1·00920	7·96360	8585	36
2·8	1·00755	7·87775	8621	36
2·9	1·00619	7·79154	8656	35
3·0	1·00507	7·70498	8690	34
3·1	1·00415	7·61808	8723	33
3·2	1·00340	7·53085	8755	32
3·3	1·00278	7·44330	8786	31
3·4	1·00227	7·35544	8816	30
3·5	1·00185	7·26728	8845	29
3·6	1·00151	7·17883	8873	28
3·7	1·00123	7·09010	8899	26
3·8	1·00100	7·00111	8926	27
3·9	1·00082	6·91185	8950	24
4·0	1·00066	6·82235	8974	24
4·1	1·00054	6·73261	8996	22
4·2	1·00044	6·64265	9019	23
4·3	1·00036	6·55246	9040	21
4·4	1·00029	6·46206	9060	20
4·5	1·00024	6·37146	9080	20
4·6	1·00019	6·28066	9099	19
4·7	1·00015	6·18967	9116	17
4·8	1·00013	6·09851	9134	18
4·9	1·00010	6·00717	9151	17
5·0	1·00008	5·91566	9167	16
5·1	1·00007	5·82399	9183	16
5·2	1·00005	5·73216	9197	14
5·3	1·00004	5·64019	9212	15
5·4	1·00004	5·54807	9226	14
5·5	1·00003	5·45581	9240	14
5·6	1·00002	5·36341	9252	12
5·7	1·00002	5·27089	9266	14
5·8	1·00002	5·17823	9277	11
5·9	1·00001	5·08546	9289	12
6·0	1·00001	4·99257		12

TABLE III. THREE ORGANS.

Values of R and R^2.

$\log_{10} n$	R	Δ	Δ^2	R^2	Δ	Δ^2
		+	−		+	(±)
1·0	2·67583		318	7·16007		51
		9196			50062	
1·1	2·76779		294	7·66069		5*
		8902			50067	
1·2	2·85681		275	8·16136		28
		8627			50039	
1·3	2·94308		253	8·66175		52
		8374			49987	
1·4	3·02682		238	9·16162		68
		8136			49919	
1·5	3·10818		219	9·66081		82
		7917			49837	
1·6	3·18735		206	10·15918		88
		7711			49749	
1·7	3·26446		193	10·65667		93
		7518			49656	
1·8	3·33964		178	11·15323		96
		7340			49560	
1·9	3·41304		169	11·64883		96
		7171			49464	
2·0	3·48475		158	12·14347		95
		7013			49369	
2·1	3·55488		149	12·63716		93
		6864			49276	
2·2	3·62352		139	13·12992		92
		6725			49184	
2·3	3·69077		133	13·62176		88
		6592			49096	
2·4	3·75669		124	14·11272		85
		6468			49011	
2·5	3·82137		119	14·60283		82
		6349			48929	
2·6	3·88486		112	15·09212		78
		6237			48851	
2·7	3·94723		106	15·58063		76
		6131			48775	
2·8	4·00854		102	16·06838		72
		6029			48703	
2·9	4·06883		95	16·55541		69
		5934			48634	
3·0	4·12817		93	17·04175		66
		5841			48568	
3·1	4·18658		88	17·52743		63
		5753			48505	
3·2	4·24411		83	18·01248		60
		5670			48445	
3·3	4·30081		81	18·49693		58
		5589			48387	
3·4	4·35670		77	18·98080		54
		5512			48333	
3·5	4·41182		74	19·46413		53
		5438			48280	
3·6	4·46620		71	19·94693		50
		5367			48230	
3·7	4·51987		68	20·42923		48
		5299			48182	
3·8	4·57286		66	20·91105		47
		5233			48135	
3·9	4·62519		63	21·39240		43
		5170			48092	
4·0	4·67689		61	21·87332		43
		5109			48049	
4·1	4·72798		59	22·35381		40
		5050			48009	
4·2	4·77848		56	22·83390		40
		4994			47969	
4·3	4·82842		56	23·31359		37
		4938			47932	
4·4	4·87780		53	23·79291		36
		4885			47896	
4·5	4·92665		51	24·27187		35
		4834			47861	
4·6	4·97499		51	24·75048		33
		4783			47828	
4·7	5·02282		47	25·22876		32
		4736			47796	
4·8	5·07018		47	25·70672		31
		4689			47765	
4·9	5·11707		46	26·18437		30
		4643			47735	
5·0	5·16350		44	26·66172		29
		4599			47706	
5·1	5·20949		43	27·13878		28
		4556			47678	
5·2	5·25505		41	27·61556		27
		4515			47651	
5·3	5·30020		42	28·09207		26
		4473			47625	
5·4	5·34493		38	28·56832		25
		4435			47600	
5·5	5·38928		39	29·04432		25
		4396			47575	
5·6	5·43324		38	29·52007		23
		4358			47552	
5·7	5·47682		36	29·99559		23
		4322			47529	
5·8	5·52004		35	30·47088		23
		4287			47506	
5·9	5·56291		35	30·94594		21
		4252			47485	
6·0	5·60543		35	31·42079		20

TABLE III. THREE ORGANS—*continued.*

Values of P and $10+log_{10}(P-1)$.

$\log_{10} n$	P	$10+\log_{10}(P-1)$	Δ	Δ^2
			–	(±)
1·0	1·17957	9·25423	8246	45
1·1	1·14852	9·17177	8219	27
1·2	1·12291	9·08958	8206	13
1·3	1·10175	9·00752	8206	0
1·4	1·08423	8·92546	8215	9
1·5	1·06971	8·84331	8233	18
1·6	1·05767	8·76098	8256	23
1·7	1·04769	8·67842	8284	28
1·8	1·03941	8·59558	8315	31
1·9	1·03254	8·51243	8350	35
2·0	1·02685	8·42893	8386	36
2·1	1·02213	8·34507	8424	38
2·2	1·01823	8·26083	8461	37
2·3	1·01500	8·17622	8500	39
2·4	1·01234	8·09122	8538	38
2·5	1·01014	8·00584	8575	37
2·6	1·00832	7·92009	8613	38
2·7	1·00682	7·83396	8649	36
2·8	1·00559	7·74747	8684	35
2·9	1·00458	7·66063	8718	34
3·0	1·00374	7·57345	8752	34
3·1	1·00306	7·48593	8784	32
3·2	1·00250	7·39809	8814	30
3·3	1·00204	7·30995	8845	31
3·4	1·00167	7·22150	8873	28
3·5	1·00136	7·13277	8900	27
3·6	1·00111	7·04377	8927	27
3·7	1·00090	6·95450	8953	26
3·8	1·00073	6·86497	8977	24
3·9	1·00060	6·77520	9000	23
4·0	1·00048	6·68520	9023	23
4·1	1·00039	6·59497	9044	21
4·2	1·00032	6·50453	9065	21
4·3	1·00026	6·41388	9085	20
4·4	1·00021	6·32303	9104	19
4·5	1·00017	6·23199	9123	19
4·6	1·00014	6·14076	9140	17
4·7	1·00011	6·04936	9157	17
4·8	1·00009	5·95779	9174	17
4·9	1·00007	5·86605	9189	15
5·0	1·00006	5·77416	9204	15
5·1	1·00005	5·68212	9219	15
5·2	1·00004	5·58993	9234	15
5·3	1·00003	5·49759	9246	12
5·4	1·00003	5·40513	9260	14
5·5	1·00002	5·31253	9273	13
5·6	1·00002	5·21980	9285	12
5·7	1·00001	5·12695	9297	12
5·8	1·00001	5·03398	9308	11
5·9	1·00001	4·94090	9319	11
6·0	1·00001	4·84771		11

On the Direct Determination of Stress in an Elastic Solid, with application to the Theory of Plates. *By* J. H. MICHELL, M.A.

Read April 13th, 1899. Received, in revised form, September 4th, 1899.

In treating the problem of an elastic solid in equilibrium under given volume- and surface-forces, some of the advantages of a direct determination of the stress are so obvious that it is surprising more attention has not been given to this mode of attack. G. B. Airy,[*] in 1862, gave a solution of the statical equations of stress in two dimensions in terms of a function which is called by Maxwell "Airy's function of stress in two dimensions." Airy did not consider the differential equation satisfied by his function. This arises from substituting in the identical strain-relation[†] of St. Venant the values of the strains in terms of the stresses.

Maxwell,[‡] in 1869, supplied this equation in an awkward form, and extended Airy's method to three dimensions by means of three functions of stress. Finally, Ibbetson[§], in 1886, gave the straightforward process for determining the equations satisfied by Maxwell's functions by substitution in the six identical strain-relations of St. Venant.

In the present paper I begin with a discussion of plane stress in an isotropic body under given volume- and surface-forces. The problem is reduced to the determination of a function ψ satisfying $\nabla^4_{xy}\psi = 0$ with ψ, and $d\psi/dn$ given over the boundary. It is shown that the stress is independent of the moduli of elasticity if there is no volume-force, and if the body is simply-connected, and that the same is true for a multiply-connected body if the resultant force (not necessarily the couple) over *each* boundary separately vanishes. Reference must here be made to a statement of Maxwell's at the bottom of p. 201 of the paper cited, which, without adequate discussion, partly anticipates this result, but appears to involve more than one oversight. I have

* *Brit. Assoc. Report*, 1862.

† See A. E. H. Love, *Elasticity*, Vol. I., § 66.

‡ *Scientific Papers*, Vol. II., p. 161.

§ *Proc. Lond. Math. Soc.*, Vol. XVII. Reference should also be made to Voigt, Wiedemann, *Annalen*, XVI., 1882.

discussed at some length the conditions to be satisfied by ψ, in order that the displacements may be single-valued, as well as the form and uniqueness of the solution to be obtained for that function.

In the second part of the paper I begin by obtaining the equations of stress in three dimensions, in a form analogous to the ordinary equations of displacement. It is not advantageous to introduce Maxwell's functions of stress, at any rate for the applications I have in view. Here, again, I have considered the surface-conditions to be satisfied by the stresses in order that the displacements deduced may be single-valued.

In the last part of the paper the stress-equations are applied to the theory of plates. A general method of solution for any distribution of force is given. The essential difference between this and previous solutions is that no assumption is here made as to the values of the stresses on planes parallel to the faces of the plate. Instead, it is shown how to begin by determining the value of the normal pressure on such planes without considering the boundary-conditions. The possibility of this rests on the fact, almost intuitive, that any local normal pressure cannot be transmitted along the plate, except to an utterly negligible extent, a distance many times the plate's thickness. It is further shown that each of the tangential stresses on planes parallel to the faces is composed of two terms, one of which depends on the form of the median plane of the plate, and the other is determined directly in terms of the applied forces. The other stresses are then expressed in terms of the curvature and stretch of the median plane and their rates of change, together with the quantities already completely determined. Differential equations of the fourth order are next obtained for the two unknown functions in terms of which the normal displacement and the stretch of the median plane are expressed.

Finally, a method of successive approximation is indicated connecting the solution here given with the ordinary approximation. The elastic solid has throughout been supposed isotropic; the method of extension to anisotropic bodies is perfectly obvious.

Plane Stress.

Under this heading we may, following Maxwell, conveniently treat two problems: (*a*) that of a long cylinder with applied forces perpendicular to its length, and the same at corresponding points along its length; (*b*) that of a thin plate with applied forces in its plane.

Case (*a*).

Adopting the notation of Thomson and Tait's *Natural Philosophy* and Love's *Elasticity*, and taking the axis of z in the direction of the length of the cylinder, we have here

$$S = T = 0,$$

$$g = \text{const.},$$

and u, v functions of x, y only.

$P = \sigma_{xx}$
$Q = \sigma_{yy}$
$R = \sigma_{zz}$
$S = \sigma_{yz}$
$T = \sigma_{zx}$
$U = \sigma_{xy}$

The equations of stress become

$$P_x + U_y = V_x,$$

$$U_x + Q_y = V_y,$$

assuming a force-potential and using suffixes to denote differentiation where no doubt can arise as to the meaning.

These equations are satisfied quite generally by

$$P - V = \psi_{yy},$$

$$Q - V = \psi_{xx},$$

$$U = -\psi_{xy},$$

where ψ is Airy's function of stress.

$e = \epsilon_{xx}$
$f = \epsilon_{yy}$
$g = \epsilon_{zz}$
$a = \epsilon_{yz}$
$b = \epsilon_{zx}$
$c = \epsilon_{xy}$

Write $\quad \Theta \equiv P + Q + R = (3\lambda + 2\mu)(e + f + g).$

Since $\quad P + Q = 2(\lambda + \mu)(e + f) + 2\lambda g,$

we have $\quad \Theta = \dfrac{3\lambda + 2\mu}{2(\lambda + \mu)}(P + Q) + \text{const.}$

The strain-relation* $\quad e_{yy} + f_{xx} = c_{xy}$

becomes $\quad P_{yy} + Q_{xx} - \dfrac{\lambda}{3\lambda + 2\mu}\nabla^2_{xy}\Theta = 2U_{xy}$

or $\quad \nabla^4_{xy}\psi + \nabla^2_{xy}V - \dfrac{\lambda}{2(\lambda + \mu)}\nabla^2_{xy}(\nabla^2_{xy}\psi + 2V) = 0,$

that is, $\quad (\lambda + 2\mu)\nabla^4_{xy}\psi = -2\mu\nabla^2_{xy}V,$

where $\quad \nabla^2_{xy} \equiv \dfrac{d^2}{dx^2} + \dfrac{d^2}{dy^2}.$

---

* Love's *Elasticity*, Vol. I., § 66.

We may now drop the suffix, since the coordinate z no longer appears, and write the equation for ψ,

$$(\lambda+2\mu)\,\nabla^4\psi = -2\mu\nabla^2 V, \tag{1}$$

where
$$\nabla^2 \equiv \frac{d^2}{dx^2}+\frac{d^2}{dy^2}.$$

Case (*b*).

We adopt for the present the ordinary approximation, viz.: taking the axis of z normal to the plate, we put

$$R = S = T = 0,$$

so that
$$\Theta = P+Q.$$

The stress-equations are solved as in Case (*a*) and the strain-relation, which is equivalent to

$$P_{yy}+Q_{xx}-\frac{\lambda}{3\lambda+2\mu}\,\nabla^2_{xy}\Theta = 2U_{xy},$$

as before, now becomes

$$\nabla_{xy}\psi+\nabla_{xy}V-\frac{\lambda}{3\lambda+2\mu}\,\nabla^2_{xy}\,(\nabla^2_{xy}\psi+2V) = 0,$$

that is
$$2\,(\lambda+\mu)\,\nabla^4_{xy}\psi = -\,(\lambda+2\mu)\,\nabla^2_{xy}V,$$

or, dropping the suffixes, the equation for ψ is now

$$2\,(\lambda+\mu)\,\nabla^4\psi = -(\lambda+2\mu)\,\nabla^2 V. \tag{2}$$

Conditions for ψ in a Multiply-connected Body.

In these equations ψ is not in general single-valued if the body is not singly-connected. The second and higher derivatives of ψ must always be single-valued if V is so, since the stresses must be so. Further, ψ must be such that the displacements are single-valued.

Since
$$u_{xy} = e_y,$$
$$u_{yy} = c_y - f_x,$$
we have
$$u_y = {}_0u_y + \int_0 \{e_y\,dx + (c_y - f_x)\,dy\}.$$

The strain-relation makes the integral vanish for a complete reducible circuit; but we must have in addition

$$\int_0^0 \{e_y\,dx + (c_y - f_x)\,dy\} = 0 \tag{3}$$

on each independent irreducible circuit.

Since $u_y + v_x = c$ is single-valued, the condition (3) will ensure that v_x is single-valued, and hence no further conditions are required for the first derivatives of the displacements. We have still to express the conditions that u, v may be single-valued. We must have

$$\int_0^0 (u_x dx + u_y dy) = 0$$

and
$$\int_0^0 (v_x dx + v_y dy) = 0$$

for each irreducible circuit.

Now $\displaystyle\int_0^0 u_x dx = \left[xu_x\right]_0^0 - \int_0^0 x\, du_x = -\int_0^0 x\,(e_x dx + e_y dy)$

and $\displaystyle\int_0^0 u_y dy = \left[yu_y\right]_0^0 - \int_0^0 y\, du_y = -\int_0^0 y\,\{e_y dx + (c_y - f_x)\, dy\},$

the condition (3) being supposed satisfied.

Hence
$$\int_0^0 \left[(xe_x + ye_y)\, dx + \{xe_y + y\,(c_y - f_x)\}\, dy\right] = 0, \tag{4}$$

and, similarly,

$$\int_0^0 \left[\{yf_x + x\,(c_x - e_y)\}\, dx + (xf_x + yf_y)\, dy\right] = 0. \tag{5}$$

We proceed to express these equations in terms of ψ.

In *Case* (*a*), $\displaystyle 2\mu e = P - \frac{\lambda}{3\lambda + 2\mu}\Theta$

$$= P - \frac{\lambda}{2\,(\lambda + \mu)}(P + Q) + \text{const.}$$

or $\quad 4\mu\,(\lambda + \mu)\, e = (\lambda + 2\mu)(P + Q) - 2\,(\lambda + \mu)\, Q + \text{const.},$

and, similarly,

$$4\mu\,(\lambda + \mu) f = (\lambda + 2\mu)(P + Q) - 2\,(\lambda + \mu)\, P + \text{const.}$$

Hence equation (3) becomes

$$(\lambda + 2\mu)\int_0^0 \left\{\frac{d}{dy}(P + Q)\, dx - \frac{d}{dx}(P + Q)\, dy\right\}$$
$$-2\,(\lambda + \mu)\int_0^0 \left\{\frac{d}{dy}(\psi_{xx})\, dx + \frac{d}{dy}(\psi_{xy})\, dy\right\}$$
$$-2\,(\lambda + \mu)\int_0^0 (V_y dx - V_x dy) = 0,$$

that is

$$-(\lambda+2\mu)\int_0^0 \frac{d}{dn}(P+Q)\,ds-2(\lambda+\mu)\left[\psi_{xy}\right]_0^0+2(\lambda+\mu)\int_0^0\frac{dV}{dn}\,ds=0,$$

where ds is an element of arc of the boundary of a section z const., dn an element of normal to that boundary.

Since ψ_{xy} is single-valued, this reduces to

$$(\lambda+2\mu)\int_0^0\frac{d}{dn}(\nabla^2\psi)\,ds+2\mu\int_0^0\frac{dV}{dn}\,ds=0, \tag{6}$$

which may also be written

$$\frac{\lambda+2\mu}{3\lambda+2\mu}\int_0^0\frac{d\Theta}{dn}\,ds=\int_0^0\frac{dV}{dn}\,ds,$$

or, again,

$$(\lambda+2\mu)\int_0^0\frac{d\theta}{dn}\,ds=\int_0^0\frac{dV}{dn}\,ds,$$

in which form it is a simple deduction from the displacement-equations

$$(\lambda+2\mu)\,\theta_x-2\mu\varpi_y=V_x,$$

$$(\lambda+2\mu)\,\theta_y+2\mu\varpi_x=V_y,$$

viz., we deduce

$$(\lambda+2\mu)\frac{d\theta}{dn}=\frac{dV}{dn}+2\mu\frac{d\varpi}{ds};$$

and, integrating around a boundary and remembering that ϖ is single-valued, the equation at once follows.

If there is no volume-force,

$$\int_0^0\frac{d\theta}{dn}\,ds=0$$

over each boundary.

The existence of the corresponding equation

$$\int\frac{d\theta}{dn}\,dS=0$$

over *each* bounding surface in a three-dimensional solid under no volume-force may here be noted. It is of importance in connexion with solutions in which a determination of θ is the first step. For example, in the problem of the stress of a cylindrical or a spherical boiler under uniform pressure it shows at once that θ is constant.

The equation (4) becomes

$$2(\lambda+\mu)\int_0^0 d\psi_x+(\lambda+2\mu)\int_0^0\left\{x\frac{d}{ds}(P+Q)-y\frac{d}{dn}(P+Q)\right\}ds$$
$$-2(\lambda+\mu)\int_0^0\left(x\frac{dV}{ds}-y\frac{dV}{dn}\right)ds=0,$$

or

$$2(\lambda+\mu)\left[\psi_x\right]_0^0+(\lambda+2\mu)\int_0^0\left(x\frac{d}{ds}\nabla^2\psi-y\frac{d}{dn}\nabla^2\psi\right)ds$$
$$+2\mu\int_0^0\left(x\frac{dV}{ds}-y\frac{dV}{dn}\right)ds=0, \tag{7}$$

and, similarly, equation (5) becomes

$$2(\lambda+\mu)\left[\psi_y\right]^0+(\lambda+2\mu)\int_0^0\left(y\frac{d}{ds}\nabla^2\psi+x\frac{d}{dn}\nabla^2\psi\right)ds$$
$$+2\mu\int_0^0\left(y\frac{dV}{ds}+x\frac{dV}{dn}\right)ds=0. \tag{8}$$

In *Case* (*b*), $$2\mu e=P-\frac{\lambda}{3\lambda+2\mu}(P+Q),$$

or $$2\mu(3\lambda+2\mu)e=2(\lambda+\mu)(P+Q)-(3\lambda+2\mu)Q$$

and $$2\mu(3\lambda+2\mu)f=2(\lambda+\mu)(P+Q)-(3\lambda+2\mu)P,$$

so that the appropriate equations in this case are derived from those of *Case* (*a*) by substituting for λ according to the equation

$$\frac{\lambda'}{3\lambda'+2\mu}=\frac{\lambda}{2(\lambda+\mu)},$$

that is, putting $$\lambda+2\mu=\frac{4\mu(\lambda'+\mu)}{\lambda'+2\mu},$$

$$\lambda+\mu=\frac{\mu(3\lambda'+2\mu)}{\lambda'+2\mu}.$$

The meanings of $\left[\psi_x\right]_0^0$, $\left[\psi_y\right]_0^0$ will appear from the next section.

The Stresses at the Boundary.

With the usual notation,

$$lP+mU=F,$$
$$lU+mQ=G,$$

where $$l=dy/ds,\quad m=-dx/ds,$$

and F, G are the x, y components of the external stress.

Hence* $$F = \frac{dy}{ds}(\psi_{yy}+V)+\frac{dx}{ds}\psi_{xy} = \frac{d}{ds}(\psi_y)+V\frac{dy}{ds}$$

and $$G = -\frac{d}{ds}(\psi_x)-V\frac{dx}{ds},$$

so that $$\left.\begin{aligned} \psi_x &= -\int_0^s G\,ds-\int_0^s V\frac{dx}{ds}\,ds+\alpha \\ \psi_y &= \int_0^s F\,ds-\int_0^s V\frac{dy}{ds}\,ds+\beta \end{aligned}\right\}, \tag{9}$$

where α, β are constants.

We may therefore write

$$\psi_x = H+\alpha,$$
$$\psi_y = K+\beta,$$

where H, K are known functions for each boundary, and α, β are unknown constants, different of course, in general, for each boundary. Hence

$$\left.\begin{aligned} \psi &= \int_0^s\left(H\frac{dx}{ds}+K\frac{dy}{ds}\right)ds+\alpha x+\beta y+\gamma \\ \frac{d\psi}{dn} &= H\frac{dy}{ds}-K\frac{dx}{ds}+\alpha\frac{dy}{ds}-\beta\frac{dx}{ds} \end{aligned}\right\}, \tag{10}$$

where γ is another unknown constant to be determined for each boundary.

If the body is singly-connected, the values of α, β, γ for the single boundary do not affect the result, for the addition of a solution

$$\psi = -\alpha x-\beta y-\gamma,$$

which does not affect the stresses, makes the constants disappear. The same is no longer true if the body is multiply-connected. We must then apply the equations (6), (7), (8) to determine the *three* constants corresponding to each boundary. Of course, the three constants corresponding to one of the boundaries are still arbitrary.

It appears from equations (9) that, if there is no volume force,

$$[\psi_x]_0^\sigma = -\int_0^\sigma G\,ds,$$

$$[\psi_y]_0^\sigma = \int_0^\sigma F\,ds,$$

---

* *Cf.* Maxwell, *loc. cit.*, p. 193. Maxwell's process is, I think, erroneous.

so that the quantities on the left are the resultant forces in the directions of the two axes, on the boundary considered. If these resultants vanish for each boundary, the function ψ is determined by the following conditions :—

$$\nabla^4\psi = 0 \text{ throughout,} \tag{11}$$

$$\left.\begin{aligned} \psi &= \int_0^s (H\,dx + K\,dy) + \alpha x + \beta y + \gamma \\ \frac{d\psi}{dn} &= H\frac{dy}{ds} - K\frac{dx}{ds} + \alpha\frac{dy}{ds} - \beta\frac{dx}{ds} \end{aligned}\right\} \tag{12}$$

at each point of each boundary, and

$$\left.\begin{aligned} &\int_0^0 \frac{d}{dn}(\nabla^2\psi)\,ds = 0 \\ &\int_0^0 \left\{ x\frac{d}{ds}(\nabla^2\psi) - y\frac{d}{dn}(\nabla^2\psi) \right\} ds = 0 \\ &\int_0^0 \left\{ y\frac{d}{ds}(\nabla^2\psi) + x\frac{d}{dn}(\nabla^2\psi) \right\} ds = 0 \end{aligned}\right\} \tag{13}$$

for each boundary.

Form of Solution.

If the constants α, β, γ are fixed, the equations (11), (12) will determine ψ uniquely. The properties of the solutions of such equations have been discussed by Mathieu* in connexion with other problems. Mathieu considers only singly-connected regions or single-valued functions, but we can easily extend his result in the present connexion. If ψ_1, ψ_2 are two solutions, $\phi = \psi_1 - \psi_2$ will satisfy $\nabla^4\phi = 0$ and make $\phi = 0$, $d\phi/dn = 0$ over the boundaries. The function ϕ is therefore a single-valued function† in the region, and Mathieu's proof shows that $\phi = 0$, or that the two assumed solutions are identical. The solution having been obtained with arbitrary constants α, β, γ, their actual values are found by substituting the solution in equations (6), (7), (8), or, in the particular case above, in equations (13). Now the moduli of elasticity do not appear in

* *Journal de Math.*, T. XIV., 2me sér., 1869, p. 391.

† Make the region simply-connected by cross-cuts. The first derivatives of ϕ have the same value (zero) at an end of a cross-cut on its two sides, and the second derivatives are single-valued; therefore the first derivatives have the same values on the two sides along the whole cross-cut. Similarly, ϕ itself is the same on the two sides of the cross-cut.

equations (13). *Hence the stresses are independent of the moduli, provided the resultant force on each boundary vanishes.* This condition is, of course, always satisfied where there is only a single boundary, so that the stresses are always independent of the moduli in this case. It must be remembered that we have assumed the absence of volume-force.

Supposing, now, that the resultant forces on the boundaries do not vanish, we can add any convenient type of stress over the boundaries to reduce the resultants to zero, so that, if there are n boundaries, the function ψ will be of the form

$$c_1\psi_1 + c_2\psi_2 + \dots + c_{2n-2}\psi_{2n-2} + \psi',$$

where $\psi_1, \psi_2, \dots, \psi_{2n-2}$ involve the moduli of elasticity and depend only on the form of the body and the types of stress added, not on the given distribution of surface stress, while $c_1, c_2, \dots, c_{2n-2}$ are constants depending only on the magnitudes of the resultant forces on the boundaries. Finally, ψ' is independent of the moduli. We can proceed further in this direction. Add any $(n-1)$ convenient types of stress which reduce the couples on the boundaries also to zero. We may then write

$$\psi = c_1\psi_1 + \dots + c_{3n-3}\psi_{3n-3} + \psi'',$$

where ψ'' is now single-valued, and $\psi_{2n-1} \dots \psi_{3n-3}$ are functions depending on the types of stress chosen and independent of the moduli. To prove this it is only necessary to show that, if the forces on each boundary equilibrate, ψ is single-valued. Now, since in this case

$$[\psi_x]_0^0 = -\int_0^0 G\,ds = 0,$$

$$[\psi_y]_0^0 = \int_0^0 F\,ds = 0,$$

ψ_x, ψ_y are single-valued. And

$$\begin{aligned}
[\psi]_0^0 &= \int_0^0 (H\,dx + K\,dy) \\
&= [Hx + Ky]_0^0 - \int_0^0 \left(x\frac{dH}{ds} + y\frac{dK}{ds}\right) ds \\
&= \int_0^0 (Gx - Fy)\,ds \\
&= 0,
\end{aligned}$$

since the couple of the stresses on s vanishes. Hence ψ is single-valued in this case.

Introduction of Curvilinear Coordinates.

For application to curved boundaries the expressions for the stresses in curvilinear coordinates are required. These may be readily obtained as follows. We have seen that the components of stress, in the directions of the axes, across any element of arc ds, are

$$F = \frac{d}{ds}(\psi_y),$$

$$G = -\frac{d}{ds}(\psi_x),$$

giving for the stress normal to the arc

$$\begin{aligned} N &= \frac{dy}{ds}\,\frac{d}{ds}(\psi_y) + \frac{dx}{ds}\,\frac{d}{ds}(\psi_x) \\ &= \frac{d}{ds}\left(\frac{dy}{ds}\psi_y + \frac{dx}{ds}\psi_x\right) - \left(\frac{d^2y}{ds^2}\psi_y + \frac{d^2x}{ds^2}\psi_x\right) \\ &= \frac{d^2\psi}{ds^2} + \kappa\frac{d\psi}{dn}, \end{aligned}$$

where κ is the curvature of the arc and dn is an element of normal to it, in the direction opposite to that in which the curvature is measured.

The tangential stress is

$$\begin{aligned} M &= \frac{dx}{ds}\,\frac{d}{ds}(\psi_y) - \frac{dy}{ds}\,\frac{d}{ds}(\psi_x) \\ &= \frac{d}{ds}\left(\frac{dx}{ds}\psi_y - \frac{dy}{ds}\psi_x\right) - \left(\frac{d^2x}{ds^2}\psi_y - \frac{d^2y}{ds^2}\psi_x\right) \\ &= -\frac{d}{ds}\left(\frac{d\psi}{dn}\right) + \kappa\frac{d\psi}{ds}. \end{aligned}$$

Introducing orthogonal curvilinears, so that

$$ds = h_2\,d\eta,$$

$$dn = h_1\,d\xi,$$

$$\kappa = \frac{1}{h_1 h_2}\,\frac{dh_2}{d\xi},$$

these expressions give at once for the corresponding elements of stress

$$\left.\begin{aligned} P' &= \frac{1}{h_2^2}\frac{d^2\psi}{d\eta^2} - \frac{1}{h_2^3}\frac{dh_2}{d\eta}\frac{d\psi}{d\eta} + \frac{1}{h_1^2 h_2}\frac{dh_2}{d\xi}\frac{d\psi}{d\xi} \\ Q' &= \frac{1}{h_1^2}\frac{d^2\psi}{d\xi^2} - \frac{1}{h_1^3}\frac{dh_1}{d\xi}\frac{d\psi}{d\xi} + \frac{1}{h_2^2 h_1}\frac{dh_1}{d\eta}\frac{d\psi}{d\eta} \\ U' &= -\frac{1}{h_1 h_2}\frac{d^2\psi}{d\xi\, d\eta} + \frac{1}{h_1^2 h_2}\frac{dh_1}{d\eta}\frac{d\psi}{d\xi} + \frac{1}{h_2^2 h_1}\frac{dh_2}{d\xi}\frac{d\psi}{d\eta} \end{aligned}\right\}. \qquad (14)$$

For example, in plane polars,

$$P' = \frac{1}{r^2}\frac{d^2\psi}{d\theta^2} + \frac{1}{r}\frac{d\psi}{dr},$$

$$Q' = \frac{d^2\psi}{dr^2},$$

$$U' = -\frac{d}{dr}\left(\frac{1}{r}\frac{d\psi}{d\theta}\right),$$

and the general form of solution for a cylindrical shell is

$$\begin{aligned} \psi = {} & A_0 r^2 + B_0 r^2(\log r - 1) + C_0 \log r + D_0\theta \\ & + (A_1 r + B_1 r^{-1} + B_1'\theta r + C_1 r^3 + D_1 r\log r)\cos\theta \\ & + (E_1 r + F_1 r^{-1} + F_1'\theta r + G_1 r^3 + H_1 r\log r)\sin\theta \\ & + \sum_2^\infty (A_n r^n + B_n r^{-n} + C_n r^{n+2} + D_n r^{-n+2})\cos n\theta \\ & + \sum_2^\infty (E_n r^n + F_n r^{-n} + G_n r^{n+2} + H_n r^{-n+2})\sin n\theta. \end{aligned}$$

I will conclude this part of the paper by remarking that Betti's "Reciprocal Theorems"* here take the form of generalized Green theorems for ψ, some of which are given by Mathieu.† The particular case of a circular boundary would, I think, repay a detailed treatment.‡

Stress Equations in Three Dimensions.

Writing the displacement equations in the form

$$(\lambda+\mu)\,\theta_x + \mu\nabla^2 u = -X, \text{ \&c.}, \qquad (15)$$

---

* Love, *Elasticity*, Vol. I., ch. viii. † *Loc. cit.*

‡ See Borchardt, "Ueber Deformationen elasticher isotroper Körper," &c., *Berlin Monatsberichte*, 1873; Hertz, *Miscellaneous Papers*, p. 261.

we have
$$(\lambda+2\mu)\,\nabla^2\theta = -\,(X_x+Y_y+Z_z)$$
$$= -\Lambda, \text{ say.}$$

Since
$$P = \lambda\theta+2\mu e,$$
$$\nabla^2 P = 2\mu\nabla^2 e-\Lambda\lambda/(\lambda+2\mu).$$

Differentiating (15) with respect to x,
$$(\lambda+\mu)\,\theta_{xx}+\mu\nabla^2 e = -\,X_x$$

therefore
$$\nabla^2 P+2\,(\lambda+\mu)\,\theta_{xx} = -\,2X_x-\Lambda\lambda/(\lambda+2\mu),$$

or similarly,
$$\left.\begin{aligned} \nabla^2 P+\kappa\Theta_{xx} &= -\,2X_x-\nu\Lambda \\ \nabla^2 Q+\kappa\Theta_{yy} &= -\,2Y_y-\nu\Lambda \\ \nabla^2 R+\kappa\Theta_{zz} &= -\,2Z_z-\nu\Lambda \end{aligned}\right\}, \tag{16}$$

where
$$\kappa = 2\,(\lambda+\mu)/(3\lambda+2\mu),$$
$$\nu = \lambda/(\lambda+2\mu).$$

Equations (16) involve
$$\nabla^2\Theta = -\,(2\nu+1)\,\Lambda. \tag{17}$$

Again, from (15),
$$2\,(\lambda+\mu)\,\theta_{yz}+\mu\nabla^2\,(w_y+v_z) = -\,Z_y-Y_z,$$

that is, similarly,
$$\left.\begin{aligned} \nabla^2 S+\kappa\Theta_{yz} &= -\,Z_y-Y_z \\ \nabla^2 T+\kappa\Theta_{zx} &= -\,X_z-Z_x \\ \nabla^2 U+\kappa\Theta_{xy} &= -\,Y_x-X_y \end{aligned}\right\}, \tag{18}$$

We proceed to show that equations (16) and (18), together with
$$\left.\begin{aligned} P_x+U_y+T_z &= -\,X \\ U_x+Q_y+S_z &= -\,Y \\ T_x+S_y+R_z &= -\,Z \end{aligned}\right\}, \tag{19}$$

imply the existence of the identical strain-relations. Take first the type
$$e_{yy}+f_{xx} = c_{xy}. \tag{20}$$

From (19), we have
$$2U_{xy}+P_{xx}+Q_{yy}-R_{zz} = 2Z_z-\Lambda,$$

[redacted]

so that (20) becomes

$$2\mu e_{yy}+2\mu f_{xx}+P_{xx}+Q_{yy}-R_{zz}=2Z_z-\Lambda,$$

or $$P_{yy}+Q_{xx}-\lambda\,(\theta_{xx}+\theta_{yy})+P_{xx}+Q_{yy}-R_{zz}=2Z_z-\Lambda,$$

or $$\nabla^2\,(P+Q)-\Theta_{zz}-\lambda\,(\theta_{xx}+\theta_{yy})=2Z_z-\Lambda,$$

which, using (16), becomes

$$-\kappa\,(\Theta_{xx}+\Theta_{yy})-\Theta_{zz}-(1-\kappa)(\Theta_{xx}+\Theta_{yy})=2Z_z-\Lambda+2\,(X_x+Y_y)+2\nu\Lambda$$
$$=(2\nu+1)\,\Lambda,$$

so that (20) becomes $\nabla^2\Theta=-\,(2\nu+1)\,\Lambda,$

which is (17).

Take now the second type of strain-relation

$$2e_{yz}+a_{xx}=b_{xy}+c_{zx}, \tag{21}$$

or $$P_{yz}-\lambda\theta_{yz}+S_{xx}=T_{xy}+U_{zx}.$$

Now, from (19),

$$T_{xy}+U_{zx}+S_{yy}+S_{zz}+R_{yz}+Q_{yz}=-Z_y-Y_z,$$

so that (21) becomes

$$P_{yz}-(1-\kappa)\,\Theta_{yz}+\nabla^2S+Q_{yz}+R_{yz}=-Z_y-Y_z,$$

that is $$\nabla^2S+\kappa\Theta_{yz}=-Z_y-Y_z,$$

which is one of equations (18).

The strain-relations are therefore satisfied, and conversely they lead by a reversal of the above proofs to the stress-equations obtained. Hence the *complete* stress-equations are

$$\nabla^2P+\kappa\Theta_{xx}=-2X_x-\nu\Lambda,$$
$$\nabla^2Q+\kappa\Theta_{yy}=-2Y_y-\nu\Lambda,$$
$$\nabla^2R+\kappa\Theta_{zz}=-2Z_z-\nu\Lambda,$$
$$\nabla^2S+\kappa\Theta_{yz}=-\;Z_y-Y_z,$$
$$\nabla^2T+\kappa\Theta_{zx}=-\;X_z-Z_x,$$
$$\nabla^2U+\kappa\Theta_{xy}=-\;Y_x-X_y,$$

$$P_x+U_y+T_z=-X,$$
$$U_x+Q_y+S_z=-Y,$$
$$T_x+S_y+R_z=-Z,$$

where $$\Theta = P+Q+R.$$

Of course similar equations can be obtained for anisotropic bodies.

As in the case of plane-stress, the satisfaction of the strain-relations does not ensure that the displacements and their first derivatives are single-valued in the case of a multiply-connected solid. The additional conditions are obtained as in two dimensions. We have

$$du_y = e_y dx + (c_y - f_x)\, dy + \tfrac{1}{2}(c_z + b_y - a_x)\, dz.$$

Hence for each irreducible circuit

$$\int_0^0 \{e_y dx + (c_y - f_x)\, dy + \tfrac{1}{2}(c_z + b_y - a_x)\, dz\} = 0.$$

There are two similar conditions derived from a consideration of v_z and w_x.

Further $$du = u_x dx + u_y dy + u_z dz.$$

Hence $$\int_0^0 (u_x dx + u_y dy + u_z dz) = 0$$

for each irreducible circuit. This is equivalent to

$$\int_0^0 (x\, du_x + y\, du_y + z\, du_z) = 0,$$

if the conditions already obtained are satisfied. Hence

$$\int^0 \big[(xe_x + ye_y + ze_z)\, dx + \{xe_y + y(c_y - f_x) + \tfrac{1}{2}z(c_z + b_y - a_x)\}\, dy + \{xe_z + \tfrac{1}{2}y(c_z + b_y - a_x) + z(b_z - g_x)\}\, dz\big] = 0.$$

There are two similar equations derived from v and w.

We have therefore *six* conditions which can be at once expressed as stress-conditions by substitution for the strains.

Theory of Plates.

Let the faces of the plate be $z = \pm h$, and suppose at first that it extends to infinity in all directions. We may then find the stresses in the plate in the following manner. From the equations

$$\nabla^2 R + \kappa \Theta_{zz} = -2Z_z - \nu\Lambda,$$

$$\nabla^2 \Theta = -(2\nu + 1)\Lambda,$$

we obtain $$\nabla^4 R = -2\nabla^2 Z_z - \nu \nabla^2 \Lambda + (\nu + 1)\Lambda_{zz}, \qquad (22)$$

[redacted]

remembering that $$2\nu+1 = \frac{3\lambda+2\mu}{\lambda+2\mu},$$

so that $$(2\nu+1)\,\kappa = \frac{2(\lambda+\mu)}{\lambda+2\mu} = \nu+1.$$

Over the faces $z = \pm h$ we have R, S, T given; let them be H, F, G over $z = h$, and H', F', G' over $z = -h$. Since, throughout the plate,

$$T_x + S_y + R_z = -Z,$$

we have $$R_z = -Z - T_x - S_y,$$

also known over $$z = \pm h.$$

Now R is completely determined by (22) when R and dR/dz are given over the faces of the plate, and its value can be written down, *e.g.*, by means of Fourier-integrals. But, without entering into the different ways of obtaining the solution, we may now assume R known. This is the fundamental point of the present theory. It may be noted that, if there is no volume-force, or if, more generally,

$$-2\nabla^2 Z_z - \nu\nabla^2\Lambda + (\nu+1)\,\Lambda_{zz} = 0 \qquad (23)$$

throughout, the stress R is independent of the moduli of elasticity. If (23) holds and if R and $-Z-T_x-S_y$ vanish over the faces, then $R = 0$ throughout, and *these are the necessary and sufficient conditions that R may vanish throughout the plate*, a point about which there has been so much discussion.

Taking now $$\kappa\Theta_{zz} = -2Z_z - \nu\Lambda - \nabla^2 R,$$

we have $$\kappa\Theta = -\int_0^z\!\!\int_0^z (2Z_z + \nu\Lambda + \nabla^2 R)\,dz^2 + \kappa z\,{}_0\Theta_z + \kappa\Theta_0, \qquad (24)$$

where ${}_0\Theta_z$, Θ_0 are the values of Θ_z, Θ at $(x, y, 0)$. Also

$$\nabla^2\Theta = -(2\nu+1)\,\Lambda,$$

and therefore $$\nabla^2\Theta_z = -(2\nu+1)\,\Lambda_z,$$

so that
$$\begin{aligned}\nabla^2_{xy}\Theta_z &= -(2\nu+1)\,\Lambda_z - \Theta_{zzz}\\ &= -(2\nu+1)\Lambda_z + \frac{2}{\kappa}Z_{zz} + \frac{\nu}{\kappa}\Lambda_z + \frac{1}{\kappa}\nabla^2 R_z\\ &= -\frac{1}{\kappa}\Lambda_z + \frac{2}{\kappa}Z_{zz} + \frac{1}{\kappa}\nabla^2 R_z.\end{aligned}$$

Hence $$\kappa\nabla^2_{xy}\,{}_0\Theta_z = 2\,{}_0Z_{zz} - {}_0\Lambda_z + (\nabla^2 R_z)_0, \qquad (25)$$

12

and, similarly, $$\kappa\nabla^2_{xy}\Theta_0 = 2\,{}_0Z_z - \Lambda_0 + (\nabla^2 R)_0. \qquad (26)$$

Further $$\nabla^2 S + \kappa\Theta_{yz} = -Y_z - Z_y\,;$$

therefore $$\nabla^2(\nabla^2_{xy}S) + \kappa(\nabla^2_{xy}\Theta)_{yz} = -\nabla^2_{xy}(Y_z + Z_y)\,;$$

so that $$\nabla^2(\nabla^2_{xy}S) = \Lambda_{yz} - 2Z_{yzz} - \nabla^2 R_{yz} - \nabla^2_{xy}(Y_z + Z_y), \qquad (27)$$

and, since $\nabla^2_{xy}S$ is known over $z = \pm h$, being obtained by differentiation of the known values of S, equation (27) determines $\nabla^2_{xy}S$ throughout the plate. Also

$$\nabla^2 S_z + \kappa\Theta_{yzz} = -Y_{zz} - Z_{yz}$$

and $$\kappa\Theta_{yzz} = -2Z_{yz} - \nu\Lambda_y - \nabla^2 R_y,$$

so that $$S_{zzz} = -\nabla^2_{xy}S_z + \nu\Lambda_y + \nabla^2 R_y - Y_{zz} + Z_{yz}\,;$$

and therefore S_{zzz} is known throughout the plate. Hence

$$S = A + Bz + Cz^2 + \int_0^z\int_0^z\int_0^z S_{zzz}\,dz^3$$

$$= A + Bz + Cz^2 + S', \text{ say,}$$

where $$F = A + Bh + Ch^2 + S'(h),$$

$$F' = A - Bh + Ch^2 + S'(-h)$$

are known, giving

$$Bh = \tfrac{1}{2}\{F - F' - S'(h) + S'(-h)\}$$

and $$A + Ch^2 = \tfrac{1}{2}\{F + F' - S'(h) - S'(-h)\},$$

so that we may write

$$S = \sigma(z^2 - h^2) + S'',$$

where S'' is known, but σ is an unknown function of x, y.

Similarly, $$T = \tau(z^2 - h^2) + T'',$$

where T'' is known, but τ is an unknown function of x, y.

Substituting these values in

$$\nabla^2 S + \kappa\Theta_{yz} = -Y_z - Z_y,$$

$$\nabla^2 T + \kappa\Theta_{zx} = -Z_x - X_z,$$

we get $$2\sigma = -\kappa\Theta_{yz} - Y_z - Z_y - S''_{zz} - \nabla^2_{xy}S\,;$$

or, putting $z = 0$, $$\sigma = -\tfrac{1}{2}\kappa({}_0\Theta_z)_y + \sigma',$$

where σ' is a known function of x, y.

Similarly, $$\tau = -\tfrac{1}{2}\kappa\,({}_0\Theta_z)_x + \tau',$$
where τ' is known. Hence
$$\left.\begin{aligned} S &= -\tfrac{1}{2}\kappa\,({}_0\Theta_z)_y\,(z^2-h^2) + S''' \\ T &= -\tfrac{1}{2}\kappa\,({}_0\Theta_z)_x\,(z^2-h^2) + T''' \end{aligned}\right\}, \tag{28}$$
where
$$S''' = S'' + \sigma'\,(z^2-h^2),$$
$$T''' = T'' + \tau'\,(z^2-h^2),$$
and S''', T''' are known.

Now, take the equation
$$\{R-(1-\kappa)\,\Theta\}_{xx} + \{P-(1-\kappa)\,\Theta\}_{zz} = 2T_{zx},$$
which is the strain-relation
$$g_{xx} + e_{zz} = b_{zx},$$
expressed in terms of the stresses.

It gives, using equation (24),
$$P_{zz} = -2\kappa\,({}_0\Theta_z)_{xx}z + (1-\kappa)({}_0\Theta_z)_{xx}z + (1-\kappa)(\Theta_0)_{xx} + P'_{zz},$$
where P'_{zz} is known. Hence
$$P = (1-3\kappa)\,\frac{z^3}{6}\,({}_0\Theta_z)_{xx} + (1-\kappa)\,\frac{z^2}{2}\,(\Theta_0)_{xx} + z\,{}_0P_z + P_0 + P'', \tag{29}$$
where
$$P'' = \int_0^z\!\!\int_0^z P'_{zz}\,dz^2,$$
but ${}_0P_z$ and P_0 are undetermined functions of x, y.

Similarly,
$$\left.\begin{aligned} Q &= (1-3\kappa)\,\frac{z^3}{6}\,({}_0\Theta_z)_{yy} + (1-\kappa)\,\frac{z^2}{2}(\Theta_0)_{yy} + z\,{}_0Q_z + Q_0 + Q'' \\ U &= (1-3\kappa)\,\frac{z^3}{6}\,({}_0\Theta_z)_{xy} + (1-\kappa)\,\frac{z^2}{2}\,(\Theta_0)_{xy} + z\,{}_0U_z + U_0 + U'' \end{aligned}\right\}, \tag{29}$$
the last being derived from the equation
$$\{R-(1-\kappa)\,\Theta\}_{xy} + U_{zz} - S_{zx} - T_{zy} = 0,$$
which is the equivalent of the strain-relation
$$2g_{xy} + c_{zz} - a_{zx} - b_{zy} = 0.$$
It remains to connect and find differential equations for the unknowns ${}_0\Theta_z$, Θ_0, ${}_0P_z$, P_0, ${}_0Q_z$, Q_0, ${}_0U_z$, U_0.

Now, first, from the equations

$$P_x + U_y + T_z = -X,$$
$$U_x + Q_y + S_z = -Y,$$

we have
$$(P_0)_x + (U_0)_y = -X_0 - {}_0T_z''',$$
$$(U_0)_x + (Q_0)_y = -Y_0 - {}_0S_z''';$$

and hence
$$P_0 = \psi_{yy} + V,$$
$$Q_0 = \psi_{xx} + W,$$
$$U_0 = -\psi_{xy};$$

and therefore
$$\Theta_0 = \nabla_{xy}^2 \psi + V + W + R_0,$$

where V, W are known.

Substituting in $\kappa \nabla_{xy}^2 \Theta_0 = 2\,{}_0Z_z - \Lambda_0 + (\nabla^2 R)_0,$

we have
$$\kappa \nabla_{xy}^4 \psi = 2\,{}_0Z_z - \Lambda_0 + (\nabla^2 R)_0 - \kappa \nabla_{xy}^2 (V + W + R_0), \tag{30}$$

which is the differential equation for ψ.

Similarly,
$$P_x + U_y + T_z = -X,$$
$$U_x + Q_y + S_z = -Y$$

give
$$({}_0P_z)_x + ({}_0U_z)_y = -{}_0X_z - {}_0T_{zz}''' + \kappa\,({}_0\Theta_z)_x,$$
$$({}_0U_z)_x + ({}_0Q_z)_y = -{}_0Y_z - {}_0S_{zz}''' + \kappa\,({}_0\Theta_z)_y.$$

Hence
$${}_0P_z - \kappa\,{}_0\Theta_z = \phi_{yy} + V',$$
$${}_0Q_z - \kappa\,{}_0\Theta_z = \phi_{xx} + W',$$
$${}_0U_z = -\phi_{xy};$$

and therefore
$$(1-2\kappa)\,{}_0\Theta_z = \nabla_{xy}^2 \phi + V' + W' + {}_0R_z,$$

where V', W' are known.

Substituting in the equation

$$\kappa \nabla_{xy}^2\,{}_0\Theta_z = 2\,{}_0Z_{zz} - {}_0\Lambda_z + (\nabla^2 R_z)_0,$$

we obtain

$$\frac{\kappa}{1-2\kappa} \nabla_{xy}^4 \phi = 2\,{}_0Z_{zz} - {}_0\Lambda_z + (\nabla^2 R_z)_0 - \frac{\kappa}{1-2\kappa} \nabla_{xy}^2 (V' + W' + {}_0R_z) \tag{31}$$

or $(\nu+1)\,\nabla_{xy}^4 \phi = -2\,{}_0Z_{zz} + {}_0\Lambda_z - (\nabla^2 R_z)_0 - (\nu+1)\,\nabla_{xy}^2 (V' + W' + {}_0R_z).$

The unknown functions are now reduced to two functions ϕ, ψ of x, y for which the differential equations (30), (31) hold. The

differential equation for the normal displacement of the median plane is derived at once.

We have $$w_{zx} = (u_z + w_x)_x - e_x.$$

Hence $$2\mu w_{zx} = 2T_z - \{P - (1-\kappa)\Theta\}_z$$

and $$2\mu\, {}_0w_{xx} = +\kappa h^2 ({}_0\Theta_z)_{xx} + 2\, {}_0T_z''' + (1-2\kappa)\, {}_0\Theta_z - \phi_{yy} - V'$$

$$= \kappa h^2 ({}_0\Theta_z)_{xx} + 2\, {}_0T_x''' + \phi_{xx} + W' + {}_0R_z.$$

Similarly,

$$2\mu\, {}_0w_{yy} = \kappa h^2 ({}_0\Theta_z)_{yy} + 2\, {}_0S_y''' + \phi_{yy} + V' + {}_0R_z,$$

$$2\mu\, {}_0w_{xy} = \kappa h^2 ({}_0\Theta_z)_{xy} + {}_0S_x''' + {}_0T_y''' + \phi_{xy}.$$

Therefore $$2\mu\, w_0 = \kappa h^2\, {}_0\Theta_z + \phi + \Omega$$

$$= \frac{\kappa h^2}{1-2\kappa} (\nabla_{xy}^2 \phi + V' + W' + {}_0R_z) + \phi + \Omega,$$

where Ω is a known function of x, y; a linear function of x, y denoting a rigid body displacement of the median plane being neglected.

Also

$$2\mu \nabla_{xy}^2 w_0 = \kappa h^2 \nabla_{xy}^2 ({}_0\Theta_z) + 2\, ({}_0T_x''' + {}_0S_y''') + \nabla_{xy}^2 \phi + V' + W' + 2\, {}_0R_z.$$

This simplifies very much by use of the equation

$$T_x + S_y + R_z = -Z,$$

which gives $${}_0T_x''' + {}_0S_y''' + {}_0R_z + \tfrac{1}{2}\kappa h^2 \nabla_{xy}^2 ({}_0\Theta_z) = -Z_0.$$

Hence $$2\mu\, \nabla_{xy}^2 w_0 = \nabla_{xy}^2 \phi + V' + W' - 2Z_0$$

$$= (1-2\kappa)\, {}_0\Theta_z - {}_0R_z - 2Z_0,$$

so that $\phi = 2\mu w_0 + 2\mu\, (\nu+1)\, h^2 \nabla_{xy}^2 w_0 + (\nu+1)\, h^2\, ({}_0R_z + 2Z_0) - \Omega,$

and w_0 is known in terms of ϕ and conversely. The differential equation for w_0 is therefore

$$2\mu\kappa \nabla_{xy}^4 w_0 = (1-2\kappa)\, \{2\, {}_0Z_{zz} - {}_0\Lambda_z + (\nabla^2 R_z)_0\} - \kappa \nabla_{xy}^2 ({}_0R_z + 2Z_0).$$

This result is more easily obtained from the equation

$$(\lambda + \mu)\, \theta_z + \mu \nabla^2 w = -Z$$

or $$\kappa\Theta_z + 2\mu\, \nabla_{xy}^2 w + R_z - (1-\kappa)\, \Theta_z = -Z.$$

Putting $z = 0$, we have

$$2\mu\, \nabla_{xy}^2 w_0 = (1-2\kappa)\, {}_0\Theta_z - {}_0R_z - 2Z_0,$$

the same equation as before.

Application to Finite Plates.

If the plate is finite, the equation

$$\nabla^4 R = -2\nabla^2 Z_z - \nu\nabla^2\Lambda + (2\nu+1)\,\kappa\Lambda_{zz},$$

with R and R_z given over $z = \pm h$, does not theoretically completely determine R. The problem is of exactly the same nature as that of the ordinary plate condenser. The potential between the plates is not completely determined by $\nabla^2 V = 0$, with V given over $z = \pm h$. In each case R or V is practically determined with great accuracy by the conditions, except for points at a distance less than a small multiple of the thickness $2h$ from the edge. The same remark applies to the determination of $\nabla^2_{xy} S$ and $\nabla^2_{xy} T$. The above solution then applies without modification to the finite plate. The edge conditions for ϕ, ψ are written down as in the Thomson-Boussinesq theory, using the principle of equipollent loads, and there is no occasion to enter on their discussion here.

The nature of the transmission of stress in the plate will perhaps appear more plainly if we suppose that there is no applied force on part of it. The solutions for R, $\nabla^2_{xy} S$, $\nabla^2_{xy} T$ in that part are then simply

$$R = \nabla^2_{xy} S = \nabla^2_{xy} T = 0$$

throughout. Hence

$$S = -\tfrac{1}{2}\kappa\,({}_0\Theta_z)_y\,(z^2-h^2),$$

$$T = -\tfrac{1}{2}\kappa\,({}_0\Theta_z)_x\,(z^2-h^2),$$

$$P = (1-3\kappa)\,\frac{z^3}{6}\,({}_0\Theta_z)_{xx} + (1-\kappa)\,\frac{z^2}{2}\,(\Theta_0)_{xx} + z\,(\kappa\,{}_0\Theta_z + \phi_{yy}) + \psi_{yy},$$

$$Q = (1-3\kappa)\,\frac{z^3}{6}\,({}_0\Theta_z)_{yy} + (1-\kappa)\,\frac{z^2}{2}\,(\Theta_0)_{yy} + z\,(\kappa\,{}_0\Theta_z + \phi_{xx}) + \psi_{xx},$$

$$U = (1-3\kappa)\,\frac{z^3}{6}\,({}_0\Theta_z)_{xy} + (1-\kappa)\,\frac{z^2}{2}\,(\Theta_0)_{xy} - z\phi_{xy} - \psi_{xy},$$

where

$$(1-2\kappa)\,{}_0\Theta_z = \nabla^2_{xy}\phi,$$

$$\Theta_0 = \nabla^2_{xy}\psi,$$

$$\nabla^2_{xy}\,{}_0\Theta_z = \nabla^2_{xy}\Theta_0 = 0,$$

$$\phi = 2\mu w_0 + 2\mu\,(\nu+1)\,h^2\nabla^2_{xy} w_0,$$

$$\nabla^4_{xy} w_0 = 0,$$

and hence $$\nabla^2_{xy}\phi = 2\mu\nabla^2_{xy} w_0.$$

Introducing the curvatures ϖ_1, ϖ_2 and the twist τ, we have

$$\varpi_1 = {}_0w_{xx},$$
$$\varpi_2 = {}_0w_{yy},$$
$$\tau = {}_0w_{xy},$$
$$\nabla^2_{xy}(\varpi_1+\varpi_2) = 0,$$

and hence $$S = \mu(\nu+1)(\varpi_1+\varpi_2)_y (z^2-h^2),$$
$$T = \mu(\nu+1)(\varpi_1+\varpi_2)_x (z^2-h^2),$$
$$\begin{aligned} P = 2\mu\frac{(1-3\kappa)}{1-2\kappa}\frac{z^3}{6}(\varpi_1+\varpi_2)_{xx} &- 2\mu z\{(\nu+1)\varpi_1+\nu\varpi_2\} \\ &+ 2\mu(\nu+1)zh^2(\varpi_1+\varpi_2)_{yy} \\ &+ (1-\kappa)\frac{z^2}{2}(P_0+Q_0)_{xx}+P \end{aligned}$$
$$\begin{aligned} = &-2\mu z\{(\nu+1)\varpi_1+\nu\varpi_2\} \\ &+2\mu\left\{(\nu+2)\frac{z^3}{6}-(\nu+1)zh^2\right\}(\varpi_1+\varpi_2)_{xx} \\ &+P_0+(1-\kappa)\frac{z^2}{2}(P_0+Q_0)_{xx}. \end{aligned}$$

Similarly for Q, and

$$\begin{aligned} U = -2\mu z\tau + 2\mu(\nu+2)\frac{z^3}{6}(\varpi_1+\varpi_2)_{xy} &- 2\mu(\nu+1)zh^2\nabla^2_{xy}\tau \\ &+U_0+(1-\kappa)\frac{z^2}{2}(P_0+Q_0)_{xy}. \end{aligned}$$

Approximate Solution.

Taking the unit of length as of the same order as the thickness of the plate, if we assume that the rates of variation of the functions of the forces in the equations and conditions for R, $\nabla^2_{xy}S$, $\nabla^2_{xy}T$, parallel to the plate, are small, we may apply the method of successive approximations to find those quantities to any order of accuracy. To keep the algebra within bounds, let us consider the particular case in which

$$X = Y = 0, \quad Z_z = 0,$$
$$F = F' = G = G' = 0,$$

so that the plate is under normal pressures on its faces, and normal volume force. For a first approximation, we write

$$\nabla^4 R = \frac{d^4 R}{dz^4},$$

and hence in this case $\qquad \dfrac{d^4 R}{dz^4} = 0,$

$$R = A + Bz + Cz^2 + Dz^3,$$

where

$$H = A + Bh + Ch^2 + Dh^3,$$

$$H' = A - Bh + Ch^2 - Dh^3.$$

$$R_z(h) = -Z = B + 2Ch + 3Dh^2,$$

$$R_z(-h) = -Z = B - 2Ch + 3Dh^2.$$

Hence

$$C = 0,$$

$$B + 3Dh^2 = -Z,$$

$$2A = H + H',$$

$$2Bh + 2Dh^3 = H - H',$$

$$4Bh = 3(H - H') + 2hZ,$$

$$4Dh^3 = -(H - H') - 2hZ,$$

and $\quad R = \tfrac{1}{2}(H + H') + \dfrac{1}{4h^3}(H - H')(3zh^2 - z^3) + \dfrac{1}{2h^3} Z(zh^2 - z^3).$

This is the first approximation for R; for a further approximation we must substitute this value in the terms of $\nabla^4 R$ previously neglected, and repeat the above process. Contenting ourselves with the above value of R, we have

$$\nabla^2(\nabla^2_{xy} S) = -\nabla^2 Z_y - \nabla^2 R_{yz}$$

or

$$\nabla^2(\nabla^2_{xy} S + Z_y + R_{yz}) = 0.$$

Now, $\qquad S = 0, \quad R_z + Z = 0,$

when

$$z = \pm h,$$

so that

$$\nabla^2_{xy} S + Z_y + R_{yz} = 0,$$

when

$$z = \pm h.$$

Hence the equation gives

$$\nabla^2_{xy} S + Z_y + R_{yz} = 0$$

throughout, and this is true whenever

$$X = Y = F = F' = G = G' = 0,$$

so that in this case the trouble of successive approximations is connected with the determination of R alone.

Since
$$\nabla^2_{xy} S_z + R_{yzz} = 0,$$
we have
$$\frac{d^3 S}{dz^3} = +\,2R_{yzz} = -\frac{3}{h^3}(H_y - H'_y)\,z - \frac{6}{h^3}\,Z_y z,$$
neglecting
$$\nabla^2_{xy} R_y.$$
Hence
$$S = A + Bz + Cz^2 - \frac{1}{8h^3}(H_y - H'_y)\,z^4 - \frac{1}{4h^3}Z_y z^4,$$
so that
$$S = \sigma(z^2 - h^2) - \frac{1}{8h^3}(H_y - H'_y)(z^4 - h^4) - \frac{1}{4h^3}Z_y\,(z^4 - h^4),$$
and, similarly,
$$T = \tau\,(z^2 - h^2) - \frac{1}{8h^3}(H_x - H'_x)(z^4 - h^4) - \frac{1}{4h^3}Z_x\,(z^4 - h^4),$$
where
$$\begin{aligned} 2\sigma &= -\kappa\,({}_0\Theta_z)_y - Z_y - \nabla^2_{xy} S_0 \\ &= -\kappa\,({}_0\Theta_z)_y + ({}_0R_z)_y \\ &= -\kappa\,({}_0\Theta_z)_y + \frac{3}{4h}(H_y - H'_y) + \tfrac{1}{2}Z_y, \end{aligned}$$
and, similarly,
$$2\tau = -\kappa\,({}_0\Theta_z)_x + \frac{3}{4h}(H_x - H'_x) + \tfrac{1}{2}Z_x,$$
so that
$$S = -\tfrac{1}{2}\kappa\,({}_0\Theta_z)_y\,(z^2 - h^2) + \frac{1}{8h^3}(H_y - H'_y)(z^2 - h^2)(2h^2 - z^2) - \frac{1}{4h^3}Z_y\,(z^2 - h^2)\,z^2,$$
and
$$T = -\tfrac{1}{2}\kappa\,({}_0\Theta_z)_x\,(z^2 - h^2) + \frac{1}{8h^3}(H_x - H'_x)(z^2 - h^2)(2h^2 - z^2) - \frac{1}{4h^3}Z_x\,(z^2 - h^2)\,z^2.$$

The differential equation for w_0 is, therefore, neglecting the second and higher differential coefficients of the applied forces with respect to x, y,
$$2\mu\kappa\nabla^4_{xy} w_0 = (1 - 2\kappa)\frac{d^4 R}{dz^4} = -(1 - 2\kappa)\left\{\frac{3}{h^2}Z + \frac{3}{2h^3}(H - H')\right\},$$
or
$$4\mu\,(\nu + 1)\,h^3\nabla^4_{xy} w_0 = 3\left\{2hZ + H - H'\right\}.$$

Since
$$\nu+1=\frac{2(\lambda+\mu)}{\lambda+2\mu},$$
this is the equation which would be obtained according to the ordinary approximation, $2hZ+H-H'$ being the normal force per unit area of plate.

The processes here developed are obviously well suited to the treatment of the problem of Cerutti and Boussinesq, viz., to determine the stress through an infinite solid, bounded by an infinite plane, on which the forces are given. As a matter of fact, the processes lead directly to a potential-solution of the form given by the authors named, but the present is hardly a fit occasion to give a mere revision of this famous problem.

The Stress in a Rotating Lamina. *By* J. H. Michell, M.A.

Read April 13th, 1899. Received, in revised form, September 4th, 1899.

In a paper* recently presented to the Society I have given a general theory of plates under any forces. I propose, in the present note, to apply the theory to the case of a lamina rotating about a fixed axis perpendicular to its plane. The notation is that of the paper referred to.

Taking the axis of z normal to the lamina, let ϖ be the angular velocity of the lamina, ρ its density, so that the component forces are
$$X=\rho\varpi^2x,$$
$$Y=\rho\varpi^2y,$$
and hence
$$X_x+Y_y=\Lambda=2\rho\varpi^2,$$
so that
$$\nabla^4R=0.$$

Since R and dR/dz vanish on the faces $z=\pm h$, supposed free from stress, it follows that $R=0$ throughout the plate (except, of course, at points close to the edges, as explained in the former paper).

---

* "On the Direct Determination of Stress in an Elastic Solid, &c.," pp. 100–124.

Further, $$\kappa\Theta_{zz} = -\nu\Lambda - \nabla^2 R$$
$$= -2\nu\rho\varpi^2,$$

so that $$\kappa\Theta = -\nu\rho\varpi^2 z^2 + \kappa\Theta_0, \tag{1}$$

since, by symmetry, $${}_0\Theta_z = 0.$$

Substituting in the equation
$$\nabla^2\Theta = -(2\nu+1)\Lambda$$
$$= -2(2\nu+1)\rho\varpi^2, \tag{2}$$

we find $$\kappa\nabla^2_{xy}\Theta_0 = -2\rho\varpi^2, \tag{3}$$

remembering that $\kappa(2\nu+1) = \dfrac{2(\lambda+\mu)}{\lambda+2\mu} = \nu+1.$

Also, since in the equations
$$\nabla^2 S + \kappa\Theta_{yz} = 0,$$
$$\nabla^2 T + \kappa\Theta_{zx} = 0,$$
$$\Theta_{yz} = \Theta_{zx} = 0,$$

we have $$\nabla^2 S = \nabla^2 T = 0;$$

and therefore, since $$S = T = 0$$

over the faces, $$S = T = 0$$

throughout the plate. We see, therefore, that, in this case, the famous equations
$$R = S = T = 0$$

are satisfied, except, of course, close to the edges.

The equation
$$\{R-(1-\kappa)\Theta\}_{xx} + \{P-(1-\kappa)\Theta\}_{zz} - 2T_{zx} = 0$$

now gives $$P_{zz} = (1-\kappa)\Theta_{zz} + (1-\kappa)\,{}_0\Theta_{xx}$$
$$= -\frac{2\nu(1-\kappa)}{\kappa}\rho\varpi^2 + (1-\kappa)\,{}_0\Theta_{xx}$$

or $$P = P_0 - \frac{\nu(1-\kappa)}{\kappa}\rho\varpi^2 z^2 + \tfrac{1}{2}(1-\kappa)\,{}_0\Theta_{xx} z^2,$$

since ${}_0P_z = 0$, by symmetry.

Similarly, $Q = Q_0 - \dfrac{\nu(1-\kappa)}{\kappa}\rho\varpi^2 z^2 + \tfrac{1}{2}(1-\kappa)\,{}_0\Theta_{yy}\,z^2,$

$$U = U_0 + \tfrac{1}{2}(1-\kappa)\,{}_0\Theta_{xy} z^2,$$

the last being derived from the equation

$$U_{zz} + \{R - (1-\kappa)\,\Theta\}_{xy} = S_{zx} + T_{yz}.$$

Further, from
$$P_x + U_y + T_z = -\rho\varpi^2 x,$$
$$U_x + Q_y + S_z = -\rho\varpi^2 y,$$

we derive
$${}_0P_x + {}_0U_y = -\rho\varpi^2 x,$$
$${}_0U_x + {}_0Q_y = -\rho\varpi^2 y,$$

by putting
$$z = 0,$$

and hence
$$P_0 = \psi_{yy} - \tfrac{1}{2}\rho\varpi^2(x^2+y^2),$$
$$Q_0 = \psi_{xx} - \tfrac{1}{2}\rho\varpi^2(x^2+y^2),$$
$$U_0 = -\psi_{xy},$$

which, on substitution in

$$\kappa\nabla^2_{xy}\Theta_0 = -2\rho\varpi^2,$$

gives
$$\kappa\nabla^4_{xy}\psi = 2(2\kappa-1)\rho\varpi^2,$$

while
$$\Theta_0 = \nabla^2_{xy}\psi - \rho\varpi^2(x^2+y^2).$$

It is necessary to introduce curvilinear coordinates in applications of these formulæ. Let the curvilinear cylindrical coordinates be ξ, η, z, and write

$$ds = h_1\,d\xi, \quad d\sigma = h_2\,d\eta$$

for the elements of arc along the two curves η, z; ξ, z. The x, y components of stress per unit area across the element $d\eta\,dz$ are

$$\left.\begin{aligned} F &= lP + mU \\ &= \frac{d}{d\sigma}(\psi_y) + \tfrac{1}{2}(1-\kappa)z^2\frac{d}{ds}({}_0\Theta_x) - \tfrac{1}{2}l\rho\varpi^2 r^2 - \frac{\nu(1-\kappa)}{\kappa}l\rho\varpi^2 z^2 \\ G &= -\frac{d}{d\sigma}(\psi_x) + \tfrac{1}{2}(1-\kappa)z^2\frac{d}{ds}({}_0\Theta_y) - \tfrac{1}{2}m\rho\varpi^2 r^2 - \frac{\nu(1-\kappa)}{\kappa}m\rho\varpi^2 z^2 \end{aligned}\right\}, \tag{4}$$

where
$$r^2 = x^2 + y^2.$$

Hence the normal and tangential stresses across the same element are

$$P' = \frac{dy}{d\sigma}\frac{d}{d\sigma}(\psi_y) + \frac{dx}{d\sigma}\frac{d}{d\sigma}(\psi_x)$$
$$+\tfrac{1}{2}(1-\kappa)z^2\left\{\frac{dx}{ds}\frac{d}{ds}({}_0\Theta_x) + \frac{dy}{ds}\frac{d}{ds}({}_0\Theta_y)\right\}$$
$$-\tfrac{1}{2}\rho\varpi^2r^2 - \frac{\nu(1-\kappa)}{\kappa}\rho\varpi^2z^2$$
$$= \frac{d^2\psi}{d\sigma^2} + \kappa_2\frac{d\psi}{ds}$$
$$+\tfrac{1}{2}(1-\kappa)z^2\left\{\frac{d^2\Theta_0}{ds^2} - \kappa_1\frac{d\Theta_0}{d\sigma}\right\}$$
$$+\tfrac{1}{2}\rho\varpi^2r^2 - \frac{\nu(1-\kappa)}{\kappa}\rho\varpi^2z^2.$$

$$U' = -\frac{dy}{ds}\frac{d}{d\sigma}(\psi_y) - \frac{dx}{ds}\frac{d}{d\sigma}(\psi_x)$$
$$+\tfrac{1}{2}(1-\kappa)z^2\left\{\frac{dx}{d\sigma}\frac{d}{ds}({}_0\Theta_x) + \frac{dy}{d\sigma}\frac{d}{ds}({}_0\Theta_y)\right\}$$
$$= -\frac{d}{d\sigma}\left(\frac{d\psi}{ds}\right) + \kappa_2\frac{d\psi}{d\sigma}$$
$$+\tfrac{1}{2}(1-\kappa)z^2\left\{\frac{d^2\Theta_0}{ds\,d\sigma} - \kappa_1\frac{d\Theta_0}{ds}\right\},$$

where κ_1, κ_2 are the curvatures of the arcs ds, $d\sigma$ reckoned positively from x to y.

Hence, substituting the curvilinear coordinates, the elements of stress are

$$P' = \frac{d}{h_2\,d\eta}\left(\frac{d\psi}{h_2\,d\eta}\right) + \frac{1}{h_1^2h_2}\frac{dh_2}{d\xi}\frac{d\psi}{d\xi}$$
$$+\tfrac{1}{2}(1-\kappa)z^2\left\{\frac{d}{h_1\,d\xi}\left(\frac{d\Theta_0}{h_1\,d\xi}\right) + \frac{1}{h_1h_2^2}\frac{dh_1}{d\eta}\frac{d\Theta_0}{d\eta}\right\}$$
$$-\tfrac{1}{2}\rho\varpi^2r^2 - \frac{\nu(1-\kappa)}{\kappa}\rho\varpi^2z^2,$$

$$Q' = \frac{d}{h_1\,d\xi}\left(\frac{d\psi}{h_1\,d\xi}\right) + \frac{1}{h_1h_2^2}\frac{dh_1}{d\eta}\frac{d\psi}{d\eta}$$
$$+\tfrac{1}{2}(1-\kappa)z^2\left\{\frac{d}{h_2\,d\eta}\left(\frac{d\Theta_0}{h_2\,d\eta}\right) + \frac{1}{h_2h_1^2}\frac{dh_2}{d\xi}\frac{d\Theta_0}{d\xi}\right\}$$
$$-\tfrac{1}{2}\rho\varpi^2r^2 - \frac{\nu(1-\kappa)}{\kappa}\rho\varpi^2z^2,$$

$$U' = -\frac{1}{h_2 d\eta}\left(\frac{d\psi}{h_1 d\xi}\right) + \frac{1}{h_1 h_2^2}\frac{dh_2}{d\xi}\frac{d\psi}{d\eta}$$
$$+\tfrac{1}{2}(1-\kappa)z^2\left\{\frac{d}{h_1 d\xi}\left(\frac{d\Theta_0}{h_2 d\eta}\right) - \frac{1}{h_1^2 h_2}\frac{dh_1}{d\eta}\frac{d\Theta_0}{d\xi}\right\}.$$

Thus, *e.g.*, in plane polars,

$$P' = \frac{1}{r^2}\frac{d^2\psi}{d\theta^2} + \frac{1}{r}\frac{d\psi}{dr} + \tfrac{1}{2}(1-\kappa)z^2\frac{d^2\Theta_0}{dr^2} - \tfrac{1}{2}\rho\varpi^2 r^2 - \frac{\nu(1-\kappa)}{\kappa}\rho\varpi^2 z^2,$$

$$Q' = \frac{d^2\psi}{dr^2} + \tfrac{1}{2}(1-\kappa)z^2\left(\frac{d^2\Theta_0}{r^2 d\theta^2} + \frac{1}{r}\frac{d\Theta_0}{dr}\right)$$
$$-\tfrac{1}{2}\rho\varpi^2 r^2 - \frac{\nu(1-\kappa)}{\kappa}\rho\varpi^2 z^2,$$

$$U' = -\frac{d}{dr}\left(\frac{1}{r}\frac{d\psi}{d\theta}\right) + \tfrac{1}{2}(1-\kappa)z^2\frac{d}{dr}\left(\frac{1}{r}\frac{d\Theta_0}{d\theta}\right).$$

The boundary conditions can now be written down. That the resultant force per unit length across the edge η may vanish, we must have

$$\frac{1}{h_1 d\xi}\left(\frac{d\psi}{h_1 d\xi}\right) + \frac{1}{h_1 h_2^2}\frac{dh_1}{d\eta}\frac{d\psi}{d\eta} + \tfrac{1}{6}(1-\kappa)h^2\left\{\frac{d}{h_2 d\eta}\left(\frac{d\Theta_0}{h_2 d\eta}\right) + \frac{1}{h_1^2 h_2}\frac{dh_2}{d\xi}\frac{d\Theta_0}{d\xi}\right\}$$
$$-\tfrac{1}{2}\rho\varpi^2 r^2 - \tfrac{1}{3}\frac{\nu(1-\kappa)}{\kappa}\rho\varpi^2 h^2 = 0,$$

$$-\frac{1}{h_2 d\eta}\left(\frac{d\psi}{h_1 d\xi}\right) + \frac{1}{h_1 h_2^2}\frac{dh_2}{d\xi}\frac{d\psi}{d\eta}$$
$$+\tfrac{1}{6}(1-\kappa)h^2\left\{\frac{d}{h_1 d\xi}\left(\frac{d\Theta_0}{h_2 d\eta}\right) - \frac{1}{h_1^2 h_2}\frac{dh_1}{d\eta}\frac{d\Theta_0}{d\xi}\right\} = 0.$$

If the lamina is not singly-connected, we must add the conditions which ensure that the displacements are single-valued. We may suppose the irreducible circuits taken in the plane $x = 0$. The question is then reduced to that treated in the previous paper, and the conditions are, for each boundary,

$$\left.\begin{aligned}(\nu+1)\int_0^0 \frac{d}{dn}(\nabla^2\psi)\,ds + \int_0^0 \frac{dV}{dn}\,ds &= 0\\ (2\nu+1)\,[\psi_x]_0^0 + (\nu+1)\int_0^0\left(x\frac{d}{ds}\nabla^2\psi - y\frac{d}{dn}\nabla^2\psi\right)ds + \int_0^0\left(x\frac{dV}{ds} - y\frac{dV}{dn}\right)ds &= 0\\ (2\nu+1)\,[\psi_y]_0^0 + (\nu+1)\int_0^0\left(y\frac{d}{ds}\nabla^2\psi + x\frac{d}{dn}\nabla^2\psi\right)ds + \int_0^0\left(y\frac{dV}{ds} + x\frac{dV}{dn}\right)ds &= 0\end{aligned}\right\},\tag{5}$$

where $V = -\frac{1}{2}\rho\varpi^2 r^2$,

$$[\psi_x]_0^0 = -\int_0^0 G\,ds + \tfrac{1}{2}(1-\kappa)\,z^2\int_0^0 \frac{d}{dn}({}_0\Theta^y)\,ds$$
$$-\tfrac{1}{2}\rho\varpi^2\int_0^0 mr^2 ds - \frac{\nu(1-\kappa)}{\kappa}\rho\varpi^2 z^2\int m\,ds,$$

taking for the moment the circuit s in the plane z; and this, on integrating with respect to z, gives

$$[\psi_x]_0^{0'} = \tfrac{1}{6}(1-\kappa)\,h^2\int_0^0 \frac{d}{dn}({}_0\Theta_y)\,ds - \tfrac{1}{2}\rho\varpi^2\int_0^0 mr^2 ds,$$

the edge being supposed free from stress. Similarly,

$$[\psi_y]_0^0 = -\tfrac{1}{6}(1-\kappa)\,h^2\int_0^0 \frac{d}{dn}({}_0\Theta_x)\,ds + \tfrac{1}{2}\rho\varpi^2\int_0^0 lr^2 ds.$$

Consider, in particular, the case of a circular disc of internal and external radii a, b respectively. Here Θ_0 will be a function of r only, so that

$$\int_0^0 \frac{d}{dn}({}_0\Theta_x)\,ds \quad\text{and}\quad \int_0^0 \frac{d}{dn}({}_0\Theta_y)\,ds$$

will vanish, and the conditions (5) just investigated reduce to the single one

$$(\nu+1)\frac{d}{dr}(\nabla^2\psi) = \rho\varpi^2 r. \tag{6}$$

The equation $\kappa\nabla^4\psi = 2(2\kappa-1)\rho\varpi^2$

is $$\kappa\frac{1}{r}\frac{d}{dr}r\frac{d\nabla^2\psi}{dr} = 2(2\kappa-1)\rho\varpi^2,$$

giving $$\kappa\nabla^2\psi = \tfrac{1}{2}(2\kappa-1)\rho\varpi^2 r^2 + A\log r + B.$$

The condition (6) gives

$$(2\kappa-1)(\nu+1)\rho\varpi^2 r + (\nu+1)A/r = \kappa\rho\varpi^2 r,$$

and, since $$2\kappa-1 = \frac{\lambda+2\mu}{3\lambda+2\mu} = \kappa\frac{\lambda+2\mu}{2(\lambda+\mu)},$$

so that $$(2\kappa-1)(\nu+1) = \kappa,$$

this gives $$A = 0.$$

Thus $$\kappa\left(\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} + \frac{1}{r^2}\frac{d^2\psi}{d\theta^2}\right) = \tfrac{1}{2}(2\kappa-1)\rho\varpi^2 r^2 + B,$$

which gives $\kappa\psi = \frac{1}{32}(2\kappa-1)\rho\varpi^2 r^4 + \frac{1}{4}Br^2 + C\log r + D,$

remembering that P', Q' must be single-valued, and that from symmetry $U' = 0$.

The stress-conditions at the boundary give

$$\tfrac{1}{8}(2\kappa-1)\rho\varpi^2a^2 - \tfrac{1}{2}\kappa\rho\varpi^2a^2 + \tfrac{1}{2}B + C/a^2 - \tfrac{1}{6}(1-\kappa)h^2\rho\varpi^2 - \tfrac{1}{3}\nu(1-\kappa)h^2\rho\varpi^2 = 0$$

or

$$\tfrac{1}{2}B + C/a^2 = \tfrac{1}{8}(2\kappa+1)\rho\varpi^2a^2 + \tfrac{1}{6}\nu h^2\rho\varpi^2,$$

$$\tfrac{1}{2}B + C/b^2 = \tfrac{1}{8}(2\kappa+1)\rho\varpi^2b^2 + \tfrac{1}{6}\nu h^2\rho\varpi^2.$$

Hence

$$C = -\tfrac{1}{8}(2\kappa+1)\rho\varpi^2a^2b^2,$$

$$B = \tfrac{1}{4}(2\kappa+1)\rho\varpi^2(a^2+b^2) + \tfrac{1}{3}\nu h^2\rho\varpi^2;$$

so that

$$\kappa\psi = \tfrac{1}{32}(2\kappa-1)\rho\varpi^2r^4 + \tfrac{1}{16}(2\kappa+1)\rho\varpi^2(a^2+b^2)r^2 - \tfrac{1}{8}(2\kappa+1)\rho\varpi^2a^2b^2\log r + \tfrac{1}{12}\nu\rho\varpi^2h^2r^2 + D$$

and

$$\kappa\Theta_0 = -\tfrac{1}{2}\rho\varpi^2r^2 + \tfrac{1}{4}(2\kappa+1)\rho\varpi^2(a^2+b^2) + \tfrac{1}{3}\nu h^2\rho\varpi^2.$$

The elements of stress are therefore given by

$$\kappa P' = \tfrac{1}{8}(2\kappa+1)\rho\varpi^2(r^2-a^2)(b^2-r^2)/r^2 + \tfrac{1}{6}\nu\rho\varpi^2(h^2-3z^2),$$

$$\kappa Q' = \tfrac{1}{8}\rho\varpi^2\left\{(2\kappa-3)r^2 + (2\kappa+1)a^2b^2/r^2 + (2\kappa+1)(a^2+b^2)\right\} + \tfrac{1}{6}\nu\rho\varpi^2(h^2-3z^2),$$

$$U' = 0.$$

These expressions for the stresses agree with those given by Chree in his paper "On the Theory of Rotating Isotropic Disks," *Proc. Camb. Phil. Soc.*, 1891.

The Uniform Torsion and Flexure of Incomplete Tores, with application to Helical Springs. By J. H. MICHELL, M.A. Read April 13th, 1899. Received, in revised form, September 4th, 1899.

The object of the present paper is to give a theory of the uniform torsion and flexure of incomplete tores, corresponding to that which St. Venant has long since given for cylinders. The theory can be applied to stout helical springs of small pitch, and serves to de-

termine the degree of approximation arrived at in the wire-theory of springs. I have considered in detail a particular family of tores of approximately circular section in order to make some numerical comparisons. The general result is that the *stiffness* of the spring of circular section under axial force is given in all ordinary cases, with sufficient accuracy, by the wire-theory, but that the magnitude of the stresses in very stout springs may be considerably in excess of those given by that theory. The tore of rectangular section is also considered, but not numerically. The fundamental equations for the torsion of straight bars of varying circular section are deduced from those for a tore, but I have not entered on a detailed treatment of the problem in the present paper.

I. *Torsion.*

The uniform torsion* of an incomplete tore, isotropic or of cylindrical æolotropy of a certain degree of generality, is of the following nature. Each meridian section shifts, without deformation in its own plane, a distance along the axis of the tore proportional to the angular distance of the section from a fixed meridian. At the same time each meridian section receives the same distortion at right angles to its plane. The problem is therefore reduced to the determination of a single function in terms of which this distortion is expressed. The magnitude of the torsion may be defined, in terms of the axial shift, as follows.

Let r be the radius of the circle of centroids, which we may call the central line of the tore, λ the geometrical torsion of this line after deformation, k the axial shift per unit angle; then, since the lines of a meridian section are unrotated in that section, one may define the torsion τ by the equations

$$\tau = \lambda = \frac{\sin\alpha\cos\alpha}{r},$$

where α is the small angle of the helix formed by the central line. Since

$$\sin\alpha = k/r,$$

this gives

$$\tau = k/r^2,$$

neglecting the square of α.

* [A comment by the referee, on the use of the word "torsion" here, suggests a reference to Thomson and Tait, *Natural Philosophy*, §§ 604–608.]

We use cylindrical coordinates r, θ, z, the axis of z being the axis of the tore. The strain-elements corresponding to these coordinates are denoted by e, f, g, a, b, c; the stress elements by P, Q, R, S, T, U; the displacements by u, v, w, as in Love's *Elasticity*, Vol. I., p. 216.

The deformation described is expressed in symbols by

$$u = 0,$$
$$v = v(r, z),$$
$$w = k\theta,$$

which gives for the strain-elements

$$e = \frac{du}{dr} = 0,$$

$$f = \frac{1}{r}\frac{dv}{d\theta} + \frac{u}{r} = 0,$$

$$g = \frac{dw}{dz} = 0,$$

$$a = \frac{1}{r}\frac{dw}{d\theta} + \frac{dv}{dz} = \frac{dv}{dz} + \frac{k}{r},$$

$$b = \frac{du}{dz} + \frac{dw}{dr} = 0,$$

$$c = r\frac{d}{dr}\left(\frac{v}{r}\right) + \frac{1}{r}\frac{du}{d\theta} = r\frac{d}{dr}\left(\frac{v}{r}\right),$$

and for the stress-elements

$$P = Q = R = T = 0,$$

$$S = La = L\left(\frac{dv}{dz} + \frac{k}{r}\right),$$

$$U = Nc = Nr\frac{d}{dr}\left(\frac{v}{r}\right),$$

where the energy-function W is supposed of the form

$$2W = (A, B, C, F, G, H \between e, f, g)^2 + La^2 + Mb^2 + Nc^2;$$

so that we have cylindrical æolotropy.

The volume-equations reduce to the single equation

$$\frac{dU}{dr} + \frac{2U}{r} + \frac{dS}{dz} = 0$$

or
$$\frac{d}{dr}(r^3U)+\frac{d}{dz}(r^3S)=0\,;$$

so that we may put
$$r^3U=Nr^3c=-\frac{d\phi}{dz},$$
$$r^3S=Lr^3a=\frac{d\phi}{dr}.$$

Now
$$\frac{dc}{dz}=r\frac{d}{dr}\frac{1}{r}\frac{dv}{dz}$$
$$=r\frac{d}{dr}\frac{1}{r}\left(a-\frac{k}{r}\right);$$

therefore
$$-\frac{1}{N}\frac{d}{dz}\frac{1}{r^3}\frac{d\phi}{dz}=r\frac{d}{dr}\frac{1}{r}\left(\frac{1}{Lr^3}\frac{d\phi}{dr}-\frac{k}{r}\right)$$

or
$$\frac{1}{L}\left(\frac{d^2\phi}{dr^2}-\frac{3}{r}\frac{d\phi}{dr}\right)+\frac{1}{N}\frac{d^2\phi}{dz^2}+2k=0.$$

The surface-conditions over the curved boundary reduce to
$$U\,dz-S\,dr=0$$

or
$$\frac{d\phi}{dz}dz+\frac{d\phi}{dr}dr=0,$$

that is, $$\phi=\text{const.}$$

If we write $$\phi=N(\phi'-kz^2),$$

we have
$$N\left(\frac{d^2\phi'}{dr^2}-\frac{3}{r}\frac{d\phi'}{dr}\right)+L\frac{d^2\phi'}{dz^2}=0,$$

with $$\phi'=kz^2+\text{const.}$$

over the boundary.

The differential equation for ϕ' is derived from the generalized Laplace's equation
$$N\left(\frac{d^2V}{dr^2}+\frac{1}{r}\frac{dV}{dr}+\frac{1}{r^2}\frac{d^2V}{d\theta^2}\right)+L\frac{d^2V}{dz^2}=0,$$

by putting
$$V=\frac{\phi'}{r^2}\cos 2\theta\,;$$

so that the general solution for ϕ' requires no discussion here.

We shall now suppose the coefficients L, N equal, and write μ for each of them.

[redacted]

The problem then reduces to the solution of

$$\frac{d^2\phi}{dr^2}-\frac{3}{r}\frac{d\phi}{dr}+\frac{d^2\phi}{dz^2}+2\mu k=0,$$

with the condition $\phi = \text{const.}$

over the curved surface; or of

$$\frac{d^2\phi'}{dr^2}-\frac{3}{r}\frac{d\phi'}{dr}+\frac{d^2\phi'}{dz^2}=0,$$

with $\phi' = kz^2+\text{const.}$

over the surface.

The resultant of the stresses across any meridian section must be a single force Z in the axis of the tore. For, since $Q = 0$, the stress is all in the plane of the section, and therefore reduces to a single force or a single couple in that plane. Since the part of the tore between two meridian sections must be in equilibrium under the pair of forces or couples across those sections, the result at once follows.

The analytical investigation of the resultant is easy. We have

$$Z=\iint S\,dr\,dz$$

$$=\iint\frac{1}{r^3}\frac{d\phi}{dr}\,dr\,dz,$$

and this may be written

$$Z=2\iint\frac{\phi}{r^3}\,dr\,dz,$$

on integrating by parts, if $\phi = 0$ over the boundary.

Also
$$\iint U\,dr\,dz=-\iint\frac{1}{r^3}\frac{d\phi}{dz}\,dr\,dz$$

$$=-\int\frac{1}{r^3}[\phi]_{z_1}^{z_2}\,dr$$

$$=0$$

and
$$\iint(zU-rS)=-\iint\left(\frac{z}{r^3}\frac{d\phi}{dz}+\frac{1}{r}\frac{d\phi}{dr}\right)dr\,dz$$

$$=\iint\phi\left(\frac{d}{dz}\frac{z}{r^3}+\frac{d}{dr}\frac{1}{r}\right)dr\,dz$$

$$=0.$$

Particular Cases.

Any solution of the differential equation for ϕ will give a solution of the problem in which any one of the curves $\phi = \text{const.}$, not cutting the axis of z, may be taken as the meridian section of the tore, with an obvious limitation as to the nature of ϕ within the curve chosen. And, of course, two of the curves $\phi = \text{const.}$ may be taken as meridian sections of the boundaries of a toroidal shell for which the torsion problem is solved.

A very simple solution will give us a section sufficiently close to the circular form in all ordinary proportions to enable us to avoid the direct consideration of the circular form by toroidal functions which are not yet tabulated.

Consider the solution

$$\phi = pr^4 + 2qr^2 + 4qz^2 - \mu kz^2,$$

where p, q, k are independent constants.

The possible boundaries are

$$(r^2 + q/p)^2 + \frac{4q - \mu k}{p} z^2 = a^4,$$

or, writing
$$q/p = -b^2,$$
$$4q - \mu k = pc^2,$$
they are
$$(r^2 - b^2)^2 + c^2z^2 = a^4,$$
where
$$\phi = p\left\{(r^2 - b^2)^2 + c^2z^2\right\}.$$

If $b^2 > a^2$, the bounding curve is an oval not cutting the axis of z. This oval is inscribed in the rectangle

$$z = \pm a^2/c,$$
$$r = \sqrt{b^2 \pm a^2}.$$

The lines $r = b$, $z = 0$, we may call the axes, of lengths $2a^2/c$, $\sqrt{b^2+a^2} - \sqrt{b^2-a^2}$, which we write 2α, 2β respectively.

Since
$$b^2 + a^2 - (b^2 - a^2) = 2a^2,$$
we have
$$\sqrt{b^2+a^2} + \sqrt{b^2-a^2} = a^2/\beta;$$

and therefore
$$2\sqrt{b^2+a^2} = a^2/\beta+2\beta,$$
$$2\sqrt{b^2-a^2} = a^2/\beta-2\beta,$$
$$a^4 = 4\beta^2(b^2-\beta^2),$$
$$b^4-a^4 = (b^2-2\beta^2)^2,$$
$$1-\sqrt{1-a^4/b^4} = 2\beta^2/b^2 \quad [b^2 > 2\beta^2].$$

Also since,
$$q/p = -b^2$$
and
$$4q-\mu k = pc^2,$$
$$\begin{aligned} \mu k/p &= -(4b^2+c^2) \\ &= -(4b^2+a^4/\alpha^2) \\ &= -4\left\{b^2(\alpha^2+\beta^2)-\beta^4\right\}/\alpha^2. \end{aligned}$$

The area of the oval is
$$\begin{aligned} A &= 2\int z\,dr \\ &= 2\,\frac{a^4}{c}\int_0^{\frac{1}{2}\pi}\frac{\sin^2 2\theta\,d\theta}{\sqrt{(b^2-a^2)\sin^2\theta+(b^2+a^2)\cos^2\theta}}, \end{aligned}$$
putting $r^2 = (b^2-a^2)\sin^2\theta+(b^2+a^2)\cos^2\theta.$

Hence $$A = \tfrac{4}{3}\,\frac{\sqrt{b^2+a^2}}{c}\left[b^2E_1(\kappa)-(b^2-a^2)\,F_1(\kappa)\right],$$
where
$$\kappa = \sqrt{2a^2/(b^2+a^2)}$$
and F_1, E_1 are the complete elliptic integrals usually so denoted.

The centroid is at a distance $\bar{r}$ from the axis given by
$$\begin{aligned} \bar{r} &= \frac{2\int zr\,dr}{A} \\ &= 2\,\frac{a^4}{cA}\int_0^{\frac{1}{2}\pi}\sin 2\theta\,d\theta \\ &= \frac{\pi a^4}{2cA}. \end{aligned}$$

The resultant axial force is

$$Z = \iint \frac{1}{r^2}\,\frac{d\phi}{dr}\,dr\,dz$$

$$= 4p \iint \frac{r^2-b^2}{r}\,dr\,dz$$

$$= 8p \int (r^2-b^2)\,\frac{z}{r}\,dr$$

$$= 8p\,\frac{a^4}{c}\int_0^{\frac{1}{2}\pi} \sin^2 2\theta\,d\theta - 8pb^2\,\frac{a^4}{c}\int_0^{\frac{1}{2}\pi} \frac{\sin^2 2\theta\,d\theta}{(b^2-a^2)\sin^2\theta + (b^2+a^2)\cos^2\theta}$$

$$= -2\pi p\,\frac{a^4}{c}\left(\frac{b^2}{a^2} - \sqrt{\frac{b^4}{a^4}-1}\right)^2$$

$$= \pi\mu k\,\frac{\alpha^3\beta^3}{\sqrt{b^2-\beta^2}\,\{b^2(\alpha^2+\beta^2)-\beta^4\}}.$$

If b is large compared with a, this is approximately

$$\frac{\pi\mu\kappa}{b^3}\,\frac{\alpha^3\beta^3}{\alpha^2+\beta^2};$$

or, introducing the torsion, $\dfrac{\pi\mu\tau}{b}\,\dfrac{\alpha^3\beta^3}{\alpha^2+\beta^2}$,

so that the moment of the force about the centre of the section approximates to

$$\pi\mu\tau\,\frac{\alpha^3\beta^3}{\alpha^2+\beta^2}.$$

This is the ordinary formula for the torsion-couple of an elliptic prism of axes 2α, 2β, as, of course, it should be, since the section approximates to such an ellipse when the curvature of the tore approaches zero.

Numerical Examples.

We proceed to make application of these formulæ to the case of the circular section, and in particular to compare the actual stiffness and the greatest stress or strain with those which the wire-method gives. The approximately circular sections are obtained by putting $\alpha = \beta$, which makes the axial force

$$Z = \pi\mu k\,\frac{a^4}{\sqrt{b^2-a^2}\,(2b^2-a^2)}.$$

The maximum stress or strain is got by making a maximum

$$\sqrt{S^2+U^2} = \frac{2p}{r^3}\sqrt{4r^2(r^2-b^2)^2+c^2z^2}.$$

This is a maximum for a given value of r when z is a maximum, so that the points of greatest stress or strain lie on the boundary. Inserting the value of z for a point on the boundary, we have

$$\sqrt{S^2+U^2}$$
$$= \frac{\mu k}{2b^2-a^2}\sqrt{r^2+b^2(3b^2-a^2)/r^2-(b^2-2a^2)^2(b^2-a^2)/r^4-(3b^2-a^2)}.$$

Denoting the maximum value of the square root by M, the maximum shearing stress is

$$\frac{\mu k}{2b^2-a^2}M,$$

and the ratio of this to the average axial force per unit area of cross-section is

$$R = \frac{\sqrt{b^2-a^2}\,AM}{\pi a^4}.$$

Let $\bar{a}$ be the radius of the circle of the same area as the actual section. The maximum shearing stress due to torsion $\tau = k/\bar{r}^3$ of a prism of this section is

$$\mu\tau\bar{a} = \mu k\bar{a}/\bar{r}^3,$$

and the couple of torsion is $\frac{1}{2}\pi\mu\tau\bar{a}^4$; so that the corresponding axial force given by the wire-method is

$$Z' = \tfrac{1}{2}\pi\mu k\bar{a}^4/\bar{r}^3,$$

and the axial force per unit area is

$$\tfrac{1}{2}\mu k\bar{a}^2/\bar{r}^3.$$

If, then, we adopt the common assumption of the wire-method that the maximum shear is the maximum due to torsion, together with the average axial force, we have for this maximum

$$\mu k\bar{a}/\bar{r}^3+\tfrac{1}{2}\mu k\bar{a}^2/\bar{r}^3,$$

and the ratio of this to the average axial force is

$$R' = 2\bar{r}/\bar{a}+1;$$

so that
$$\frac{R}{R'} = \frac{\sqrt{b^2-a^2}\,\bar{a}AM}{\pi a^4\,(2\bar{r}+\bar{a})}.$$

The calculations have been performed for the cases

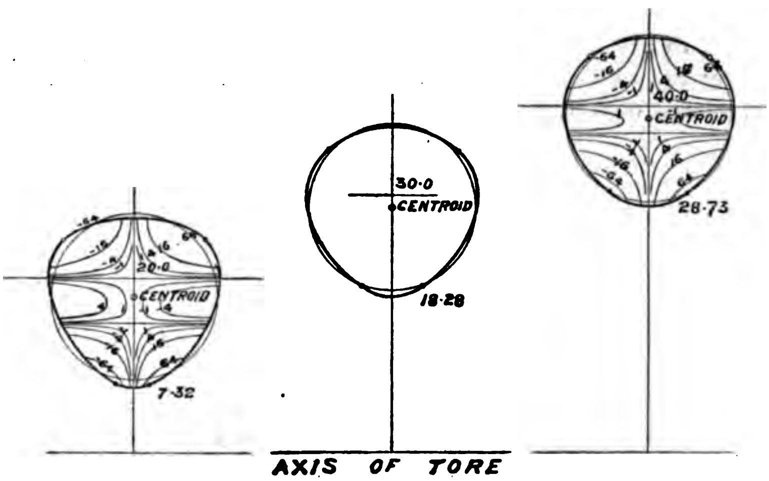
$$b = 2a,\ 3a,\ \dots,\ 7a,$$

which are sufficiently close to cover all cases not given with close approximation by the common assumption.

The diagrams represent the cases

$$b = 2a,\ 3a,\ 4a,$$

where, in the distances assigned, $a = 10$. The points of greatest



stress have been indicated by small black circles; those of least stress by small white circles. The curves numbered from 1 to 64 are curves of equal normal displacement relative to a plane section turned through the angle k/b, so as to be normal to the helix formed by the circle $r = b$, $z = 0$ after deformation. The numbers attached to the curves are proportional to the displacements along them. The formula for such displacements is

$$v' = -\frac{k}{b}\,z\,(r-b)\,\frac{rb-b^2+a^2}{r\,(2b^2-a^2)};$$

so that the displacement of the parts outside $r = b$ and inside $r = b - a^2/b$ is greater than that corresponding to the rotation k/b.

The light circular boundary in each figure is the circle of equal area described with the centroid as centre.

b	$\bar{r}$	πa^2	A	R/R'	Z/Z'
20	18·09	314·2	300·8	1·69	1·06
30	28·73	314·2	309·2	1·29	1·02
40	39·02	314·2	311·8	1·19	1·00
50	49·28	314·2	312·2	1·12	1·00
60	59·39	314·2	312·9	1·09	1·00
70	69·40	314·2	313·6	1·08	1·00

It appears that the wire-method gives a sufficiently close approximation to the stiffness for ordinary purposes of design, especially having regard to the uncertainty of the value of μ arising from irregularity of temper.

Rectangular Section.

Let
$$r = a, \quad r = b,$$
$$z = 0, \quad z = h,$$
be the sides of the rectangular section. Put
$$\phi = \mu kz\,(h-z) + \phi'.$$
Taking $\phi = 0$ as the boundary condition, we have
$$\phi' = 0$$
over
$$z = 0, \quad z = h,$$
and we may put
$$\phi' = r^2 \Sigma \left\{ C_n I_2 \left(\frac{n\pi r}{h}\right) + D_n K_2 \left(\frac{n\pi r}{h}\right) \right\} \sin \frac{n\pi z}{h},$$
where n is an integer, and I_2, K_2 are the Bessel's functions so denoted by Gray and Mathews, pp. 66, 67, viz.,
$$I_2(x) = -J_2(ix)$$
and
$$K_2(x) = \frac{3}{x^2} \int_0^\infty \frac{\cos(x \sinh \phi)}{\cosh^4 \phi}\, d\phi.$$

We have then $$\phi' = -\mu kz\,(h-z)$$
over $$r = a, \quad r = b;$$
and therefore

$$C_n I_2\left(\frac{n\pi a}{h}\right)+D_n K_2\left(\frac{n\pi a}{h}\right) = -\frac{2\mu k}{a^2 h}\int_0^h z\,(h-z)\sin\frac{n\pi z}{h}\,dz,$$

$$C_n I_2\left(\frac{n\pi b}{h}\right)+D_n K_2\left(\frac{n\pi b}{h}\right) = -\frac{2\mu k}{b^2 h}\int_0^h z\,(h-z)\sin\frac{n\pi z}{h}\,dz;$$

so that $$C_{2n} = D_{2n} = 0$$

and

$$C_{2n+1} I_2\left\{\frac{(2n+1)\,\pi a}{h}\right\}+D_{2n+1} K_2\left\{\frac{(2n+1)\,\pi a}{h}\right\} = -\frac{8\mu k h^2}{(2n+1)^3\pi^3 a^2},$$

$$C_{2n+1} I_2\left\{\frac{(2n+1)\,\pi b}{h}\right\}+D_{2n+1} K_2\left\{\frac{(2n+1)\,\pi b}{h}\right\} = -\frac{8\mu k h^2}{(2n+1)^3\pi^3 b^2},$$

which determine the constants.

The resultant axial force is

$$Z = 2\iint (\phi/r^3)\,dr\,dz.$$

Now $$\int \frac{1}{r} I_2\left(\frac{n\pi r}{h}\right) dr = \frac{h}{n\pi r} I_1 + \text{const.},$$

$$\int \frac{1}{r} K_2\left(\frac{n\pi r}{h}\right) dr = \frac{h}{n\pi r} K_1 + \text{const.}$$

Hence

$$Z = \tfrac{1}{6}\mu k h^3\left(\frac{1}{a^2}-\frac{1}{b^2}\right)$$

$$+4\Sigma\frac{h^2}{(2n+1)^2\pi^2}\Bigg[C_{2n+1}\left\{\frac{1}{b} I_1\left(\frac{2(n+1)\,\pi b}{h}\right)-\frac{1}{a} I_1\left(\frac{2(n+1)\,\pi a}{h}\right)\right\}$$

$$+D_{2n+1}\left\{\frac{1}{b} K_1\left(\frac{2(n+1)\,\pi b}{h}\right)-\frac{1}{a} K_1\left(\frac{2(n+1)\,\pi a}{h}\right)\right\}\Bigg],$$

which determines the stiffness of the tore.

Torsion of a Rod of Varying Circular Section.

If in the general solution for the torsion of tores we put $k = 0$, we obtain the general solution for the torsion of a straight rod of *varying*

circular section. We have here

$$u = w = 0,$$

$$v = v(r, z),$$

$$e = f = g = b = 0,$$

$$a = \frac{dv}{dz},$$

$$c = r\frac{d}{dr}\left(\frac{v}{r}\right),$$

$$r^2 U = Nr^3 c = -\frac{d\phi}{dz},$$

$$r^3 S = Lr^3 a = \frac{d\phi}{dr},$$

where ϕ satisfies the equation

$$N\left(\frac{d^2\phi}{dr^2} - \frac{3}{r}\frac{d\phi}{dr}\right) + L\frac{d^2\phi}{dz^2} = 0$$

and is constant over the curved surfaces.

The stress on an end z is given by

$$S = \frac{1}{r^3}\frac{d\phi}{dr},$$

the resultant couple on the end being

$$2\pi\int_{r_1}^{r_2}\frac{d\phi}{dr}\,dr = 2\pi(\phi_2 - \phi_1),$$

and the rate of torsion

$$\frac{dv}{dz} = a = -\frac{1}{L}\frac{1}{r^3}\frac{d\phi}{dr},$$

which is, in general, a function of both r and z, as might have been expected.

The application of this theory to shafting of varying diameter should lead to information of value; but on this I do not enter at present.

II. *Flexure.*

We here consider the clearly possible uniform flexure of the tore into a tore of slightly different radius. The fundamental fact of this kind of deformation is that the meridian sections remain plane and

become meridian sections in the new tore. This result is derived at once from a consideration of symmetry, as in the case of the flexure of a cylindrical beam. Suppose, for the sake of definiteness, that the new tore is coaxial with the old. The displacements are then

$$u = u\,(r,\, z),$$
$$v = \beta r\theta,$$
$$w = w\,(r,\, z),$$

where β is a constant, and we have supposed the new position such that $v = 0$, when $\theta = 0$.

The strains are then

$$e = \frac{du}{dr},$$
$$f = \beta + u/r,$$
$$g = \frac{dw}{dz},$$
$$b = \frac{du}{dz} + \frac{dw}{dr},$$
$$a = c = 0;$$

so that the stresses are

$$P = \lambda\left(\frac{1}{r}\,\frac{dru}{dr} + \frac{dw}{dz} + \beta\right) + 2\mu\,\frac{du}{dr},$$

$$Q = \lambda\left(\frac{1}{r}\,\frac{dru}{dr} + \frac{dw}{dz} + \beta\right) + 2\mu\,(\beta + u/r),$$

$$R = \lambda\left(\frac{1}{r}\,\frac{dru}{dr} + \frac{dw}{dz} + \beta\right) + 2\mu\,\frac{dw}{dz},$$

$$T = \mu\left(\frac{du}{dz} + \frac{dw}{dr}\right),$$

$$S = U = 0,$$

where the solid is supposed isotropic, to simplify as much as possible.

The stress equations are

$$\frac{dP}{dr} + \frac{dT}{dz} + \frac{P - Q}{r} = 0,$$

$$\frac{1}{r}\,\frac{dTr}{dr} + \frac{dR}{dz} = 0.$$

The latter is satisfied by putting

$$Tr = \frac{d\phi}{dz} = \mu r\left(\frac{du}{dz} + \frac{dw}{dr}\right),$$

$$Rr = -\frac{d\phi}{dr} = r\left\{\lambda\left(\beta + \frac{1}{r}\frac{dru}{dr}\right) + (\lambda+2\mu)\frac{dw}{dz}\right\}.$$

The equations
$$\frac{d}{dz}\left(\frac{\phi}{r} - \mu u\right) = \mu\frac{dw}{dr},$$

$$\frac{d}{dr}(\phi + \tfrac{1}{2}\lambda\beta r^2 + \lambda ru) = -(\lambda+2\mu)\frac{drw}{dz}$$

are satisfied by putting

$$\frac{\phi}{r} - \mu u = \frac{d\vartheta}{dr},$$

$$\mu w = \frac{d\vartheta}{dz},$$

$$\phi + \tfrac{1}{2}\lambda\beta r^2 + \lambda ru = \frac{d\chi}{dz},$$

$$(\lambda+2\mu)\, rw = -\frac{d\chi}{dr},$$

which make
$$(\lambda+2\mu)\frac{d\vartheta}{dz} = -\mu\frac{d\chi}{dr},$$

leading to
$$r\vartheta = \mu\frac{d\psi}{dr},$$

$$\chi = -(\lambda+2\mu)\frac{d\psi}{dz},$$

and these introduce the function ψ in terms of which all the other quantities can be expressed.

We now have

$$\phi = \mu\left(ru + r\frac{d}{dr}\frac{1}{r}\frac{d\psi}{dr}\right),$$

$$(\lambda+\mu)\, ru = -(\lambda+2\mu)\frac{d^2\psi}{dz^2} - \mu r\frac{d}{dr}\frac{1}{r}\frac{d\psi}{dr} - \tfrac{1}{2}\lambda\beta r^2,$$

$$w = \frac{1}{r}\frac{d^2\psi}{dr\,dz},$$

so that the strains are given by

$$(\lambda+\mu)\, e = -(\lambda+2\mu)\frac{d^3}{dz^2\,dr}\left(\frac{\psi}{r}\right) - \mu\frac{d^2}{dr^2}\,\frac{1}{r}\,\frac{d\psi}{dr} - \tfrac{1}{2}\lambda\beta,$$

$$(\lambda+\mu)\, f = -(\lambda+2\mu)\frac{1}{r^2}\,\frac{d^2\psi}{dz^2} - \mu\frac{1}{r}\,\frac{d}{dr}\,\frac{1}{r}\,\frac{d\psi}{dr} + \tfrac{1}{2}(\lambda+2\mu)\,\beta,$$

$$g = \frac{1}{r}\,\frac{d^3\psi}{dr\,dz^2},$$

$$(\lambda+\mu)\, b = -(\lambda+2\mu)\frac{1}{r}\,\frac{d^3\psi}{dz^3} + \lambda\frac{d}{dr}\,\frac{1}{r}\,\frac{d^2\psi}{dr\,dz},$$

$$a = c = 0.$$

It remains to find the equation satisfied by ψ by substituting in the stress-equation

$$\frac{dP}{dr} + \frac{dT}{dz} + \frac{P-Q}{r} = 0.$$

Now the dilatation is given by

$$(\lambda+\mu)\,\delta = -\mu\frac{1}{r}\,\frac{d}{dr}\left(r\frac{d}{dr}\,\frac{1}{r}\,\frac{d\psi}{dr} + \frac{d^2\psi}{dz^2}\right) + \mu\beta$$

$$= -\mu\frac{1}{r}\,\frac{d\Delta^2\psi}{dr} + \mu\beta$$

$$= -\mu\nabla^2\frac{1}{r}\,\frac{d\psi}{dr} + \mu\beta,$$

where

$$\Delta^2 \equiv r\frac{d}{dr}\,\frac{1}{r}\,\frac{d}{dr} + \frac{d^2}{dz^2},$$

$$\nabla^2 \equiv \frac{1}{r}\,\frac{d}{dr}\,r\,\frac{d}{dr} + \frac{d^2}{dz^2}.$$

Hence, writing

$$P = (\lambda+2\mu)\,\delta - 2\mu\,(f+g),$$

$$Q = \lambda\delta + 2\mu f,$$

$$P - Q = 2\mu\,(\delta - 2f - g),$$

the equation becomes

$$(\lambda+\mu)(\lambda+2\mu)\frac{d\delta}{dr} + 2\mu\,(\lambda+\mu)\frac{\delta}{r}$$

$$+ (\lambda+2\mu)\,2\mu\frac{1}{r}\,\frac{d^3\psi}{dz^2\,dr} + 2\mu^2\frac{1}{r^2}\,\frac{d}{dr}\,r\,\frac{d}{dr}\,\frac{1}{r}\,\frac{d\psi}{dr} - 2\mu\,(\lambda+2\mu)\,\beta/r$$

$$-(\lambda+\mu)\,2\mu\frac{1}{r}\,\frac{d^4\psi}{dr^2\,dz^2}$$

$$-(\lambda+2\mu)\,\mu\frac{1}{r}\,\frac{d^4\psi}{dz^4} + \lambda\mu\frac{d}{dr}\,\frac{1}{r}\,\frac{d^3\psi}{dr\,dz^2} = 0,$$

that is,

$$(\lambda+\mu)(\lambda+2\mu)\frac{d\delta}{dr}+2\mu(\lambda+\mu)\frac{\delta}{r}$$

$$-(\lambda+2\mu)\,\mu\,\frac{1}{r}\,\frac{d^2}{dz^2}\left(\frac{d^2\psi}{dr^2}-\frac{1}{r}\,\frac{d\psi}{dr}+\frac{d^2\psi}{dz^2}\right)$$

$$+2\mu^2\,\frac{1}{r^2}\,\frac{d}{dr}\left(\frac{d^2\psi}{dr^2}-\frac{1}{r}\,\frac{d\psi}{dr}+\frac{d^2\psi}{dz^2}\right)-2\mu(\lambda+2\mu)\,\beta/r=0;$$

or, inserting the value of δ,

$$(\lambda+2\mu)\,\Delta^4\psi+2\,(\lambda+\mu)\,\beta=0,$$

where

$$\Delta^4\equiv\left(r\,\frac{d}{dr}\,\frac{1}{r}\,\frac{d}{dr}+\frac{d^2}{dz^2}\right)^2,$$

and the identity

$$\frac{1}{r}\,\frac{d}{dr}\,\Delta^2\equiv\nabla^2\,\frac{1}{r}\,\frac{d}{dr}$$

must be borne in mind.

The surface-conditions reduce to

$$P\,dz-T\,dr=0,$$
$$T\,dz-R\,dr=0,$$

or

$$\left\{\lambda\,\frac{1}{r}\,\frac{d\Delta^2\psi}{dr}+2\,(\lambda+2\mu)\,\frac{d^3}{dz^2dr}\left(\frac{\psi}{r}\right)+2\mu\,\frac{d^2}{dr^2}\,\frac{1}{r}\,\frac{d\psi}{dr}\right\}dz$$
$$-\left\{(\lambda+2\mu)\,\frac{1}{r}\,\frac{d^3\psi}{dz^3}-\lambda\,\frac{d}{dr}\,\frac{1}{r}\,\frac{d^2\psi}{dr\,dz}\right\}dr=0,$$

$$\left\{\lambda\,\frac{d}{dr}\,\frac{1}{r}\,\frac{d^2\psi}{dr\,dz}-(\lambda+2\mu)\,\frac{1}{r}\,\frac{d^3\psi}{dz^3}\right\}dz$$
$$+\left\{\lambda\,\frac{1}{r}\,\frac{d\Delta^2\psi}{dr}-2\,(\lambda+\mu)\,\frac{1}{r}\,\frac{d^3\psi}{dr\,dz^2}-\lambda\beta\right\}dr=0.$$

If we put $\beta=0$, we have the reduction of the symmetrical deformation of a solid of revolution to the determination of a single function ψ corresponding to the reduction already made in the case of plane stress.

No attempt is here made to solve the equation for ψ in special cases.

Thursday, June 8th, 1899.

Lord KELVIN, G.C.V.O., F.R.S., President, in the Chair.

Present, twenty-two members.

The President announced that the Council had awarded the De Morgan Medal to Prof. W. Burnside for his researches in mathematics, particularly in the theory of groups of finite order. Prof. Burnside briefly thanked the Society for the honour conferred upon him.

The Secretary (Mr. Tucker) announced a recent loss the Society had sustained through the premature death of Mr. S. O. Roberts, who was elected a member of the Society, January 8th, 1885.

Prof. Mittag-Leffler was admitted into the Society, and then made an interesting communication (in French) "On the Convergency of Series." Messrs. Elliott, Hobson, and Love offered some remarks to which Prof. Mittag-Leffler briefly replied. At the request of the meeting he promised to put his communication into the form of a paper for the *Proceedings.*

The President next spoke on "Solitary Waves, Equivoluminal and Irrotational, in an Elastic Solid." Prof. Love expressed himself as having been much interested in the President's diagram and remarks. He then gave a sketch of a paper by Mr. J. H. Michell, "On the Transmission of Stress across a Plane of Discontinuity in an Isotropic Elastic Solid and the Potential Solutions for a Plane Boundary."

The following papers were taken as read:—

On several Classes of Simple Groups: Dr. G. A. Miller.

On Theta Differential Equations and Expansions: Rev. M. M. U. Wilkinson.

Finite Current Sheets: J. H. Jeans.

(1) On a Congruence Theorem having reference to an Extensive Class of Coefficients; (2) On a Set of Coefficients analogous to the Eulerian Numbers: Dr. Glaisher.

(1) On the Reduction of a Linear Substitution to its Canonical Form; (2) On the Integration of Systems of Total Differential Equations: Prof. A. C. Dixon.

The following presents were made to the Library:—

"Educational Times," June, 1899.

"Indian Engineering," Vol. xxv., Nos. 16–19, April 22–May 13, 1899.

"Memoirs of the National Academy of Sciences," Vol. VIII. (3rd Memoir); Washington, 1899.

"Periodico di Matematica," Serie II., Tomo I., Fasc. 6; Livorno, 1899.

The following exchanges were received:—

"Proceedings of the Royal Society," Vol. LXV., Nos. 413, 414, 1899.

"Bulletin of the American Mathematical Society," Vol. V., No. 8; New York, May, 1899.

"Monatshefte für Mathematik und Physik," Jahrgang X., Pt. 3; Wien, 1899.

"Atti dell' Accademia delle Scienze Fisiche e Matematiche," Serie 2, Vol. IX.; Napoli, 1899.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Serie 3, Vol. V., Fasc. 4; Napoli, April, 1899.

"Journal für die reine und angewandte Mathematik," Band CXX., Heft 2; Berlin, 1899.

"Annali di Matematica," Serie 3, Tomo II., Fasc. 2, 3; Milano, April, 1899.

"Archives Néerlandaises," Série 2, Tome II., Livr. 5; La Haye, 1899.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 1, Vol. VIII., Fasc. 8, 9; Roma, 1899.

"Nyt Tidsskrift for Matematik," Aargang X., A, Nr. 5, B, Nr. 1; Copenhagen, 1899.

"Vierteljahrsschrift der Naturforschenden Gesellschaft in Zurich," Jahrgang XXXIV., Heft 4, February 15, 1898.

"Proceedings of the Physical Society," Vol. XVI., Pt. 5; May, 1899.

"Sitzungsberichte der Königl. Preuss. Akademie der Wissenschaften zu Berlin," I.–XXII.; 1899.

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. XLIII., Pt. 1; 1899.

"Proceedings of the Royal Irish Academy," Vol. V., No. 2; Dublin, 1899.

"Proceedings of the Canadian Institute," Vol. II., Pt. 1, No. 7; Toronto, 1899.

On Several Classes of Simple Groups. *By* G. A. Miller.

Received June 8th, 1899. Read June 8th, 1899.

Let G be any primitive substitution group of degree kp, p being any prime number, and of order $g = pm$, m being prime to p. Since a primitive group cannot contain any intransitive self-conjugate sub-group, every self-conjugate sub-group of G that differs from identity must include all its substitutions of order p. It is easy to prove that these substitutions of order p generate a simple group.

If this group were compound, it would have to contain a self-conjugate sub-group whose order could not be divisible by p. This sub-group would therefore be intransitive, and hence it could not be self-conjugate in G. Hence the sub-groups of order p that would be contained in G would generate a characteristic sub-group of G which would not coincide with G. If this characteristic sub-group were compound, it would have to contain a system of largest self-conjugate sub-groups whose orders would not be divisible by p. As this is clearly impossible, we have the

THEOREM I.—*If a primitive group is of degree kp (p being any prime number) and of order mp (m being prime to p), all of its sub-groups of order p generate a simple group. When k exceeds unity this simple group must always be of a composite order.*

COROLLARY I.—*If the degree of a primitive group is divisible by 2, its order must be divisible by 4.**

COROLLARY II.—*The sub-groups of order p that are contained in any transitive group of degree p generate a simple group.*†

The quotient group of G with respect to the given simple sub-group has a $1, pn$ isomorphism with the group formed by all its substitutions that transform any one of its sub-groups of order p into itself, np being the number of these substitutions that are found in the simple sub-group. Hence we observe that G cannot have more than one composite factor of composition whenever k is less than 5. When $k=1$ the given quotient group must be cyclical. A given simple group can therefore be self-conjugate in only one transitive group of degree p and of a given order. To determine all the transitive groups of degree p that contain a given simple group as a self-conjugate sub-group, it is only necessary to determine the largest sub-group of the metacyclic group which contains any one of the sub-groups of order p that is included in the given simple group, and transforms this simple group into itself.

It is known that a cyclical substitution of order p cannot occur in any primitive group that does not include the alternating group unless the degree of the group is p, $p+1$, or $p+2$.‡ The case when the degree is p is included in the preceding considerations. When

* *Cf.* Frobenius, *Berliner Sitzungsberichte*, 1895, p. 180.
† *Bulletin of the American Mathematical Society*, Vol. IV., 1898, p. 139.
‡ *Ibid.*, p. 143.

the degree is $p+2$* the group must be triply transitive. We proceed to prove that all the substitutions of order p that are contained in such a group generate a simple group of composite order. If the triply transitive group which is generated by the substitutions were compound, it would contain a transitive self-conjugate sub-group whose order could not exceed $(p+1)(p+2)$, since every self-conjugate sub-group of a transitive group of degree p includes substitutions of order p.

Since the maximal sub-group of degree $p+1$ would be doubly transitive, it could not contain any intransitive self-conjugate sub-group. Hence the given transitive self-conjugate sub-group would be either of order $(p+1)(p+2)$ or of order $p+2$. In either case it would involve just $p+1$ substitutions of degree $p+2$. This is clearly impossible since the group contains substitutions of order p. Hence we have the

THEOREM II.—*If a primitive group of degree* $p+2$ *contains substitutions of order* p, *all of these substitutions generate a simple group whose order is divisible by* $p\,(p+1)(p+2)$.

When a primitive group of degree $p+1$ contains substitutions of order p these substitutions must generate either the entire group or a characteristic sub-group. Since this group is doubly transitive, it cannot contain an intransitive self-conjugate sub-group. If it is compound, it must therefore involve the regular group of order $p+1$ as self-conjugate sub-group. Since this regular group cannot contain any sub-group besides identity that is commutative to one of the given substitutions of order p, it must involve p conjugate sub-groups; *i.e.*, $p+1$ must be a power of 2. Hence the

THEOREM III.—*If a primitive group of degree* $p+1$ *contains substitutions of order* p, *all of these substitutions must generate a simple group whose order is divisible by* $p\,(p+1)$, *whenever* $p+1$ *is not a power of* 2. *When* $p+1$ *is a power of* 2 *the group generated by these substitutions of order* p *cannot contain any self-conjugate sub-group except the regular group of order* $p+1$ *which contains* p *sub-groups of order* 2 *and identity.*

COROLLARY I.—*If a primitive group contains substitutions of degree and order* p, *it cannot have more than one composite factor of composition.*

COROLLARY II.—*If the substitutions of order* p *that are contained in such a primitive group do not generate the entire group, they generate a*

---

* In what follows we shall assume that $p > 2$.

characteristic sub-group with respect to which the entire group is isomorphic with a cyclical group whose order is a divisor of $p-1$.

Most of these results can be readily applied to the theory of abstract groups. If a group contains a maximal sub-group that does not include any self-conjugate sub-group of the entire group besides identity, and if the order of this maximal sub-group is the quotient obtained by dividing the order of the entire group by n, then it is possible to represent the entire group as a primitive substitution group of degree n.* Hence it follows from what precedes that the operators of order p which are contained in a group in which n may be equal to p or $p+2$ generate a simple group, and that such a group cannot have more than one composite factor of composition. Similar remarks can evidently be made in regard to most of the other results.

Finite Current-Sheets. *By* J. H. Jeans. Received May 30th, 1899. Communicated by G. T. Walker, M.A., June 8th, 1899.

1. The theory of the currents induced in an infinite current sheet has been completely worked out by Maxwell (*Elect. and Mag.*, §§ 647–666). I have attempted to develop a similar theory for the case of a finite current-sheet.

As in the case of the infinite current-sheet, the mathematical discussion falls naturally into two parts: in the former it is necessary to determine the currents induced by an instantaneous change in the inducing field; in the latter to examine the dissipation of these currents under the action of resistance and self-induction.

In §§ 4 and 5 I have discussed these two questions for the particular case of a semi-infinite plane, but the method is quite general, as I have tried to show in § 8. § 7 contains the extension of former results to the two cases of an infinite strip with parallel edges, and an infinite plane from which such a strip has been removed. In § 6 Maxwell's theory for an infinite plane is deduced

---

* Dyck, *Mathematische Annalen*, Vol. xxii., 1883, p. 102.

from the results obtained for the semi-infinite plane bounded by a straight edge by supposing the straight edge to be at infinity.

§ 2 merely contains the transformation of the ordinary electrical equations into a form suitable for dealing with any finite current-sheet.

2. We start with a current-function Φ defined as by Maxwell (§ 648). A knowledge of Φ at every point will give us a complete knowledge of the currents in the sheet. The magnetic effect of the current-sheet at any external point is the same as that of a magnetic shell of strength $C+\Phi$, where C is a constant so chosen that $C+\Phi$ vanishes at the boundary.

In dealing with plane finite current-sheets we shall suppose Φ measured from a point on the boundary, so that $C=0$.

We next introduce Maxwell's function P, defined for all points in space by the equation

$$P = \iint \frac{\Phi}{r} dx' dy', \tag{1}$$

where $dx'dy'$ is an element of the current-sheet at which the current-function is Φ, and r is the distance from this element to the point at which we are evaluating P.

P is the gravitational potential due to a distribution of density Φ over the surface of the sheet, and therefore the magnetic potential Ω due to the currents in the sheet is given by

$$\Omega = -\frac{dP}{dz}. \tag{2}$$

Also, since the value of P is symmetrical as regards the two surfaces of the sheet, we have at the positive surface

$$\Omega = -\frac{dP}{dz} = 2\pi\Phi,$$

and at the negative surface

$$\Omega = -\frac{dP}{dz} = -2\pi\Phi.$$

If F, G, H are the components of the vector potential due to these currents, we may take

$$F = \frac{dP}{dy}, \quad G = -\frac{dP}{dx}, \quad H = 0,$$

for these satisfy the equations

$$\left.\begin{aligned}\frac{dH}{dy}-\frac{dG}{dz}&=-\frac{d\Omega}{dx}\\ \frac{dF}{dz}-\frac{dH}{dx}&=-\frac{d\Omega}{dy}\\ \frac{dG}{dx}-\frac{dF}{dy}&=-\frac{d\Omega}{dz}\end{aligned}\right\},$$

in virtue of equation (2).

Let F', G', H' be the components of the vector potential of the inducing field, and P' a quantity analogous to P, defined by

$$\Omega' = -\frac{dP'}{dz}, \tag{3}$$

where Ω' is the scalar potential of the inducing field.

P' is, so far, defined except for an arbitrary function of x and y, and this can be so chosen that P' shall be the potential of matter none of which is indefinitely near to the current-sheet. For a value of P which satisfies equation (3) is the potential of thin cylinders of matter starting from each pole of the external field, and going to infinity in the direction of the positive axis of z, the line-density of any cylinder being equal to the strength of the pole from which it started. Take as the additional arbitrary function of x and y the potential due to a series of cylinders coinciding with such of the above cylinders as pass through the current-sheet, and of equal but opposite strength. Then P' will satisfy the condition mentioned above.

Exactly as in the former case, we may take

$$F' = \frac{dP'}{dy},\quad G' = -\frac{dP'}{dx},\quad H' = 0.$$

It is now possible to write down the electrical equations for the current-sheet in the form

$$\left.\begin{aligned}X &= \quad\sigma\frac{d\phi}{dy} = -\frac{d}{dt}(F+F') - \frac{d\psi}{dx}\\ Y &= -\sigma\frac{d\phi}{dx} = -\frac{d}{dt}(G+G') - \frac{d\psi}{dy}\end{aligned}\right\}. \tag{4}$$

Let us now introduce two new functions χ, χ' defined for all space by the equations

$$\left.\begin{aligned} \chi &= \frac{\sigma}{2\pi}\frac{dP}{dz} - \frac{d}{dt}(P+P') \\ \chi' &= -\frac{\sigma}{2\pi}\frac{dP}{dz} - \frac{d}{dt}(P+P') \end{aligned}\right\}. \tag{5}$$

At the positive surface of the current-sheet

$$\frac{d\chi}{dy} = -\sigma\frac{d\phi}{dy} - \frac{d}{dt}(F+F')$$

$$= \frac{d\psi}{dx}, \text{ by equation (4),}$$

$$\frac{d\chi}{dx} = -\sigma\frac{d\phi}{dx} + \frac{d}{dt}(G+G')$$

$$= -\frac{d\psi}{dy}, \text{ by the same equation.}$$

Hence, at all points on the positive side of the current-sheet

$$\frac{d^2\chi}{dx^2} + \frac{d^2\chi}{dy^2} = 0,$$

and, similarly, at all points on the negative surface

$$\frac{d^2\chi'}{dx^2} + \frac{d^2\chi'}{dy^2} = 0.$$

But χ, χ' are potentials due to gravitating matter none of which lies just outside either surface of the sheet, and therefore at either surface

$$\frac{d^2\chi}{dx^2} + \frac{d^2\chi}{dy^2} + \frac{d^2\chi}{dz^2} = 0,$$

$$\frac{d^2\chi'}{dx^2} + \frac{d^2\chi'}{dy^2} + \frac{d^2\chi'}{dz^2} = 0.$$

The electrical equations can therefore be put into the form

$$\frac{d^2\chi}{dz^2} = 0$$

at the positive surface of the current-sheet, and

$$\frac{d^2\chi'}{dz^2} = 0$$

at the negative surface of the current-sheet.

The expression of the fact that Φ vanishes at all points on the boundary takes the simple form

$$\chi = \chi'$$

at the boundary.

3. Following the method adopted by Maxwell, we begin by considering the currents induced by a sudden change in the external field.

Writing R for $\frac{\sigma}{2\pi}$, we have seen that at the positive surface of the sheet

$$\frac{d}{dt}\left[\frac{d^2}{dz^2}(P+P')\right] = R\frac{d^3P}{dz^3}, \tag{6a}$$

and at the negative surface

$$\frac{d}{dt}\left[\frac{d^2}{dz^2}(P+P')\right] = -R\frac{d^3P}{dz^3}. \tag{6b}$$

Integrating either equation with respect to the time throughout the indefinitely short period during which the change is supposed to occur,

$$\left[\frac{d^2}{dz^2}(P+P')\right] = \pm R\int_0^\tau \frac{d^3P}{dz^3}\,d\tau$$
$$= 0,$$

if τ is supposed to vanish in the limit, since $\frac{d^3P}{dz^3}$ is finite.

Now $\left[\frac{d}{dz}(P+P')\right]$ or $-\left[\Omega+\Omega'\right]$ is the potential due to an unknown distribution on the current-sheet and a known distribution in external space. Denoting this potential by ϖ, the electrical equations express that $\frac{d\varpi}{dz}$ vanishes at both surfaces of the current-sheet. Also the poles of ϖ are known in the space external to the current-sheet, being identical with those of $-[\Omega']$, and ϖ vanishes at infinity.

Hence sufficient is known about ϖ to uniquely determine its value for all space. It is, in fact, the velocity potential in an incompressible fluid in which the current-sheet is a rigid boundary, and in which a system of sources and sinks exists such that the velocity potential due to it in a free unbounded fluid would be $-[\Omega']$.

If the hydrodynamical problem of finding this velocity potential can be solved, we can find the currents induced by a sudden change

in the external magnetic field at the moment immediately after the change. If the current-sheet is perfectly conducting, the solution thus found will represent the currents at any time.

The next two sections deal with the case of a semi-infinite current-sheet bounded by a straight edge; the further discussion of the general case is postponed until § 8.

4. In dealing with a semi-infinite plane, let us take cylindrical coordinates ξ, r, ϕ. Let the axis of ξ coincide with the free edge of the current-sheet, the positive side of its surface coinciding with the sectorial plane $\phi = 0$, and the negative side with $\phi = 2\pi$.

To find the value of the potential ϖ, we shall use the method of multiform potentials introduced by Dr. Sommerfeld.*

Let us construct a Riemann's space of two windings, such that the branch-line coincides with the edge of our current-sheet, and let the position of a point in this space be determined by the cylindrical coordinates ξ, r, ϕ, so that the point returns to its original position if ϕ is increased by 4π, but not, of course, if it is increased by 2π. Moreover, let that sheet† of the space in which ϕ varies from 0 to 2π be identical with our original real space, in which ϕ varied from 0 to 2π.

It can be shown that u, a function of position in Riemann's space, is uniquely determined throughout any region of that space which does not include a branch-line, by the following conditions:—

(i.) Its infinities in this region must be known.

(ii.) It must be finite and continuous and satisfy $\nabla^2 u = 0$ at every point of this region, except at the infinities.

(iii.) $\dfrac{du}{dn}$ must be given at every point of the boundary, and u must vanish at infinity (supposing the region to extend to infinity), to a degree at least equal to that of the inverse distance.

The proof depends on Sommerfeld's theorem‡ that Green's theorem can be applied without alteration to Riemann's space. Thus, if u_1, u_2

* "Über verzweigte Potentiale im Raum," *Proc. Lond. Math. Soc.*, Vol. XXVIII., p. 395.

† *Exemplar.*

‡ *L.c. ante*, p. 403.

be two values of u satisfying the above conditions,

$$\iiint \left\{ \left(\frac{d(u_1-u_2)}{dx}\right)^2 + \left(\frac{d(u_1-u_2)}{dy}\right)^2 + \left(\frac{d(u_1-u_2)}{dz}\right)^2 \right\} dv$$

$$= \iint (u_1-u_2)\frac{d}{dn}(u_1-u_2)\,dS - \iiint (u_1-u_2)\,\nabla^2(u_1-u_2)\,dv$$

$$= 0,$$

for the surface integral vanishes by the third condition, and the volume integral by the second.

Thus $u_1-u_2 = 0$ at every point, and therefore any solution is unique.

Let us, then, try to find a solution for that region of our Riemann's space which extends from $\phi = 0$ to $\phi = 2\pi$, such as shall satisfy the following special conditions:—

(i.) Its infinities must be the same as those of $-[\Omega']$.

(ii.) It must satisfy $\nabla^2 u = 0$ except at these infinities.

(iii.) It must vanish at infinity and $\frac{du}{dn}$ must vanish over the boundaries $\phi = 0$ and $\phi = 2\pi$.

Then, by what we have already proved, u will at all points within this region be identical with the potential which has been denoted by ϖ.

It is easily seen that a value of u which satisfies the specified conditions may be obtained as follows.

In the section of the Riemann's space in which ϕ varies from -2π to 0 (or, what is the same thing, from 2π to 4π) place a magnetic system which shall be a positive image with respect to the plane $\phi = 0$ of the system whose potential is $-[\Omega']$, and calculate the potential due to the combined systems in the manner appropriate to a Riemann's space (Sommerfeld, *l.c. ante*, p. 405). This will have the required infinities between $\phi = 0$ and $\phi = 2\pi$, it will be a solution of Laplace's equation, and its normal differential coefficients are easily seen to vanish both at $\phi = 0$ and at $\phi = 2\pi$.

This, then, is the solution sought, and it may be immediately identified with ϖ.

In Sommerfeld's paper the potential function is specially calculated for the case of the space of two windings with which we are dealing. If ξ, r, ϕ be the coordinates of the point P at which the

potential is to be estimated, ξ', r', ϕ' those of a single pole of unit strength at Q, then the potential at P due to this pole is shown to be

$$\frac{1}{R}\,\frac{2}{\pi}\tan^{-1}\sqrt{\frac{\sigma+\tau}{\sigma-\tau}},$$

where $$R^2 = PQ^2 = (\xi-\xi')^2+r^2+r'^2-2rr'\cos(\phi-\phi'),$$

$$\tau = \cos\frac{\phi-\phi'}{2},$$

and $$\sigma = \cos\frac{ia}{2},$$

where $$\cos ia = \frac{(\xi-\xi')^2+r^2+r'^2}{2rr'}.$$

In the above expression, the symbol $\tan^{-1}$ is to denote that particular value which lies between 0 and $\pi/2$, the radical being always taken positively. It will be noticed that a and therefore σ are the same for the image of a pole as for the pole itself. Thus, since $\frac{1}{R}$ can be written in the form $\frac{1}{2\sqrt{rr'(\sigma^2-\tau^2)}}$, the combined potential of a pole of strength m and its image is

$$\frac{m}{\pi\sqrt{rr'}}\left\{\frac{1}{\sqrt{\sigma^2-\tau^2}}\tan^{-1}\sqrt{\frac{\sigma+\tau}{\sigma-\tau}}+\frac{1}{\sqrt{\sigma^2-\tau'^2}}\tan^{-1}\sqrt{\frac{\sigma+\tau'}{\sigma-\tau'}}\right\},$$

where $$\tau' = \cos\frac{\phi+\phi'}{2},$$

and the complete value of ϖ is obtained from this expression by a summation extending to all the poles of the original system.

The potential $-[\Omega']$ is, however, given by

$$\begin{aligned}-[\Omega'] &= \Sigma\frac{m}{R}\\ &= \Sigma\frac{m}{2\sqrt{rr'}}\,\frac{1}{\sqrt{\sigma^2-\tau^2}}\\ &= \Sigma\frac{m}{\pi\sqrt{rr'}}\,\frac{1}{\sqrt{\sigma^2-\tau^2}}\left\{\tan^{-1}\sqrt{\frac{\sigma+\tau}{\sigma-\tau}}+\tan^{-1}\sqrt{\frac{\sigma-\tau}{\sigma+\tau}}\right\}. \quad (7)\end{aligned}$$

Incidentally we have shown that the potential in ordinary space due to a pole at ξ, r, ϕ is equal to the value of the Riemann's potential in either region, due to a pole at ξ, r, ϕ and an equal pole at ξ, r, $\phi+2\pi$.

Subtracting the above value for $-[\Omega']$ from that already found for ϖ,

$$-[\Omega] = \Sigma \frac{m}{\pi\sqrt{rr'}} \left\{ \frac{1}{\sqrt{\sigma^2-\tau'^2}} \tan^{-1}\sqrt{\frac{\sigma+\tau'}{\sigma-\tau'}} - \frac{1}{\sqrt{\sigma^2-\tau^2}} \tan^{-1}\sqrt{\frac{\sigma-\tau}{\sigma+\tau}} \right\}. \tag{8}$$

At the positive surface of the current-sheet $\phi = 0$; and therefore

$$\tau = \tau' = \cos \phi/2.$$

If Φ be the current function due to the currents arising from the instantaneous change under consideration, $2\pi\Phi$ is equal to the value of $[\Omega]$ at the positive surface of the current-sheet, and this

$$= \Sigma \frac{m}{\pi\sqrt{rr'}\sqrt{\sigma^2-\tau^2}} \left\{ \tan^{-1}\sqrt{\frac{\sigma-\tau}{\sigma+\tau}} - \tan^{-1}\sqrt{\frac{\sigma+\tau}{\sigma-\tau}} \right\};$$

therefore
$$\Phi = \Sigma \frac{m}{2\pi R} \left\{ 1 - \frac{4}{\pi} \tan^{-1}\sqrt{\frac{\sigma+\tau}{\sigma-\tau}} \right\}. \tag{9}$$

This expression gives Φ at every point of the current-sheet. The boundary condition that Φ shall vanish at the edge of the sheet has been implicitly involved throughout in the continuity of $[\Omega]$ at the edge of the sheet. It is, however, easy to verify that the above equation does give a zero value of Φ at the edge.

For at a point on the edge, $r = 0$, $\sigma = \infty$; and therefore

$$\tan^{-1}\sqrt{\frac{\sigma+\tau}{\sigma-\tau}},$$

in accordance with the restrictions already laid upon the meaning of the inverse symbol, must be taken equal to $\pi/4$, and this gives the expected result.

Before proceeding further, it will be convenient to express the results arrived at in Cartesian coordinates, and in a somewhat symbolical form.

Let x, y, z be the coordinates of any point in ordinary space, the axes of reference being so chosen that the current-sheet is that part of the plane of xy for which x is positive. This is identical with the first region of the Riemann's space in which ϕ varies from 0 to 2π. When the point is supposed to lie in the second region, for which $2\pi < \phi < 4\pi$, its coordinates will be denoted by x', y', z'. Here $x = x'$, $y = y'$, $z = z'$, if the sign $(=)$ is regarded as simply asserting the algebraical equality of the magnitudes which it separates.

Any system of poles in ordinary space can be denoted by $f(x, y, z)$, and any system in the second region of the Riemann's space by $f(x', y', z')$. In dealing with our Riemann's space of two windings the positive image of $f(x, y, z)$ in the current-sheet ($\phi = 0$) will be $f(x', y', -z')$.

The Newtonian potential of the system $f(x, y, z)$ will be denoted by $\Pi\{f(x, y, z)\}$ and the Riemann's potential by $P\{f(x, y, z)\}$. In future the word potential will refer to Newtonian potentials in ordinary space, unless the contrary is stated.

As has already been noticed (p. 158), the following equation holds for any system, viz.,

$$\Pi\{f(x, y, z)\} = P\{f(x, y, z)\} + P\{f(x', y', z')\}.$$

Identifying $f(x, y, z)$ with the magnetic system whose potential is $-[\Omega']$, we have

$$\text{(i.)} \qquad -[\Omega'] = \Pi\{f(x, y, z)\}$$

$$= P\{f(x, y, z)\} + P\{f(x', y', z')\},$$

$$\text{(ii.)} \quad -[\Omega+\Omega'] = P\{f(x, y, z)\} + P\{f(x', y', -z')\},$$

$$\text{(iii.)} \qquad -[\Omega] = P\{f(x', y', -z')\} - P\{f(x', y', z')\}, \tag{8a}$$

$$\text{(iv.)} \qquad \Phi = \frac{1}{2\pi}\left[P\{f(x', y', z')\} - P\{f(x', y', -z')\}\right], \tag{9a}$$

the last equation having reference only to points for which $\phi = 0$.

5. We have thus arrived at a complete knowledge of the currents induced by a sudden change in the external field; let us now examine how these currents become dissipated under the influence of their own induction and the resistance of the sheet, no external field being supposed to act.

It is impossible to follow Maxwell's method any further: his discussion of this case depends on the fact that

$$\frac{d}{dz}\Pi\{f(x_0, y_0, z_0)\} = \Pi\left\{\frac{d}{dz_0}f(x_0, y_0, z_0)\right\},$$

an equation which is not true if P be substituted for Π.

$\frac{d}{dz}f(x, y, z)$ will, of course, be the symbolical expression for the system obtained by changing each pole of $f(x, y, z)$ into a doublet of the same strength whose axis is parallel to the positive axis of z.

To avoid the confusion which might arise as to the meaning of $\frac{d}{dz}f(x, y, -z)$, let the system $\frac{d}{dz}f(x, y, z)$ be denoted by $f'(x, y, z)$. Then, if V be the potential of $f(x, y, z)$, $\frac{dV}{dz}$ is the potential of $f'(x, y, z)$, and so on. The *positive* image of $f'(x, y, z)$ with respect to the plane $z = 0$ will be, in ordinary space, $-f'(x, y, -z)$.

In the case now under discussion the electrical equations ($6a$, $6b$) take the form

$$\frac{d}{dt}\left[\frac{d\Omega}{dz}\right] = R\left[\frac{d^2\Omega}{dz^2}\right] \tag{10a}$$

at the positive surface,

$$\frac{d}{dt}\left[\frac{d\Omega}{dz}\right] = -R\left[\frac{d^2\Omega}{dz^2}\right] \tag{10b}$$

at the negative surface.

In the last section we evaluated ϖ or $-[\Omega+\Omega']$ at the moment immediately following a sudden change in the inducing field. We can, in a similar manner, find $\frac{d\varpi}{dz}$ or $-\left[\frac{d\Omega}{dz}+\frac{d\Omega'}{dz}\right]$, since the electrical equations express that this potential vanishes at the boundary.

The result* is easily seen to be

$$-\left[\frac{d\Omega}{dz}+\frac{d\Omega'}{dz}\right] = P\{f'(x, y, z)\}+P\{f'(x', y', -z')\},$$

and therefore, since

$$-\left[\frac{d\Omega'}{dz}\right] = P\{f'(x, y, z)\}+P\{f'(x', y', z')\},$$

we have, by subtraction,

$$\left[\frac{d\Omega}{dz}\right] = P\{f'(x', y', z')\}-P\{f'(x', y', -z')\}. \tag{11}$$

The solution of equations ($10a$) and ($10b$) in terms of initial conditions is unique. For, knowing $\left[\frac{d\Omega}{dz}\right]$ at any instant, we are

* This problem is discussed by Sommerfeld (*l.c. ante*, p. 414). His result [equation (3), p. 417] is stated inaccurately: the corrected form is given in the *Proc. Lond. Math. Soc.*, Vol. XXX., p. 163.

explicitly given the rate of increase of $\left[\frac{d\Omega}{dz}\right]$ all over the boundary, and therefore, since this rate of increase is itself a potential, at every point of space. Hence $\left[\frac{d\Omega}{dz}\right]$ is known at the next instant, and so on.

I shall now show that, corresponding to the initial condition expressed by equation (11), a solution which satisfies (10*a*) and (10*b*) is

$$\left[\frac{d\Omega}{dz}\right] = P\{f'(x', y', z'+Rt)\} - P\{f'(x', y', -z'-Rt)\}, \quad (12)$$

the equation being supposed to hold for all space external to the current-sheet.

For at the positive surface

$$\begin{aligned}
\frac{d}{dt}\left[\frac{d\Omega}{dz}\right] &= P\left\{\frac{d}{dt}f'(x', y', z'+Rt)\right\} - P\left\{\frac{d}{dt}f'(x', y', -z'-Rt)\right\} \\
&= R\left[P\{f''(x', y', z'+Rt)\} + P\{f''(x', y', -z'-Rt)\}\right] \\
&= R\left[P\{f''(x', y', z'+Rt)\} + P\{f''(x, y, z+Rt)\}\right] \\
&= R\Pi\{f''(x, y, z+Rt)\} \\
&= R\frac{d}{dz}\{\Pi f'(x, y, z+Rt)\} \\
&= R\frac{d}{dz}\left[P\{f'(x, y, z+Rt)\} + P\{f'(x', y', z'+Rt)\}\right] \\
&= R\frac{d}{dz}\left[-P\{f'(x', y', -z'-Rt)\} + P\{f'(x', y', z'+Rt)\}\right] \\
&= R\frac{d}{dz}\left[\frac{d\Omega}{dz}\right] \\
&= R\left[\frac{d^2\Omega}{dz^2}\right].
\end{aligned}$$

Since the solution contained in equation (12) makes $\left[\frac{d\Omega}{dz}\right]$ symmetrical about the surface of the sheet, it follows at once that at the negative surface

$$\frac{d}{dt}\left[\frac{d\Omega}{dz}\right] = -R\left[\frac{d^2\Omega}{dz^2}\right].$$

A verification exactly similar to that following equation (9) will show that this solution makes Φ vanish at the boundary for all time. For we have already verified that (12) satisfies the boundary

conditions when $t = 0$, and the verification cannot depend on t, since z' is entirely arbitrary.

Thus equation (12) contains the solution, for all space and time, of the potential due to the currents induced by a single sudden change in the external field. We pass at once to the case of a continuously changing field by a method exactly analogous to Maxwell's "Trail of Images."* Clearly, if the pair of images (in the second sheet of the Riemann's space) of which the Riemann's potential represents $\left[\frac{d\Omega}{dz}\right]$ [equation (11)] be formed at every instant, and made to pass away each with velocity R, the one whose Cartesian coordinates are equal to those of the original system parallel to the positive axis of z, and the other parallel to the negative axis of z, then the Riemann's potential due to the two continuous trails of images so formed will at any time and at any point in real space be equal to $\frac{d\Omega}{dz}$, where Ω is the magnetic potential due to the currents in the sheet.

The same theory can be put in a more symbolical form in the following manner :—

Let t, τ both be used to express the time: they are to be independent variables as regards differentiation. The magnetic system at any instant will be a function (the word being used in the same extended sense as the symbol f) of the time as well as the coordinates, and may properly be denoted by $f(x, y, z, \tau)$.

The system previously denoted by $f(x, y, z)$ will now be

$$\frac{d}{d\tau} f(x, y, z, \tau)\, \delta\tau,$$

and after a time t its image will be

$$\frac{d}{d\tau} f(x, y, z+Rt, \tau)\, d\tau.$$

Equation (12) therefore becomes

$$\frac{d}{d\tau}\left(\frac{d\Omega}{dz}\right) \delta\tau = \left[P\left\{\frac{d}{d\tau} f(x', y', z'+Rt, \tau)\right\}\right.$$
$$\left. -P\left\{\frac{d}{d\tau} \overline{f'}(x', y', -z'-Rt, \tau)\right\}\right] d\tau, \qquad (13)$$

* "On the Induction of Electric Currents, &c.," *Proc. Roy. Soc.*, Vol. xx., p. 160, §§ 2–10.

where, if we are evaluating $\frac{d\Omega}{dz}$ at the origin of time, t must be put equal to $-\tau$ after differentiation.

Dividing by $\delta\tau$ and integrating with respect to τ from $-\infty$ to 0,

$$\frac{d\Omega}{dz}=\int_{-\infty}^{0}\left[P\left\{\frac{d}{d\tau}f\,(x', y', z'+Rt, \tau)\right\}\right.$$
$$\left.-P\left\{\frac{d}{d\tau}f'\,(x', y', -z'-Rt, \tau)\right\}\right]\delta\tau. \tag{13}$$

Integrating with respect to z,

$$\Phi=-\frac{1}{2\pi}\int_{0}^{\infty}dz\int_{-\infty}^{0}d\tau\left[P\left\{\frac{d}{d\tau}f'\,(x', y', z'+Rt, \tau)\right\}\right.$$
$$\left.-P\left\{\frac{d}{d\tau}f'\,(x', y', -z'-Rt, \tau)\right\}\right]. \tag{14}$$

The first step towards the evaluation of Φ will be the evaluation of $P\{f'(x, y, z)\}$, where the system $f(x, y, z)$ consists of a single pole, for the right-hand member of equation (14) can be found from this by summation and integration. In other words, we require the Riemann's potential due to a doublet.

That due to a single pole has been seen to be

$$\frac{1}{\pi\sqrt{rr'}\sqrt{\sigma^2-\tau^2}}\tan^{-1}\sqrt{\frac{\sigma+\tau}{\sigma-\tau}}.$$

Denoting this by p, and the corresponding potential for a doublet by p',

$$p'=\frac{dp}{d\xi}\quad(\text{where } \xi=r'\sin\phi')$$

$$=\frac{dp}{r'd\phi'}\cos\phi'+\frac{dp}{dr'}\sin\phi'$$

$$=\frac{\partial p}{\partial\tau}\,\frac{\cos\phi'\sin\dfrac{\phi-\phi'}{2}}{8r'\tau}-p\,\frac{\sin\phi'}{2r'}$$

$$-\frac{\partial p}{\partial\sigma}\sin\phi'\,\frac{r^2-r'^2+(\xi-\xi')^2}{8\sigma rr'^2}.$$

[redacted]

After a certain amount of reduction this takes the form

$$\frac{1}{\pi R r'} \sin \phi' \tan^{-1} \sqrt{\frac{\sigma+\tau}{\sigma-\tau}}$$

$$- \frac{1}{2\pi R^2} \sqrt{\frac{r}{r'}} \left\{ \tfrac{1}{2} \cos\phi' \tan \frac{\phi-\phi'}{2} + \sin\phi' \cos \frac{\phi-\phi'}{2} \left(\frac{r^2-r'^2+(\xi-\xi')^2}{(r+r')^2+(\xi-\xi')^2} \right) \right\}$$

$$+ \frac{1}{\pi R^3} \tan^{-1} \sqrt{\frac{\sigma+\tau}{\sigma-\tau}} \left\{ r \cos\phi' \sin \frac{\phi-\phi'}{2} + \sin\phi' \left(\frac{r^2-r'^2+(\xi-\xi')^2}{r'} \right) \right\},$$

where R is now the distance between the points ξ, r, ϕ and ξ', r', ϕ'.

Combining this with the corresponding expression for the potential due to the image of this doublet, we get for the combined potential

$$\frac{\sin\phi'}{\pi r'} \left\{ \frac{1}{R} \tan^{-1} \sqrt{\frac{\sigma+\tau}{\sigma-\tau}} + \frac{1}{R'} \tan^{-1} \sqrt{\frac{\sigma+\tau'}{\sigma-\tau'}} \right\}$$

$$- \frac{\cos\phi'}{4\pi} \sqrt{\frac{r}{r'}} \left\{ \frac{1}{R^2} \tan \frac{\phi-\phi'}{2} - \frac{1}{R'^2} \tan \frac{\phi+\phi'}{2} \right\}$$

$$- \frac{\sin\phi'}{2\pi} \sqrt{\frac{r}{r'}} \left(\frac{r^2-r'^2+(\xi-\xi')^2}{(r+r')^2+(\xi-\xi')^2} \right) \left\{ \frac{1}{R^2} \cos \frac{\phi-\phi'}{2} + \frac{1}{R'^2} \cos \frac{\phi+\phi'}{2} \right\}$$

$$+ \frac{r\cos\phi'}{\pi} \left\{ \frac{1}{R^3} \sin \frac{\phi-\phi'}{2} \tan^{-1} \sqrt{\frac{\sigma+\tau}{\sigma-\tau}} - \frac{1}{R'^3} \sin \frac{\phi+\phi'}{2} \tan^{-1} \sqrt{\frac{\sigma+\tau'}{\sigma-\tau'}} \right\}$$

$$+ \frac{\sin\phi'}{\pi r'} \left[r^2 - r'^2 + (\xi-\xi')^2 \right] \left\{ \frac{1}{R^3} \tan^{-1} \sqrt{\frac{\sigma+\tau}{\sigma-\tau}} + \frac{1}{R'^3} \tan^{-1} \sqrt{\frac{\sigma+\tau'}{\sigma-\tau'}} \right\}.$$

The currents arising from the creation of a single pole at x, y, z can now be obtained in the following manner. Let the point ξ', r', ϕ' in the above expression be made identical with the point x, y, $z+vt$ (v is the velocity of the images, the R of § 4). Then the above expression is the value of $\frac{d\Omega}{dz}$ at the point ξ, r, ϕ. An integration with respect to z between the limits 0 and ∞ will give the value of $-2\pi\Phi$ for any values of x, y, and t.

Lastly, the above theory can be stated in a form which shall contain no reference to Riemann's space in the following manner.

Any continuous change of a magnetic system, whether arising from changes in the strength or position of the magnets, or both, can be represented by the continued creation or annihilation of poles, or, what is the same thing, by the creation of positive or negative poles.

Let each pole, immediately it is created, move parallel to the axis of z with constant velocity v. Then at any time we have a number of continuous series of poles. Substitute the cylindrical coordinates of any such pole in the expression found above, multiply by the strength of the pole, and sum the expression so found for every pole of the system.

The result gives the value of $\frac{d\Omega}{dz}$, and, as before, the current function at a point in the sheet is found by an integration with respect to z.

6. Maxwell's theory for the complete sheet may be deduced from equation (13) by supposing the edge of the semi-infinite sheet removed to infinity.

We are now dealing only with pairs of points for which

$$r = r' = \infty,$$

$$\phi - \phi' = 0 \quad \text{or} \quad 2\pi;$$

therefore $$i\alpha = 0, \quad \sigma = 1,$$

and $$\tau = 1 \quad \text{or} \quad -1,$$

according to whether the continuous path, in the Riemann's space, which joins the two points is finite or infinite.

In the former case

$$P\{f(x, y, z)\} = \Sigma \frac{m}{R} = \Pi \{f(x, y, z)\}.$$

In the latter case $$P\{f(x, y, z)\} = 0.$$

Thus, due to any system on the negative side of the sheet, we have at any point on the positive side, by equation (13),

$$\frac{d\Omega}{dz} = \int_{-\infty}^{0} \Pi \left\{ \frac{d}{d\tau} f'(x, y, z + Rt, \tau) \right\} d\tau.$$

Integrating twice with respect to z,

$$\int \Omega \, dz = \int_{-\infty}^{0} \Pi \left\{ \frac{d}{d\tau} \int f(x, y, z + Rt, \tau) \, dz \right\} d\tau,$$

and this is the same as the equation which Maxwell writes in the form

$$P = -\int_{0}^{\infty} \frac{dP'_{\tau}}{dt} d\tau$$

[*Elect. and Mag.*, § 663, equation (27)].

7. After the semi-infinite current-sheet, the case which comes next in order of simplicity is that in which the boundary consists of two parallel and infinite straight lines, giving

(i.) An infinite current-sheet from which an infinite strip has been removed.

(ii.) A conducting strip bounded by two parallel straight lines.

A complete investigation of the potential function for case (i.) is given by Sommerfeld (§ 5, p. 419), and there is no difficulty in extending it to case (ii.). I shall accordingly only quote results, slightly altering Sommerfeld's notation.

We again take cylindrical coordinates ξ, ρ, θ, where ρ, θ are connected with r, ϕ by the transformations

$$\text{(i.)}\quad \rho+i\theta = \log\left(\frac{re^{i\phi}-a}{re^{i\phi}+a}\right),$$

$$\text{(ii.)}\quad \rho+i\theta = \log\left(\frac{re^{i\phi}+a}{re^{i\phi}-a}\right),$$

the former referring to case (i.), the latter to case (ii.). The variable a is now to be given by

$$\text{(i.)}\quad \cos ia = \cos i\,(\rho-\rho')+\tfrac{1}{2}\,(\cos i\rho-\cos\theta)(\cos i\rho'-\cos\theta')(\xi-\xi')^2,$$

$$\text{(ii.)}\quad \cos ia = \cos i\,(\rho-\rho')+\tfrac{1}{2}\,(\cos i\rho+\cos\theta)(\cos i\rho'+\cos\theta')(\xi-\xi')^2.$$

σ, τ have now the same expression in terms of a, ρ, θ as they formerly had in terms of a, r, ϕ, and Sommerfeld's result [given in equation (10), p. 424, for case (i.), and easily seen to be true also for case (ii.) when the quantities concerned have the meanings assigned to them above], states that in either case the potential function is

$$\frac{1}{R}\,\frac{2}{\pi}\tan^{-1}\sqrt{\frac{\sigma+\tau}{\sigma-\tau}}.$$

On account of the identity of form between this potential function and the one already dealt with, all the results of § 4 can be immediately applied to this case.

Corresponding to equation (7), we obviously have

$$-[\Omega'] = \Sigma\frac{m}{R}\,\frac{2}{\pi}\left\{\tan^{-1}\sqrt{\frac{\sigma+\tau}{\sigma-\tau}}+\tan^{-1}\sqrt{\frac{\sigma-\tau}{\sigma+\tau}}\right\},$$

and equation (8) will apply to this case if put in the form

$$-[\Omega] = \Sigma\frac{2m}{\pi}\left\{\frac{1}{R'}\tan^{-1}\sqrt{\frac{\sigma+\tau}{\sigma-\tau}}-\frac{1}{R}\tan^{-1}\sqrt{\frac{\sigma-\tau}{\sigma+\tau}}\right\}. \qquad (15)$$

Equation (9) is true without change,

$$\Phi = \Sigma \frac{m}{2\pi R}\left\{1 - \frac{4}{\pi}\tan^{-1}\sqrt{\frac{\sigma+\tau}{\sigma-\tau}}\right\}. \tag{16}$$

It will be noticed that at the two edges of the sheet $\rho = \pm\infty$; therefore $\sigma = \infty$. The same verification as before now shows that Φ vanishes at both edges.

It will also be noticed that all the symbolical work of § 5 holds good without alteration for the cases now under discussion, and consequently the value of $\left[\frac{d\Omega}{dz}\right]$ found in equation (12) holds if we assign the altered meaning to the symbol P. The alteration is necessitated by the fact that the equations of transformation from R, σ, τ to x, y, z are no longer the same. Hence equations (13) and (14) contain the symbolical solution for the cases now under consideration.

The calculation of p' which follows equation (14) does not, of course, apply to this case. The calculation could, however, be effected in an exactly similar way, either for the potential function now under discussion or for any other potential function.

8. It will now be seen that the method adopted in these two cases is, in its main principles, quite general.

The first step towards the discussion of any given shape of current-sheet consists of the construction of a Riemann's space of two regions,* one of which is to be identical with the ordinary space containing the inducing system, and the other of which will contain certain images of the system.

The current-sheet itself must be the branch membrane† of the space, and the boundary of the current-sheet its branch line.‡

If we can find three systems of mutually orthogonal surfaces, each filling all space, such that all the surfaces of one system are bounded by the edge of the current-sheet, and one of them coincides with the current-sheet, then the calculation of the potential function for this space will consist of a series of known processes in pure mathematics. The mode of procedure is given in § 6 of Sommerfeld's paper.

Independently, however, of the form which the potential function assumes for any particular space, and independently even of the

* *Exemplar.* † *Verzweigungsmembrane.* ‡ *Verzweigungslinie.*

possibility of calculating it, the symbolical work of §§ 4 and 5 will hold good for the most general case.

For the symbolical equation

$$\Pi\left\{f(x, y, z)\right\} = P\left\{f(x, y, z)\right\} + P\left\{f(x', y', z')\right\}$$

is true for any Riemann's space of two regions.

If a formal proof of this statement is required, it is sufficient to remark that throughout real space each member of the equation satisfies the four following conditions :—

(i.) It is a solution of Laplace's equation.

(ii.) It becomes infinite at the poles of $f(x, y, z)$ to the same order and with the same residues as $\Pi\left\{f(x, y, z)\right\}$.

(iii.) It vanishes at infinity.

(iv.) It is continuous everywhere, including the branch-membrane and the branch-line of the Riemann's space.

Those conditions are sufficient to uniquely determine a function of position in real space, and therefore the validity of the above equation is established.

It only remains to remark that the only quantity entering the original equations which is not entirely symbolical is z, that differentiation with respect to z remains the same in our new system of coordinates, and, lastly, that nowhere in the course of the symbolical work of §§ 4 and 5 has any account been taken of the form of the special functions which the symbolical notation is there supposed to suggest.

This establishes the generality of the method, and therefore we conclude, since $\Pi\left\{f(x, y, z)\right\}$ has the same meaning in connexion with every problem, that, if the right interpretation be assigned to the symbol P, equations (13) and (14) will contain the solution of our problem for a plane current-sheet bounded by any given curve in the plane of xy.

The Reduction of a Linear Substitution to its Canonical Form. *By* A. C. Dixon. Received June 5th, 1899. Read June 8th, 1899.

The following is another solution of the problem discussed by Prof. Burnside (*Proc. Lond. Math. Soc.*, Vol. xxx., pp. 180–194), namely, the reduction of a linear substitution to its canonical form.

Take the substitution S as at p. 183, namely,

$$x'_s = \Sigma a_{st} x_t \quad (s, t = 1, 2, \ldots, n).$$

Let $\theta^{(1)}$ be a root of the characteristic equation

$$\Delta = \begin{vmatrix} a_{11}-\theta, & a_{12}, & a_{13}, & \ldots, & a_{1n} \\ a_{21}, & a_{22}-\theta, & & \ldots, & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1}, & a_{n2}, & & \ldots, & a_{nn}-\theta \end{vmatrix} = 0.$$

Then at least one set of quantities

$$x_s^{(1)} \qquad (s = 1, 2, \ldots, n)$$

can be found such that

$$\sum_t a_{st} x_t^{(1)} = \theta^{(1)} x_s^{(1)} \quad (s = 1, 2, \ldots, n),$$

and at least one set $y_s^{(1)}$ such that

$$\sum_t a_{ts} y_t^{(1)} = \theta^{(1)} y_s^{(1)} \quad (s = 1, 2, \ldots, n).$$

Take one such set in each of the two cases, and write

$$x_1 = X_1 x_1^{(1)}, \quad x_s = X_1 x_s^{(1)} + X_s \quad (s = 2, \ldots, n).^*$$

The equations of the substitution S then become

$$X'_1 x_1^{(1)} = X_1 \Sigma a_{1t} x_t^{(1)} + \sum_{t=2}^{t=n} a_{1t} X_t$$

$$= \theta^{(1)} x_1^{(1)} X_1 + \sum_2^n a_{1t} X_t,$$

* We may suppose $x_1^{(1)} \neq 0$; if this is not so, change the order of the variables x.

$$X_1' x_s^{(1)} + X_s' = X_1 \Sigma a_{st} x_t^{(1)} + \sum_2^n a_{st} X_t$$

$$= \theta^{(1)} x_s^{(1)} X_1 + \sum_2^n a_{st} X_t \quad (s = 2, 3, \dots, n);$$

whence
$$X_s' = \sum_2^n \left(a_{st} - \frac{x_s^{(1)}}{x_1^{(1)}} a_{1t} \right) X_t.$$

It also follows that

$$X_1' \Sigma x_s^{(1)} y_s^{(1)} + \sum_2^n X_s' y_s^{(1)} = \theta^{(1)} X_1 \Sigma x_s^{(1)} y_s^{(1)} + \sum_{t=2}^{t=n} \{ X_t \sum_s a_{st} y_s^{(1)} \}$$

$$= \theta^{(1)} \left[X_1 \Sigma x_s^{(1)} y_s^{(1)} + \sum_2^n X_t y_t^{(1)} \right].$$

There are now two cases according as $\Sigma x_s^{(1)} y_s^{(1)} = 0$ or not. Suppose, first, $\Sigma x_s^{(1)} y_s^{(1)} \neq 0$, and write ξ_1 for $X_1 \Sigma x_s^{(1)} y_s^{(1)} + \sum_2^n X_t y_t^{(1)}$, that is, $\Sigma y_s^{(1)} x_s$, ξ_s for X_s $(s = 2, 3, \dots, n)$. Thus

$$\xi_1' = \theta^{(1)} \xi_1,$$

$$\xi_s' = \sum_2^n \beta_{st} \xi_t \quad (s = 2, \dots, n),$$

where
$$\beta_{st} = a_{st} - \frac{x_s^{(1)}}{x_1^{(1)}} a_{1t},$$

that is, the substitution is broken up into two, one of which affects only ξ_1, and the other only the other $n-1$ variables. The same process of reduction may then be applied to this last substitution and repeated until the canonical form is at last reached, unless at some stage it happens that

$$\Sigma x_s^{(1)} y_s^{(1)} = 0.$$

If
$$\Sigma x_s^{(1)} y_s^{(1)} = 0,$$

the functions $\xi_1, \xi_2, \dots, \xi_n$ as just given are not linearly independent.

We may, however, put*

$$\xi_n = \Sigma y_s^{(1)} x_s = \sum_2^n X_t y_t^{(1)},$$

$$\xi_s = X_s \quad (s = 1, 2, \dots, n-1).$$

---

* It is assumed that $y_n^{(1)}$ does not vanish.

Thus
$$\xi_n' = \theta^{(1)}\xi_n,$$
$$\xi_1' = \theta^{(1)}\xi_1 + \sum_2^n \frac{a_{1t}}{x_1^{(1)}}\xi_t,$$
$$\xi_s' = \sum_2^n \gamma_{st}\xi_t \quad (s = 2, 3, \ldots, n-1).$$

The substitution has thus been transformed,* so that in the characteristic determinant Δ the first column and last row contain zeros only, except where they meet the dexter (or principal) diagonal, and the two constituents at the ends of this are each $\theta^{(1)}-\theta$.

If we suppose $\xi_n = 0$, we have a linear substitution affecting $\xi_2, \xi_3, \ldots, \xi_{n-1}$ only for which the characteristic determinant is formed by striking out the first and last rows and columns of the one just arrived at. This substitution in $n-2$ variables can be further reduced by one of the two processes given, and the restoration of ξ_n will only affect the last column of the determinant Δ.

Thus by successive reduction the substitution is brought to a form in which all constituents of the determinant below the dexter diagonal vanish, and possibly some above it.

The next step is to destroy as many as possible of the constituents above this diagonal.

The substitution may now be written
$$x_s' = \sum_s^n a_{st}x_t.$$

If $a_{11} \neq a_{22}$, we may destroy a_{12} by putting
$$X_1 = x_1 + \frac{a_{12}}{a_{11}-a_{22}}x_2,\ X\ \ = x_s\ (s = 2, \ldots, n).$$

Thus
$$X_1' = a_{11}x_1 + \sum_2^n \left(a_{1t} + \frac{a_{12}a_{2t}}{a_{11}-a_{22}}\right)x_t$$
$$= a_{11}X_1 + \sum_3^n \left(a_{1t} + \frac{a_{12}a_{2t}}{a_{11}-a_{22}}\right)X_t,$$
$$X_s' = \sum_s^n a_{st}X_t \quad (s = 2, \ldots, n).$$

In this way any constituent just above the diagonal may be destroyed,

---

* By the substitution
$$\xi_1 = x_1/x_1^{(1)},\quad \xi_s = x_s - x_1\frac{x_s^{(1)}}{x_1^{(1)}},\quad \xi_n = \Sigma y_t^{(1)}x_t \quad (s = 2, 3, \ldots, n-1).$$

unless the adjacent constituents in the diagonal are identically equal, that is, we may put

$$a_{s,\,s+1} = 0,$$

unless $$a_{ss} = a_{s+1,\,s+1}.$$

In the same way, going on to the next line parallel to the diagonal we may reduce

$$a_{s,\,s+2} \text{ to } 0,$$

unless $$a_{ss} = a_{s+2,\,s+2},$$

and so on; so that ultimately we destroy

$$a_{st},$$

unless $$a_{ss} = a_{tt}.$$

Then, by changing the order of the variables, we may gather together all the constituents in the diagonal that are identically the same, and so divide the variables into sets

$$\begin{array}{l} x_1,\ x_2,\ \ldots,\ x_p, \\ x_{p+1},\ x_{p+2},\ \ldots,\ x_{p+q}, \\ x_{p+q+1},\ \ldots,\ x_{p+q+r}, \\ \ldots\quad \ldots\quad \ldots\quad \ldots \end{array}$$

in such a way that each set is transformed independently of the others, that is,

$$\begin{aligned} x'_s &= \lambda x_s + \sum_{s+1}^{p} a_{st} x_t \qquad (s = 1, 2, \ldots, p), \\ x'_{p+s} &= \mu x_{p+s} + \sum_{p+s+1}^{p+q} a_{p+s,\,t}\, x_t \quad (s = 1, 2, \ldots, q). \\ &\ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots \end{aligned}$$

Each set may now be considered separately. Take, for instance, $x_1, x_2, \ldots, x_p$. The first step is to bring any zeros there may be in the line just above the diagonal to the lower end of that line, and, in fact, to arrange that the number of zeros immediately following the diagonal constituent in any row shall be at least as great as in the row above, unless all the constituents of both rows vanish except those in the diagonal. It is the same thing to say that, if a_{it} $(t > i)$ is the first constituent in the i^{th} row not to vanish after the i^{th}, then

$$a_{i-1,\,r} \neq 0,$$

for some value of $r < t$, but $> i-1$.

Suppose this not to be the case; then the equations of substitution include

$$x'_{i-1} = \lambda x_{i-1} + \sum_t^p a_{i-1,s} x_s,$$

$$x'_i = \lambda x_i + \sum_t^p a_{is} x_s.$$

Put
$$X_{i-1} = x_i, \quad X_i = x_{i-1} - \frac{a_{i-1,t}}{a_{it}} x_i,$$

$$X_s = x_s \quad (s \neq i-1, i).$$

This transformation will not affect the rows below the i^{th}, and it will reduce a_{is} to zero; by applying it successively, and always to the lowest pair of rows in which the desired condition is not fulfilled, we get the result. The transformation giving this result is not unique, and, in fact, the arrangement will not be interfered with by any transformation, such as

$$X_i = \beta x_i + \gamma x_j \quad (j > i, \ \beta \neq 0),$$

$$X_s = x_s \quad (s \neq i).$$

Suppose then, leaving out the diagonal, that the first constituent not vanishing in the first row is the a^{th}, in the a^{th} row the a'^{th}, in the a'^{th} row the a''^{th}, and so on. The series $1, a, a', a'', \ldots$ will come to an end, since it can contain no number $> p$.

Let b be the first number not included among $1, a, a', a'', \ldots$, and let the first constituent not vanishing in the b^{th} row be the b'^{th}, in the b'^{th} row the b''^{th}, and so on.

Let c be the first number not included among $1, a, a', \ldots, b, b', b'', \ldots$, and form with it another series $c, c', c'', \ldots$, and so on until the numbers $1, 2, \ldots, p$ are all exhausted.

Then the transformation

$$\begin{array}{ll} X_1 = x_1, & X_b = x_b, \\ X_a = x'_1 - \lambda x_1, & X_{b'} = x'_b - \lambda x_b, \\ X_{a'} = X'_a - \lambda X_a, & X_{b''} = X'_{b'} - \lambda X_{b'}, \\ X_{a''} = X'_{a'} - \lambda X_{a'}, & \\ \quad \&c., & \end{array}$$

reduces the substitution for $x_1, x_2, \ldots, x_p$ to its canonical form, except for the order of suffixes.

As an example take the substitution S on p. 191,

$$
\begin{aligned}
x_1' &= -2x_1 - x_2 - x_3 + 3x_4 + 2x_5, \\
x_2' &= -4x_1 + x_2 - x_3 + 3x_4 + 2x_5, \\
x_3' &= x_1 + x_2 - 3x_4 - 2x_5, \\
x_4' &= -4x_1 - 2x_2 - x_3 + 5x_4 + x_5, \\
x_5' &= 4x_1 + x_2 + x_3 - 3x_4.
\end{aligned}
$$

The characteristic equation is

$$(\theta+1)^2 (\theta-2)^3 = 0.$$

Take $\theta^{(1)} = 2$; then the five equations to be satisfied by $x_1^{(1)}$, $x_2^{(1)}$, $x_3^{(1)}$, $x_4^{(1)}$, $x_5^{(1)}$ reduce to three only, namely,

$$
\begin{aligned}
-4x_1 - x_2 - x_3 + 3x_4 + 2x_5 &= 0, \\
x_1 + x_2 - 2x_3 - 3x_4 - 2x_5 &= 0, \\
-4x_1 - 2x_2 - x_3 + 3x_4 + x_5 &= 0.
\end{aligned}
$$

We may then put

$$x_1^{(1)} = -x_3^{(1)} = x_4^{(1)} = 1, \quad x_2^{(1)} = x_5^{(1)} = 0.$$

In like manner, the equations for $y_1^{(1)}$ ... reduce to three only,

$$y_3 = 0, \quad y_4 = 0, \quad y_1 + y_2 - y_5 = 0.$$

Now it is desirable that, if possible, $\Sigma x_s^{(1)} y_s^{(1)}$ should not vanish; we therefore put

$$y_1^{(1)} = 1 = y_5^{(1)}, \quad y_2 = y_3 = y_4 = 0.$$

The first transformation is therefore

$$X_1 = x_1 + x_5, \quad X_2 = x_2, \quad X_3 = x_3 + x_1, \quad X_4 = x_4 - x_1, \quad X_5 = x_5.$$

The substitution S as transformed is

$$
\begin{aligned}
x_1' &= 2x_1, \\
x_2' &= x_2 - x_3 + 3x_4 + 2x_5, \\
x_3' &= -x_3, \\
x_4' &= -x_2 + 2x_4 - x_5, \\
x_5' &= x_2 + x_3 - 3x_4.
\end{aligned}
$$

We now ignore x_1 and take the root -1 of the characteristic equation. Then $x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_5^{(1)}$ satisfy the equations

$$\begin{aligned} 2x_2 - x_3 + 3x_4 + 2x_5 &= 0, \\ -x_2 \quad\;\; + 3x_4 - \;\, x_5 &= 0, \\ x_2 + x_3 - 3x_4 + \;\, x_5 &= 0; \end{aligned}$$

so that $$x_2^{(1)} = -x_5^{(1)} = 1, \quad x_3^{(1)} = x_4^{(1)} = 0.$$

In like manner, $$y_3^{(1)} = 1, \quad y_2^{(1)} = y_4^{(1)} = y_5^{(1)} = 0.$$

The new substitution is then

$$X_1 = x_1, \quad X_2 = x_3, \quad X_3 = x_4, \quad X_4 = x_2 + x_5, \quad X_5 = x_2,$$

and S becomes, when transformed by it,

$$\begin{aligned} x_1' &= 2x_1, \\ x_2' &= \quad -x_2 + 3x_3 + 2x_4 - x_5, \\ x_3' &= \qquad\qquad 2x_3 - \;\, x_4, \\ x_4' &= \qquad\qquad\qquad 2x_4 \\ x_5' &= \qquad\qquad\qquad\quad -x_5. \end{aligned}$$

To reduce this finally, put

$$X_1 = x_1, \quad X_5 = x_2 - x_3 - x_4, \quad X_4 = -x_5, \quad X_3 = x_3, \quad X_2 = -x_4,$$

and it becomes

$$x_1' = 2x_1, \quad x_2' = 2x_2, \quad x_3' = 2x_3 + x_2, \quad x_4' = -x_4, \quad x_5' = -x_5 + x_4,$$

which is the canonical form.

The successive transformations used are

$$\begin{aligned} &X_1^{(1)} = x_1 + x_5, && X_2^{(1)} = x_2, && X_3^{(1)} = x_1 + x_3, && X_4^{(1)} = x_4 - x_1, && X_5^{(1)} = x_5, \\ &X_1^{(2)} = X_1^{(1)}, && X_2^{(2)} = X_2^{(1)}, && X_3^{(2)} = X_4^{(1)}, && X_4^{(2)} = X_2^{(1)} + X_5^{(1)}, && X_5^{(2)} = X_3^{(1)}, \\ &X_1 = X_1^{(2)}, && X_2 = -X_4^{(2)}, && X_3 = X_3^{(2)}, && X_4 = -X_5^{(2)}, && X_5 = X_2^{(2)} - X_3^{(2)} - X_4^{(2)} \end{aligned}$$

the resultant transformation being

$$X_1 = x_1 + x_5, \quad X_2 = -x_2 - x_5, \quad X_3 = x_4 - x_1, \quad X_4 = -x_1 - x_3,$$
$$X_5 = x_1 - x_4 - x_5.$$

On the Integration of Systems of Total Differential Equations. *By* A. C. DIXON. Received June 5th, 1899. Read June 8th, 1899.

In general, the least possible number of equations in an integral equivalent of a system of q Pfaffian equations in $m(q+1)$ variables is mq (Forsyth, *Theory of Differential Equations*, Part I., p. 315). The principle on which the proof given by Dr. Forsyth* depends is that, if a certain number of unknown functions are connected by a greater number of equations, in part differential, conditions need to be satisfied which will destroy the generality of the expressions under consideration. This principle needs qualifying on two grounds. In the first place, the way in which the equations are formed may possibly be such that the conditions for their compatibility are necessarily satisfied; secondly, such conditions may arise even when the number of equations is not greater than that of the unknown functions.†

---

* After Biermann.

† [Dr. Forsyth's argument (p. 314) is that, since there are mn equations involving $p+nq$ quantities, therefore

$$mn \leqq p+nq, \quad \ldots$$

for otherwise conditions would need to be satisfied, restricting the generality of the forms Ω.

But, on the other hand, from any one of his equations (B) we have $m+n$ equations, and from each pair of these another can be formed, in the same way as (4) in the present article; thus, since the multipliers are not eliminated, we have in all $n \cdot \frac{1}{2}(m+n)(m+n+1)$ equations and $p+nq+n^2$, that is $p(n+1)$, unknown functions u, U, ρ. It may therefore equally well be argued that

$$\tfrac{1}{2}n(m+n)(m+n+1) \leqq p(n+1).$$

Neither argument is sound, since the deduction from the mere numbers of equations and of unknown functions does not hold universally. For instance, the two equations with one unknown u,

$$\ldots \quad \ldots \quad \cdot \quad \frac{\partial u}{\partial x} = \frac{\partial X}{\partial x}, \quad \frac{\partial u}{\partial y} = \frac{\partial X}{\partial y}, \quad \ldots \quad \ldots \quad \cdot \quad \ldots$$

can be satisfied without any restriction on the given function X; on the other hand, the two equations

$$u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}+w\frac{\partial w}{\partial x} = X, \quad u\frac{\partial u}{\partial y}+v\frac{\partial v}{\partial y}+w\frac{\partial w}{\partial y} = Y$$

with three unknowns cannot be satisfied unless X, Y fulfil a condition

$$\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}.]$$

In this paper I have sought to give a proof of the theorem which shall be free from this difficulty. The proof depends ultimately on finding solutions in series, and it is, in fact, shown how such solutions of the extended Pfaff problem may in theory be found.

Take a system of q total differential equations in $m(q+1)$ variables, say

$$\Omega^{(l)} = 0 \quad (l = 1, 2, \ldots, q), \tag{1}$$

where
$$\Omega^{(l)} = \sum_{k=1}^{k=mq+m} A_k^{(l)}\, dx_k.$$

Suppose $x_1, \ldots, x_m$ to be independent variables, and let y_r stand for x_{m+r}. Let the value $\lambda_l(x_2, x_3, \ldots, x_m)$ be assigned to y_l $(l = 1, 2, \ldots, q)$, when x_1 has the particular value a_1. Then the values of the other dependent variables y_{q+r} $(r = 1, 2, \ldots, mq-q)$ for this value of x_1 must be such as to satisfy the system of equations

$$\bar{\Omega}^{(l)} = 0 \quad (l = 1, 2, \ldots, q),$$

where $\bar{\Omega}^{(l)}$ is what $\Omega^{(l)}$ becomes when x_1 is constant and $= a_1$ and $y_1, y_2, \ldots, y_q$ have the assigned forms in terms of $x_2, \ldots, x_m$. Thus the number of variables involved in $\bar{\Omega}^{(l)}$ is $(m-1)(q+1)$ and $m-1$ of them are supposed independent; the reduced problem is therefore of the same form as the original but with a lower value of m.

Thus, if we seek a solution in which the dependent variables shall be expressed in series of powers of $x_1 - a_1$ with coefficients depending on $x_2, x_3, \ldots, x_m$, and if we assign the forms of the absolute terms in q of these expansions, the absolute terms in the rest are to be found by solving a system of equations of the same form as at first, but in fewer variables.

The same applies to the reduced system, and we are thus led to the following system of initial conditions to be imposed on a solution of the original system (1):—

$$\left.\begin{array}{lll}
\text{when } x_1 = a_1, & y_l & = \lambda_l^{(1)}(x_2, x_3, \ldots, x_m),\ (l = 1, 2, \ldots, q) \\
\text{when } x_1 = a_1,\quad x_2 = a_2, & y_{q+l} & = \lambda_{q+l}^{(2)}(x_3, \ldots, x_m) \\
\ldots \quad \ldots \quad \ldots \quad \ldots & \ldots & \ldots \quad \ldots \quad \ldots \quad \ldots \\
\text{when } x_1 = a_1, \ldots, x_i = a_i, & y_{(i-1)q+l} & = \lambda_{(i-1)q+l}^{(i)}(x_{i+1}, x_{i+2}, \ldots, x_m) \\
\text{when } x_1 = a_1, \ldots, x_m = a_m, & y_{(m-1)q+l} & = \lambda_{(m-1)q+l}^{(m)}, \text{ a constant}
\end{array}\right\}. \tag{2}$$

Now the system (1) is equivalent to the system of partial differential equations

$$\omega_i^{(l)} = 0 \quad (l = 1, 2, \ldots, q;\ i = 1, 2, \ldots, m), \tag{3}$$

where $$\omega_i^{(l)} = \sum_{k=1}^{k=mq+m} A_k^{(l)} \frac{dx_k}{dx_i}{}^{*} = A_i^{(l)} + \sum_{r=1}^{r=mq} A_{m+r}^{(l)} \frac{dy_r}{dx_i}.$$

From these may be deduced

$$\omega_{ij}^{(l)} = 0 \quad (l = 1, 2, \ldots, q;\ i, j = 1, 2, \ldots, m), \quad (4)$$

where

$$\begin{aligned}\omega_{ij}^{(l)} &= \frac{d\omega_i^{(l)}}{dx_j} - \frac{d\omega_j^{(l)}}{dx_i} \\ &= \sum_{k=1}^{k=mq+m} \sum_{n=1}^{n=mq+m} \left(\frac{\partial A_k^{(l)}}{\partial x_n} - \frac{\partial A_n^{(l)}}{\partial x_k}\right) \frac{dx_k}{dx_i} \frac{dx_n}{dx_j} \\ &= a_{ij}^{(l)} + \sum_{r=1}^{r=mq} a_{m+r,j}^{(l)} \frac{dy_r}{dx_i} + \sum_{s=1}^{s=mq} a_{i,m+s}^{(l)} \frac{dy_s}{dx_j} + \sum_{r=1}^{r=mq} \sum_{s=1}^{s=mq} a_{m+r,m+s}^{(l)} \frac{dy_r}{dx_i} \frac{dy_s}{dx_j},\end{aligned}$$

$$a_{kn}^{(l)} = \frac{\partial A_k^{(l)}}{\partial x_n} - \frac{\partial A_n^{(l)}}{\partial x_k}.$$

Now, let $\{i\}$ denote the series of equations

$$\omega_i^{(l)} = 0, \quad \omega_{ij}^{(l)} = 0$$
$$(j = i+1, i+2, \ldots, m;\ l = 1, 2, \ldots, q);$$

these contain no derivatives with respect to $x_1, \ldots, x_{i-1}$.

Take the equations $\{m\}$ which only contain derivatives with respect to x_m; in them put $x_1 = a_1, \ldots, x_{m-1} = a_{m-1}$, and assign to $y_1, \ldots, y_{mq-q}$ the forms given by (2); thus we have a system of q ordinary differential equations connecting $y_{mq-q+1}, \ldots, y_{mq}$ with x_m. These are to be integrated with the initial conditions that, when

$$x_m = a_m,$$
$$y_{mq-q+l} = \lambda_{mq-q+l}^{(m)}.$$

Suppose the integrals to be

$$y_{mq-q+l} = \lambda_{mq-q+l}^{(m-1)}(x_m) \quad (l = 1, 2, \ldots, q).$$

Then take the equations $\{m-1\}$ which are $2q$ in number and contain derivatives with respect to x_{m-1} and x_m. In them put

$$x_1 = a_1, \ldots, x_{m-2} = a_{m-2},$$

and for $y_1, y_2, \ldots, y_{mq-2q}$ the forms given by (2). We thus have $2q$ partial differential equations connecting $y_{mq-2q+1}, \ldots, y_{mq}$ with x_{m-1}, x_m.

* Here d is used to indicate that that $x_1, x_2, \ldots, x_m$ are the independent variables: for strictly partial differentiation we shall use ∂.

Suppose the integrals of these with the initial conditions that when

$$x_{m-1} = a_{m-1},$$
$$y_{mq-2q+t} = \lambda^{(m-1)}_{mq-2q+t}(x_m),$$

to be
$$y_{mq-2q+t} = \lambda^{(m-2)}_{mq-2q+t}(x_{m-1}, x_m) \quad (t = 1, 2, \ldots, 2q).$$

This process may be carried on until the equations (3), (4) are exhausted; for instance, the equations $\{i\}$ are $(m-i+1)\,q$ in number, and contain derivatives with respect to $x_i, x_{i+1}, \ldots, x_m$. In them put

$$x_1 = a_1, \quad x_2 = a_2, \quad \ldots, \quad x_{i-1} = a_{i-1},$$

and assign to $y_1, y_2, \ldots, y_{iq-q}$ the forms given by (2). Thus we have a system of $(m-i+1)\,q$ partial differential equations connecting the same number of dependent variables $y_{iq-q+1}, \ldots, y_{mq}$ with the independent variables $x_i, x_{i+1}, \ldots, x_m$. Let the integrals of these with the initial conditions that, when

$$x_i = a_i,$$
$$y_{iq-q+t} = \lambda^{(i)}_{iq-q+t}(x_{i+1}, x_{i+2}, \ldots, x_m),$$

be
$$y_{iq-q+t} = \lambda^{(i-1)}_{iq-q+t}(x_i, x_{i+1}, \ldots, x_m)$$
$$\left[t = 1, 2, \ldots (m-i+1)\,q\right].$$

Then this solution of $\{i\}$ will also satisfy $\{i+1\}, \ldots, \{m\}$. This may be proved by induction; suppose it true for $i+1$. Let j, k be any two numbers of the series $i+1, \ldots, m$; then

$$\frac{d}{dx_i}\omega_j^{(i)} = \frac{d}{dx_j}\omega_i^{(i)} - \omega_{ij}^{(i)},$$

$$\frac{d}{dx_i}\omega_{jk}^{(i)} = \frac{d}{dx_j}\omega_{ik}^{(i)} - \frac{d}{dx_k}\omega_{ij}^{(i)};$$

thus, if the equations $\{i\}$ are satisfied, $\omega_j^{(i)}$ and $\omega_{jk}^{(i)}$ are independent of x_i; but, by hypothesis, they vanish when $x_i = a_i$, and therefore they vanish always.

Hence, if
$$y_r = \lambda_r^{(0)}(x_1, x_2, \ldots, x_m) \quad (r = 1, 2, \ldots, mq) \qquad (5)$$

is the result of integrating the equations $\{1\}$ with the initial conditions that, when

$$x_1 = a_1,$$
$$y_r = \lambda_r^{(1)}(x_2, x_3, \ldots, x_m),$$

we have in (5) a solution of the system of equations (3), (4), that is, of the system (1).

The integrations necessary in the foregoing process are all of the type shown to be possible by Frau von Kowalevsky (*Crelle* 80; see also Königsberger, in *Crelle* 109), that is to say, the number of dependent variables is equal to the number of equations; it ought, however, to be shown that the exceptional case, when the method of solution in series fails, does not arise here, at least in general. We may assume that the functions A which are the coefficients in (1), and the functions λ which occur in (2), have "ordinary" points in common; then all that is necessary, say in the case of the equations $\{i\}$, is that it should be possible to solve for the derivatives of the unknown functions with respect to x_i, that is, that a certain determinant Δ_i should not vanish identically. It will be enough if we show this in a particular case. Take, for instance,

$$\Omega^{(l)} = dy_l - \sum_{i=1}^{m} p_i^{(l)}\, dx_i \quad (l = 1, 2, \dots, q),$$

and suppose $p_1^{(1)}, \dots, p_1^{(q)}$ to be explicitly known functions of

$$x_1, \dots, x_m, \quad y_1, \dots, y_q, \quad p_2^{(1)}, \dots, p_2^{(q)}, \quad \dots, \quad p_m^{(1)}, \dots, p_m^{(q)}.$$

Assign arbitrarily chosen forms to

$$\begin{array}{llll}
p_2^{(1)}, \dots, p_2^{(q)}, & \text{in terms of } x_2, \dots\dots\dots, x_m, & \text{when} & x_1 = a_1, \\
p_3^{(1)}, \dots, p_3^{(q)}, & \quad ,, \quad ,, \quad x_3, \dots\dots\dots, x_m, & ,, & x_1 = a_1,\ x_2 = a_2, \\
\dots & \dots & \dots & \dots \\
p_i^{(1)}, \dots, p_i^{(q)}, & \quad ,, \quad ,, \quad x_i, x_{i+1}, \dots, x_m, & ,, & x_1 = a_1, \dots, x_{i-1} = a_{i-1}, \\
\dots & \dots & \dots & \dots
\end{array}$$

and, lastly, arbitrary constant values to

$$y_1, \dots, y_q, \quad \text{when} \quad x_1 = a_1, \dots, x_{m-1} = a_{m-1}.$$

The equations $\{i\}$ are in this case

$$\frac{dy_l}{dx_i} = p_i^{(l)}, \quad \frac{dp_j^{(l)}}{dx_i} = \frac{dp_i^{(l)}}{dx_j}$$

$$(l = 1, 2, \dots, q;\ j = i+1, \dots, m);$$

so that the value of Δ_i here is 1.

The equations $\{1\}$ are

$$\frac{dy_l}{dx_1} = p_1^{(l)}, \quad \frac{dp_j^{(l)}}{dx_1} = \frac{dp_1^{(l)}}{dx_j}$$

$$(l = 1, 2, \dots, q;\ j = 2, \dots, m);$$

so that Δ_1 is also 1.

Hence $\Delta_1, \Delta_2, \ldots, \Delta_m$ do not vanish identically in the general case, and thus the result holds good that in general the equations (1) are equivalent to an integral system consisting of mq equations, and that this system is unique if the initial conditions (2) are imposed.

If the number of variables in the equations (1) is not a multiple of $q+1$, let it be M. Then the number of variables that may be supposed independent in a solution is in general not greater than $M/(q+1)$. For suppose the number of independent variables to be m, and let them be again $x_1, x_2, \ldots, x_m$. Then by supposing $x_1 = a_1$, and substituting for $y_1, y_2, \ldots, y_q$ (*i.e.*, $x_{m+1}, \ldots, x_{m+q}$) the values given to them in any solution for this value of x_1, the equations may again be reduced to the form

$$\overline{\Omega}^{(l)} = 0,$$

q equations in $M-(q+1)$ variables.

Let μ be the greatest integer in $M/(q+1)$; then by repeating this process we come at length to a system of q equations in $M-\mu\,(q+1)$ variables which must be satisfied by the values of $y_{\mu q+1}, y_{\mu q+2}, \ldots, y_{M-m}, x_{\mu+1}, x_{\mu+2}, \ldots, x_m$, when $x_1, \ldots, x_\mu$ are constants, and $y_1, y_2, \ldots, y_{\mu q}$ are replaced by the values they have, in virtue of the supposed solution, in terms of $x_{\mu+1}, \ldots, x_m$, when $x_1, \ldots, x_\mu$ are constants. But $M-\mu\,(q+1) \not> q$; so that now, unless conditions are fulfilled, the differentials of $y_{\mu q+1}, \ldots, y_{M-m}, x_{\mu+1}, \ldots, x_m$ must vanish; that is to say, $x_{\mu+1}, \ldots, x_m$ are constants, when $x_1, \ldots, x_\mu$ are so, which is inconsistent with the supposition that they are independent variables. It will, of course, happen in particular cases that more than μ variables may be supposed independent, but it is easy to construct an example to show that the conditions necessary for this are not generally fulfilled. For instance, take the equations

$$dy_1 = x_2\,dx_1 - x_1\,dx_2 + \sum_{i=3}^{m} p_i^{(1)}\,dx_i,$$

$$dy_l = \sum_{i=2}^{m} p_i^{(l)}\,dx_i \quad (l = 2, 3, \ldots, q).$$

Here there are q equations and $mq+m-1$ variables

$$x_1, \ldots, x_m, \quad y_1, \ldots, y_q, \quad p_3^{(1)}, \ldots, p_m^{(1)}, \quad p_2^{(2)}, \ldots, p_m^{(q)},$$

and it is not possible to form an integral system satisfying these with m independent variables.

It is clear that in any case $M-\mu\,(q+1)$ of the dependent variables may be equated to arbitrary functions of the μ independent; this reduces the problem to the original one in which $M/(q+1)$ was integral.

The Transmission of Stress across a Plane of Discontinuity in an Isotropic Elastic Solid, and the Potential Solutions for a Plane Boundary. *By* J. H. MICHELL, M.A. Received May 30th, 1899. Read June 8th, 1899.

1. The volume equations of equilibrium and the surface conditions are expressed in a form which leads directly to the potential solutions* of Boussinesq and Cerruti. The method is applied to cases where different sets of conditions hold on different parts of the plane boundary. The method of images is shown to be applicable to the determination of the stress due to volume forces in the semi-infinite solid. Finally, by the use of the method of images, a potential solution is given for the transmission of stress across a plane of discontinuity (1) when two, elastically different, isotropic solids are soldered together over the plane, (2) when there is free slipping over the plane.

2. Let a semi-infinite solid, supposed isotropic, lie on the positive side of the plane $z = 0$. The volume equations to be solved are

$$(\lambda+\mu)\,\theta_x+\mu\nabla^2 u = -X, \quad \&c.$$

The sets of boundary conditions to which the methods apply are those treated by Boussinesq, viz.:

(1) $R,\ S,\ T$;

(2) $R,\ u,\ v$;

(3) $w,\ S,\ T$;

or (4) $w,\ u,\ v$;

given over the plane $z = 0$.

We proceed to transform these equations and conditions.

If we write in $\quad \mu\nabla^2 w+(\lambda+\mu)\,\theta_z = -Z$

$$w = -\frac{\lambda+\mu}{2\mu}\,z\theta+w',$$

---

* Todhunter and Pearson, *History*, II., § 1489 *et seq.*; Love, *Elasticity*, I., chap. ix.

we get $\quad 2\mu(\lambda+2\mu)\nabla^2 w' = 2\mu(\lambda+\mu)\,z\Lambda-2(\lambda+2\mu)\,Z$;

and, in addition, we have the well known equations

$$(\lambda+2\mu)\,\nabla^2\theta = -\Lambda,$$

$$2\mu\nabla^2\varpi = X_y - Y_x,$$

where

$$\Lambda = X_x+Y_y+Z_z,$$

$$2\varpi = v_x - u_y.$$

Thus the Laplacians of θ, w', ϖ are known throughout.

On $z=0$, we have

$$\begin{aligned}
R &= \lambda\theta+2\mu w_z \\
&= -\mu\theta+2\mu w'_z, \\
R_z &= \lambda\theta_z+2\mu w_{zz} \\
&= -(\lambda+2\mu)\,\theta_z+2\mu w'_{zz} \\
&= -T_x-S_y-Z, \\
2\mu\varpi_z &= S_x-T_y. \\
\theta-w_z &= \frac{\lambda+3\mu}{2\mu}\theta-w'_z \\
&= u_x+v_y.
\end{aligned}$$

The sets of conditions (1)–(4) therefore give as follows :—

(1) R, S, T given involve

(i.) $$\left.\begin{aligned}
-\mu\theta+2\mu w'_z &= R \\
-(\lambda+2\mu)\,\theta_z+2\mu w'_{zz} &= -T_x-S_y-Z \\
2\mu\varpi_z &= S_x-T_y
\end{aligned}\right\}$$

given.

(2) R, u, v involve

(ii.) $$\left.\begin{aligned}
-\mu\theta+2\mu w'_z &= R \\
(\lambda+3\mu)\,\theta-2\mu w'_z &= 2\mu\,(u_x+v_y) \\
2\varpi &= v_x-u_y
\end{aligned}\right\}$$

given.

(3) w, S, T involve

(iii.) $$\left.\begin{aligned}
w' &= w \\
-(\lambda+2\mu)\,\theta_z+2\mu w'_{zz} &= -T_x-S_y-Z \\
2\mu\varpi_z &= S_x-T_y
\end{aligned}\right\}$$

given.

And, finally,

(4) w, u, v involve

(iv.)
$$\left.\begin{aligned} w' &= w \\ (\lambda+3\mu)\,\theta-2\mu w'_z &= 2\mu\,(u_x+v_y) \\ 2\varpi &= v_x-u_y \end{aligned}\right\}$$

given.

Now the linear functions of θ, w', and ϖ in these conditions (i.)–(iv.) are all functions whose Laplacians are known throughout the solid, and whose values can therefore be written down by means of the known solutions of the potential problem for a plane boundary.

For example, suppose there are no volume forces and consider conditions (i.). They give at once

$$-\mu\theta+2\mu w'_z = -\frac{1}{2\pi}\,\frac{d}{dz}\iint\frac{R}{r}\,dx'dy',$$

$$-(\lambda+2\mu)\,\theta+2\mu w'_z = \frac{1}{2\pi}\iint(T_{x'}+S_{y'})\,\frac{1}{r}\,dx'dy,$$

$$2\mu\varpi = -\frac{1}{2\pi}\iint(S_{x'}-T_{y'})\,\frac{1}{r}\,dx'dy',$$

where (x', y') is now a point on $z=0$, and (x, y, z) is a point in the solid. Thus

$$-(\lambda+\mu)\,\theta = \frac{1}{2\pi}\,\frac{d}{dz}\iint\frac{R}{r}\,dx'dy'+\frac{1}{2\pi}\iint(T_{x'}+S_{y'})\,\frac{1}{r}\,dx'dy',$$

$$2\mu\,(\lambda+\mu)\,w'_z = -\frac{\lambda+2\mu}{2\pi}\,\frac{d}{dz}\iint\frac{R}{r}\,dx'dy'-\frac{\mu}{2\pi}\iint(T_{x'}+S_{y'})\,\frac{1}{r}\,dx'dy'.$$

Integrating the second integral by parts, we arrive at the form of solution given by Boussinesq for this case. The values of u, v are found from those of

$$u_x+v_y = \theta-w_z \quad\text{and}\quad v_x-u_y = \varpi$$

(which are now known) by a well known process, quite independently of the boundary conditions. It need only be noted here that the logarithmic potentials of Boussinesq are introduced in order to bring the known values of $u_{xx}+u_{yy}$ and $v_{xx}+v_{yy}$ into the forms $U_{xx}+U_{yy}$, $V_{xx}+V_{yy}$, which are derived from terms containing the second differential coefficient with respect to z by the use of the equation

$$U_{zz} = -U_{xx}-U_{yy},$$

satisfied by a harmonic function. The values of u, v can then be written down by inspection.

The solutions for the other sets of conditions are obtained in exactly the same way and appear in the forms given by Boussinesq.

3. Suppose now that one set of conditions holds over part of the boundary, and another set over the rest, the volume forces being still omitted.

Taking the combinations in turn and using obvious symbolism,

$$(a)\quad \left.\begin{matrix} w, & u, & v \\ w, & S, & T \end{matrix}\right\}$$

give

$$(\alpha)\quad \left.\begin{matrix} w', & (\lambda+3\mu)\,\theta - 2\mu w'_z, & \varpi \\ w', & -(\lambda+2\mu)\,\theta_z + 2\mu w'_{zz}, & \varpi_z \end{matrix}\right\}.$$

Here w' is known all over the boundary, and its value can therefore be written down for all points by the previous method.

Further,

$$w'_{zz} = -w'_{xx} - w'_{yy}$$

is known over the boundary, and therefore the set of conditions is reducible to

$$(\alpha')\quad \left.\begin{matrix} (\lambda+3\mu)\,\theta - 2\mu w'_z, & \varpi \\ (\lambda+3\mu)\,\theta_z - 2\mu w'_{zz}, & \varpi_z \end{matrix}\right\}.$$

The problem is therefore reduced to finding a harmonic function V, such that V has a given value over part of the plane $z=0$ and V_z a given value over the rest. This is the same thing as finding the electric charge, on a disc coinciding with the first part of the plane, such that its potential together with the potential of the known distribution $-V_z/2\pi$ over the rest of the plane may have a given value over the disc. The solution is known for simple cases, such as circular or elliptic discs. It is not proposed to enter into details here. In this way $(\lambda+3\mu)\,\theta - 2\mu w'_z$ and ϖ are found. Having thus found w', θ, ϖ, the solution is completed as in § 2.

Now consider

$$(b)\quad \left.\begin{matrix} R, & u, & v \\ R, & S, & T \end{matrix}\right\},$$

which give

$$(\beta)\quad \left.\begin{matrix} -\mu\theta + 2\mu w'_z, & (\lambda+3\mu)\,\theta - 2\mu w'_z, & \varpi \\ -\mu\theta + 2\mu w'_z, & -(\lambda+2\mu)\,\theta_z + 2\mu w'_{zz}, & \varpi_z \end{matrix}\right\}.$$

Here $-\mu\theta+2\mu w'_z$ is determined throughout at once, and the conditions are reduced to

$$(\beta') \quad \left.\begin{matrix} -(\lambda+2\mu)\,\theta+2\mu w'_z, & \varpi \\ -(\lambda+2\mu)\,\theta_z+2\mu w'_{zz}, & \varpi_z \end{matrix}\right\},$$

the value of $-(\lambda+2\mu)\,\theta+2\mu w'_z$ being derived from

$$-\mu\theta+2\mu w'_z \quad \text{and} \quad (\lambda+3\mu)\,\theta-2\mu w'_z.$$

The conditions (β') are treated in the same way as the conditions (α').

$$(c) \quad \left.\begin{matrix} w, & S, & T \\ R, & S, & T \end{matrix}\right\}$$

give

$$(\gamma) \quad \left.\begin{matrix} w', & -(\lambda+2\mu)\,\theta_z+2\mu w'_{zz}, & \varpi_z \\ -\mu\theta+2\mu w'_z, & -(\lambda+2\mu)\,\theta_z+2\mu w'_{zz}, & \varpi_z \end{matrix}\right\}.$$

Hence $-(\lambda+2\mu)\,\theta+2\mu w'_z$ and ϖ are determined at once, and the conditions reduce to

$$\left.\begin{matrix} -\mu\theta_z+2\mu w'_{zz} \\ -\mu\theta+2\mu w'_z \end{matrix}\right\},$$

remembering that w'_{zz} is known when w' is known. Boussinesq* has treated the particular case

$$\left.\begin{matrix} w, & 0, & 0 \\ 0, & 0, & 0 \end{matrix}\right\}$$

for which $\quad -(\lambda+2\mu)\,\theta+2\mu w'_z=0, \quad \varpi=0,$

so that the conditions reduce to

$$\left.\begin{matrix} w'=w \\ w'_z=0 \end{matrix}\right\}$$

and

$$R=-\mu\theta+2\mu w'_z$$

$$=\frac{2\mu\,(\lambda+\mu)}{\lambda+2\mu}\,w'_z$$

over $\quad z=0,$

which gives at once Boussinesq's solution.

---

* Todhunter and Pearson, II., § 1510.

The alternative case
$$\left.\begin{matrix} 0, & 0, & 0 \\ R, & 0, & 0 \end{matrix}\right\}$$

reduces to
$$(\gamma)\quad \left.\begin{aligned} w' &= 0 \\ w'_z &= \frac{\lambda+2\mu}{2\mu(\lambda+\mu)}R \end{aligned}\right\}.$$

It is clear that Boussinesq's case, in conjunction with Case (1) of § 2, can be made to cover the whole of this section.

$$(d)\quad \left.\begin{matrix} w, & u, & v \\ R, & u, & v \end{matrix}\right\}$$

give
$$(\delta)\quad \left.\begin{matrix} w', & (\lambda+3\mu)\,\theta-2\mu w'_z, & \varpi \\ -\mu\theta+2\mu w'_z, & (\lambda+3\mu)\,\theta-2\mu w'_z, & \varpi \end{matrix}\right\},$$

whence $(\lambda+3\mu)\,\theta-2\mu w'_z$ and ϖ are written down and the conditions reduce to
$$(\delta')\quad \left.\begin{matrix} w' \\ w'_z \end{matrix}\right\}.$$

Boussinesq's solution can be made to cover this case also by use of Case (2), § 2.
$$(e)\quad \left.\begin{matrix} w, & u, & v \\ R, & S, & T \end{matrix}\right\}$$

give
$$(\epsilon)\quad \left.\begin{matrix} w', & (\lambda+3\mu)\,\theta-2\mu w'_z, & \varpi \\ -\mu\theta+2\mu w'_z, & -(\lambda+2\mu)\,\theta_z+2\mu w'_{zz}, & \varpi_z \end{matrix}\right\}.$$

This, perhaps the most interesting combination, unfortunately does not appear to admit of such a simple solution as the others.

4. Suppose now there are volume forces on the semi-infinite solid, the conditions at the boundary being one of the sets (1)–(4). By using the solutions of § 2 the given stresses or displacements over the boundary are reduced to zero, and this we suppose done in what follows.

Take, as an example, conditions (4), which are reduced to

$$u = v = w = 0,$$

and therefore give, by (iv.),

$$\left.\begin{aligned} w' &= 0 \\ (\lambda+3\mu)\,\theta - 2\mu w'_z &= 0 \\ \varpi &= 0 \end{aligned}\right\}.$$

Now,
$$2\mu\,(\lambda+2\mu)\,\nabla^2 w' = -(\lambda+\mu)\,z\Lambda - 2\,(\lambda+2\mu)\,Z,$$

$$(\lambda+2\mu)\,\nabla^2\left\{(\lambda+3\mu)\,\theta - 2\mu w'_z\right\} = (\lambda+\mu)\,z\Lambda_z - 2\mu\Lambda + 2\,(\lambda+2\mu)\,Z_z,$$

$$2\mu\nabla^2\varpi = X_y - Y_x.$$

Hence w', $(\lambda+3\mu)\,\theta - 2\mu w'_z$, and ϖ are the potentials of the distributions corresponding to the given values of their Laplacians in the semi-infinite solid together with the potentials of the negative images of those distributions in the plane $z = 0$.

Thus the values of w, θ, and ϖ are written down at once, and the solution is then completed, as before, by finding u, v.

This process is clearly applicable to each of the sets of conditions. Positive images will appear if the normal rate of change is given to vanish over $z = 0$.

5. Consider now the transmission of stress across a plane of discontinuity in an infinite solid. Let $z = 0$ be the plane, and let the *constants* referring to the region z positive be unaccented, those for the region z negative accented.

First, let there be complete union or a perfect solder over $z = 0$, so that the interfacial conditions are—

R, S, T, u, v, w continuous across the plane.

These conditions, by the preceding work, lead to the continuity of

$$\left.\begin{gathered} -\mu\theta + 2\mu w'_z \\ -(\lambda+2\mu)\,\theta_z + 2\mu w'_{zz} + Z \\ \mu\varpi_z \\ w' \\ \frac{\lambda+3\mu}{2\mu}\,\theta - w'_z \\ \varpi \end{gathered}\right\},$$

where, of course,
$$w' = w + \frac{\lambda'+\mu'}{2\mu'}\,z\theta,$$

in the region z negative.

Now, $$\mu\nabla^2 w' = -Z,$$

over $$z = 0.$$

Hence $$w'_{zz} + Z/\mu = -(w'_{xx} + w'_{yy})$$

is continuous, and therefore, if p, q are any numbers,

$$\left.\begin{array}{l}\left(-\mu+p\dfrac{\lambda+3\mu}{2\mu}\right)\theta+(2\mu-p)\,w'_z \\ -(\lambda+2\mu)\,\theta_z+(2\mu+q)\,w'_{zz}+(1+q/\mu)\,Z\end{array}\right\}$$

are continuous.

We proceed to choose p, q so that the first two terms of the latter expression may be a multiple of the z-rate of change of the former, for both z positive and z negative.

We assume accordingly

$$\frac{2\mu-p}{\mu-p\,(\lambda+3\mu)/2\mu} = \frac{2\mu+q}{\lambda+2\mu} = -\gamma \text{ (say)}$$

and $$\frac{2\mu'-p}{\mu'-p\,(\lambda'+3\mu')/2\mu'} = \frac{2\mu'+q}{\lambda'+2\mu'} = -\gamma' \text{ (say)}.$$

Then, writing $$\lambda+3\mu = 2\mu\kappa,$$

$$\lambda'+3\mu' = 2\mu'\kappa',$$

we have $$\frac{2\mu'-p}{\mu'-p\kappa'}(2\kappa'-1)\,\mu'-2\mu' = q = \frac{2\mu-p}{\mu-p\kappa}(2\kappa-1)\,\mu-2\mu,$$

and this gives

$$p^2(\kappa'\mu-\kappa\mu')-4p\left\{\kappa\kappa'(\mu'^2-\mu^2)-\kappa\mu'^2+\kappa'\mu^2\right\}$$
$$+4\left\{(\kappa'-1)\,\mu'-(\kappa-1)\,\mu\right\}\mu\mu' = 0.$$

If $\kappa = \kappa'$, *i.e.*, if Poisson's ratio is the same for the two parts, the equation takes the simpler form

$$\kappa p^2+4p\kappa\,(\kappa-1)(\mu+\mu')-4\,(\kappa-1)\,\mu\mu' = 0.$$

Since $\kappa > 3/2$ for stability of the material, this equation has two real roots; but this is immaterial.

Let p_1, p_2 be the two roots, q_1, q_2 the corresponding values of q, γ_1, γ_2 those of γ.

Then we have

$$\left.\begin{array}{l}(\kappa p_1-\mu)(\theta+\gamma_1 w'_z) \\ (\lambda+2\mu)(\theta_z+\gamma_1 w'_{zz})-(1+q_1/\mu)\, Z\end{array}\right\}$$

continuous, as well as the corresponding expressions for the second root.

Put $$(\kappa p_1-\mu)(\theta+\gamma_1 w'_z) = V$$

and $$(\lambda+2\mu)\,/\,(\kappa p_1-\mu) = K$$

then $$\left.\begin{array}{l}V \\ K\dfrac{dV}{dz}-(1+q_1/\mu)\, Z\end{array}\right\}$$

are continuous.

Now, from the volume-equations, we have

$$\nabla^2 V = -4\pi\rho,$$

where ρ is a known function of the given volume forces.

The method of images may now be applied to find V; the problem is the same as that of the electric flow across a plane of discontinuity in a conducting solid, treated, *e.g.*, by Maxwell, *Electricity and Magnetism*, I., § 315.

The terms $-(1+q_1/\mu)\, Z$, &c., of course, correspond to an interfacial layer for which the potential can be written down.

Hence, corresponding to the two values of p, we can write down the values of

$$\theta+\gamma_1 w'_z,$$

$$\theta+\gamma_2 w'_z,$$

throughout the solid.

These give θ and w'_z, and therefore also w by a simple integration with respect to z. There is no trouble in finding ϖ, for ϖ and $\mu\varpi_z$ are continuous across the interface, and the Laplacian of ϖ is known throughout; so that ϖ is found by the same method as V.

The solution is then completed by finding u, v as before.

Given discontinuities of force or displacement over the interface can plainly be treated by the same process.

6. Another interesting case is that in which there is free slipping at the interface, so that R, w are continuous, and

$$S = T = 0$$

over $$z = 0.$$

We then have $$-(\lambda+2\mu)\,\theta_z + 2\mu w'_{zz} + Z = 0$$

over the whole plane, and this determines

$$-(\lambda+2\mu)\,\theta + 2\mu w'_z$$

throughout. Also $$-\mu\theta + 2\mu w'_z$$

and $$w'_{zz} + Z/\mu$$

are continuous, and therefore also

$$\frac{\lambda+2\mu}{\mu(\lambda+\mu)}\left(-\mu\theta_z + 2\mu w'_{zz}\right) + \frac{2\lambda+3\mu}{\mu(\lambda+\mu)}\,Z.$$

Thus, writing $$-\mu\theta + 2\mu w'_z = V,$$

we have to find V from $\nabla^2 V$ given throughout, and

$$\left.\begin{array}{c} V \\ \dfrac{\lambda+2\mu}{\mu(\lambda+\mu)}\dfrac{dV}{dz} + \dfrac{2\lambda+3\mu}{\mu(\lambda+\mu)}\,Z \end{array}\right\}$$

continuous across the interface.

This is the problem of § 5.

Further, the value of ϖ is written down from the condition that $\varpi_z = 0$ over the interface. Having determined θ, w and ϖ in this manner, the solution is completed as before.

Other cases, in which the continuity extends over only part of the plane, may be treated in a similar way.

In conclusion, it may be remarked that Betti's method,* derived from his reciprocal theorem, can, with obvious modifications, be applied to the case of a surface of discontinuity of any form in an infinite solid.

---

* Love, *Elasticity*, I., § 141.

On a Congruence Theorem relating to an Extensive Class of Coefficients. By J. W. L. GLAISHER. Communicated June 8th, 1899. Received August 30th, 1899.

1. It is a known theorem enunciated without proof by Sylvester,* in 1861, and proved by Stern,† in 1874, that, if E_n be the n^{th} Eulerian number and if p be any uneven prime, then

$$(-1)^n E_n \equiv (-1)^{n'} E_{n'}, \quad \text{mod}\, p,$$

if $n-n'$ is a multiple of $\frac{p-1}{2}$. This singular theorem explains why the Eulerian numbers end in 1 and 5 alternately, and gives rise to many other properties of the numbers.

The theorem may be expressed in the form

$$E_n \equiv (-1)^{tj} E_{n-tj}, \quad \text{mod}\, p,$$

where $j = \frac{1}{2}(p-1)$ and t is any integer such that $n-tj$ is positive; so that, to mod p, any Eulerian number is congruent to one of the first $\frac{1}{2}(p-1)$ Eulerian numbers, $E_1, E_2, \ldots, E_{\frac{1}{2}(p-1)}$.

I have obtained a comparatively simple proof of this theorem by a method which is applicable to expansions of a very general character, and which shows that the property in question is not peculiar to the Eulerian numbers, but is shared by an extensive class of other numbers or coefficients.

As very little simplification is produced by considering the special case of the Eulerian numbers, I proceed at once to prove the general theorems.

* "Sur une propriété des nombres premiers qui se rattache au théorème de Fermat," *Comptes Rendus*, Vol. LII., p. 212.

† "Zur Theorie der Eulerschen Zahlen," *Crelle's Journal*, Vol. LXXIX., p. 67. It should be mentioned that Sylvester and Stern give also more general theorems in which the modulus is p^n and 2^n. In the present paper the modulus is always p or 2.

2. Let $X_0, X_1, X_2, \ldots$ be any quantities connected by the recurring relation

$$\text{(i.)}\quad (\lambda+1)\, a^n X_n + (n)_1\, a^{n-1} b X_{n-1} + (n)_2\, a^{n-2} b^2 X_{n-2} + \ldots$$
$$\ldots + (n)_{n-1}\, ab^{n-1} X_1 + (n)_n\, b^n X_0 = c_n,$$

where λ is any constant, a and b any integers, and c_n is any quantity depending upon n, and such that

$$c_r \equiv c_{r-t(p-1)}, \quad \text{mod } p,$$

p being any given uneven prime and t any integer. Thus, for example, c_r might be $A\alpha^r$, if p is not a divisor of A or α, or $A\alpha^r + B\beta^r + C\gamma^r + \ldots$, if p is not a divisor of any of the quantities $A, \alpha, B, \beta, C, \gamma, \ldots$. The notation $(n)_r$ is used to express the number of combinations of n things taken r together, *i.e.*, $(n)_r$ is the coefficient of x^r in the expansion of $(1+x)^n$. The suffix of c is always supposed to be positive.

It will now be assumed that the congruence

$$X_r \equiv X_{r-t(p-1)}, \quad \text{mod } p,$$

p being any uneven prime, and t being any positive integer, holds good for the values $p, p+1, \ldots, n-1$ of r, and by means of the above recurring relation it will be shown that, this being so, it holds good also for $r = n$.

Let $n = kp+q$, where k and q are any positive integers and $q<p$. Then

$$c_n - (\lambda+1)\, a^n X_n$$
$$= b^n X_0 + (n)_1\, ab^{n-1} X_1 + (n)_2\, a^2 b^{n-2} X_2 + \ldots + (n)_{n-1}\, a^{n-1} b X_{n-1}$$
$$= b^n X_0 + (n)_1\, ab^{n-1} X_1 + (n)_2\, a^2 b^{n-2} X_2 + \ldots + (n)_{p-1}\, a^{p-1} b^{n-p+1} X_{p-1}$$
$$+ (n)_p\, a^p b^{n-p} X_p + (n)_{p+1} a^{p+1} b^{n-p-1} X_{p+1} + \ldots + (n)_{2p-1} a^{2p-1} b^{n-2p+1} X_{2p-1}$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$+ (n)_{(k-1)p}\, a^{(k-1)p}\, b^{n-(k-1)p} X_{(k-1)p} + n_{(k-1)p+1}\, a^{(k-1)p+1} b^{n-(k-1)p-1} X_{(k-1)p+1}$$
$$+ \ldots + (n)_{kp-1}\, a^{kp-1}\, b^{n-kp+1} X_{kp-1}$$
$$+ (n)_{kp}\, a^{kp} b^{n-kp} X_{kp} + (n)_{kp+1}\, a^{kp+1}\, b^{n-kp-1}\, X_{kp+2} + \ldots$$
$$\ldots + (n)_{kp+q-1} a^{kp+q-1} b X_{kp+q-1}.$$

Now it is known* that, if p is prime and q and s are less than p, then

$$(kp+q)_{gp+s} \equiv 0, \ \bmod p, \qquad \text{if } s > q,$$

and

$$\equiv (k)_g \times (q)_s, \ \bmod p, \ \text{ if } s \leqq q.$$

Reducing the coefficients by this rule, and reducing also the X's by the congruence

$$X_r \equiv X_{r-t(p-1)}, \quad \bmod p,$$

and the powers of a and b by the congruences

$$a^r \equiv a^{r-t(p-1)}, \quad \bmod p,$$

$$b^r \equiv b^{r-t(p-1)}, \quad \bmod p,$$

we find that

$$\begin{aligned}
&c_n - (\lambda+1)\, a^n X_n \\
&\equiv b^{k+q} X_0 + (q)_1\, ab^{k+q-1} X_1 + (q)_2\, a^2 b^{k+q-2} X_2 + \ldots + (q)_q\, a^q b^k X_q \\
&\quad + (k)_1 \left\{ ab^{k+q-1} X_1 + (q)_1\, a^2 b^{k+q-2} X_2 + \ldots + (q)_q\, a^{q+1} b^{k-1} X_{q+1} \right\} \\
&\quad + (k)_2 \left\{ a^2 b^{k+q-2} X_2 + (q)_1\, a^3 b^{k+q-3} X_3 + \ldots + (q)_q\, a^{q+2} b^{k-2} X_{q+2} \right\} \\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
&\quad + (k)_{k-1} \left\{ a^{k-1} b^{q+1} X_{k-1} + (q)_1\, a^k b^q X_k + \ldots + (q)_q\, a^{k+q-1} b X_{k+q-1} \right\} \\
&\quad + (k)_k \left\{ a^k b^q X_k + (q)_1\, a^{k+1} b^{q-1} X_{k+1} + \ldots + (q)_{q-1} a^{k+q-1} X_{k+q-1} \right\}, \ \bmod p.
\end{aligned}$$

Collecting the coefficients of $X_0, X_1, X_2, \ldots,$ the right-hand side of this congruence becomes

$$\begin{aligned}
&b^{k+q} X_0 + ab^{k+q-1} \left\{ (k)_1 + (q)_1 \right\} X_1 + a^2 b^{k+q-2} \left\{ (k)_2 + (q)_1 (k)_1 + (q)_2 \right\} X_2 \\
&\qquad + a^3 b^{k+q-3} \left\{ (k)_3 + (q)_1 (k)_2 + (q)_2 (k)_1 + (q)_3 \right\} X_3 + \ldots \\
&\qquad \ldots + a^k b^q \left\{ (k)_k + (q)_1 (k)_{k-1} + \ldots + (q)_k\dagger \right\} X_k + \ldots \\
&\qquad \ldots + a^{k+q-1} b \left\{ (q)_{q-1} (k)_k + (q)_q (k)_{k-1} \right\} X_{k+q-1},
\end{aligned}$$

which $= b^{k+q} X_0 + (k+q)_1\, ab^{k+q-1} X_1 + (k+q)_2\, a^2 b^{k+q-2} X_2 + \ldots$

$$\ldots + (k+q)_{k+q-1} a^{k+q-1} b X_{k+q-1}$$

$$= c_{k+q} - (\lambda+1)\, a^{k+q} X_{k+q}, \text{ by (1).}$$

---

* *Quarterly Journal*, Vol. xxx., p. 152.

† This term is not reached unless $k \leqq q$; but we may regard all the coefficients as of the form

$$(k)_m + (q)_1 (k)_{m-1} + \ldots + (q)_{m-1} (k)_1 + (q)_m = (k+q)_m,$$

if we suppose that in general $(r)_s$ denotes zero when $s > r$.

Thus, $$c_n-(\lambda+1)\,a^n X_n \equiv c_{k+q}-(\lambda+1)\,a^{k+q}X_{k+q}, \ \text{mod}\ p.$$

Now, $$c_{kp+q}\equiv c_{k+q}, \quad a_{kp+q}\equiv a^{k+q}, \quad \text{mod}\ p,$$

so that, if a is not divisible by p, this congruence gives

$$X_n \equiv X_{n-k(p-1)}, \quad \text{mod}\ p;$$

and therefore the congruence

$$X_r \equiv X_{r-t(p-1)}, \quad \text{mod}\ p,$$

holds good also when $r=n$.

It remains to show that this congruence holds good for $r=p$. Putting $n=p$ in the original recurring equation (i.), we have

$$c_p-(\lambda+1)\,a^pX_p$$
$$= b^pX_0+(p)_1\,ab^{p-1}X_1+(p)_2\,a^2b^{p-2}X_2+\ldots+(p)_{p-1}\,a^{p-1}bX_{p-1}$$

All the coefficients on the right-hand side, except the first, are divisible by p; and therefore

$$c_p-(\lambda+1)\,a^pX_p\equiv b^pX_0, \quad \text{mod}\ p,$$

i.e., $$c_p-(\lambda+1)\,aX_p \equiv bX_0, \quad \text{mod}\ p.$$

Also, putting $n=1$ in (i.),

$$c_1-(\lambda+1)\,aX_1 = bX_0,$$

whence, since $$c_p\equiv c_1, \quad \text{mod}\ p,$$

we have $$X_p\equiv X_1, \ \text{mod}\ p;$$

so that the congruence holds good for $r=p$.

3. The preceding investigation fails if a is divisible by p; so that the prime divisors of a must be excluded from the admissible values of p. Also, no divisor of a denominator of any of the X's can be an admissible value of p. If the denominator of X_0 be m, and if the quantities c_n have in their denominators only powers of certain numbers α, β, γ, ..., then the denominator of X_n can only contain m and powers of a, $\lambda+1$, and α, β, γ, ... All prime numbers therefore which are not divisors of a, $\lambda+1$, m, α, β, γ, ... are admissible values of p.

It will be noticed that in the recurring relation (i.) we may replace the powers of a and b, a^r and b^r, by a_r and b_r, where a_r and b_r are any quantities which satisfy the same congruence as c_r, *i.e*, so that

$$a_r\equiv a_{r-t(p-1)}, \quad b_r\equiv b_{r-t(p-1)}, \quad \text{mod}\ p.$$

4. In precisely the same manner, we may show that, if $X_0, X_2, X_4, \ldots$ are quantities connected by the recurring relation

$$\text{(ii.)} \quad (\lambda+1)\, a^{2n} X_{2n} + (2n)_2\, a^{2n-2} b^2 X_{2n-2} + \ldots$$
$$\ldots + (2n)_{2n-2}\, a^2 b^{2n-2} X_2 + (2n)_{2n}\, b^{2n} X_0 = c_{2n},$$

where k, a, b are as before, and

$$c_{2n} \equiv c_{2n-t(p-1)}, \quad \text{mod } p,$$

then
$$X_{2n} \equiv X_{2n-t(p-1)}, \text{ mod } p.$$

It is convenient to introduce the quantities $X_1, X_3, X_5, \ldots$, all of which are supposed to be zero; in the case of these quantities therefore the congruence

$$X_r \equiv X_{r-t(p-1)}, \quad \text{mod } p,$$

holds good.

Thus we may write (ii.),

$$c_{2n} - (\lambda+1)\, a^{2n} X_{2n}$$
$$= b^{2n} X_0 + (2n)_1\, ab^{2n-1} X_1 + (2n)_2\, a^2 b^{2n-2} X_2 + \ldots + (2n)^{2n-1}\, a^{2n-1} b X^{2n-1}.$$

Supposing, now, that the congruence

$$X_r \equiv X_{r-t(p-1)}, \quad \text{mod } p,$$

holds good for $r = p+1, p+2, \ldots, 2n-1$, we find, by putting

$$2n = kp + q$$

and reducing as before the exponents and suffixes, that the right-hand side

$$\equiv c_{k+q} - (\lambda+1)\, a^{k+q} X_{k+q}, \quad \text{mod } p.$$

Therefore
$$X_{2n} \equiv X_{k+q}, \qquad \text{mod } p,$$

and the congruence holds good also for $r = 2n$.

Putting $2n = p+1$ in (ii.), we have

$$c_{p+1} - (\lambda+1)\, a^{p+1} X_{p+1} = b^{p+1} X_0 + (p+1)_1\, ab^p X_1 + (p+1)_2\, a^2 b^{p-1} X_2 + \ldots$$
$$\ldots + (p+1)_{-1}\, a^{p-1} b^2 X_{p-1} + (p+1)_p\, a^p b X_p$$
$$\equiv b^{p+1} X_0 + ab^p X_1 + a^p b X_p, \quad \text{mod } p,$$
$$\equiv b^2 X_0, \quad \text{mod } p,$$

since X_1 and X_p are zero; and, by putting $n = 1$ in (ii.),

$$c_2 - (\lambda+1)\, a^2 X_2 = b^2 X_0,$$

Thus $$c_{p+1}-(\lambda+1)\,a^{p+1}X_{p+1} \equiv c_2-(\lambda+1)\,a^2X_2, \quad \text{mod } p,$$

and therefore $$X_{p+1} \equiv X_2, \quad \text{mod } p\,;$$

so that the congruence $$X_r \equiv X_{r-t(p-1)}, \text{ mod } p,$$

holds good for $r = p+1$, and therefore for all higher values of r.

5. The same reasoning is applicable to the recurring relation

(iii.) $$(\lambda+1)\,a^{2n+1}X_{2n+1}+(2n+1)_2\,a^{2n-1}b^2X_{2n-1}+\dots$$
$$\dots+(2n+1)_{2n-2}a^3b^{2n-2}X_3+(2n+1)_{2n}\,ab^{2n}X_1 = c_{2n+1},$$

where $$c_{2n+1} \equiv c_{2n+1-t(p-1)}, \quad \text{mod } p.$$

For, introducing the zero quantities X_0, X_2, X_4, ..., we have

$$c_{2n+1}-(\lambda+1)\,a^{2n+1}X_{2n+1}$$
$$= b^{2n+1}X_0+(2n+1)_1\,ab^{2n}X_1+\dots+(2n+1)_{2n}\,a^{2n}bX_{2n},$$

and, if we assume, as before, that the congruence

$$X_r \equiv X_{r-t(p-1)}, \quad \text{mod } p,$$

holds good for all values of r from p to $2n$ inclusive, we find, as before, by putting $2n+1 = kp+q$, that the right-hand side

$$\equiv c_{k+q}-(\lambda+1)\,a^{k+q}X_{k+q}, \quad \text{mod } p.$$

Thus $$X_{2n+1} \equiv X_{k+q}, \quad \text{mod } p,$$

and the congruence holds good also for $r = 2n+1$.

Putting $n = p$ in (iii.), we have

$$c_p-(\lambda+1)\,a^pX_p = (p)_0\,b^pX_0+(p)_1\,ab^{p-1}X_1+\dots+(p)_{p-1}a^{p-1}bX_{p-1},$$
$$\equiv 0, \quad \text{mod } p,$$

since $X_0 = 0$; and, putting $2n+1 = 1$ in (iii.),

$$c_1-(\lambda+1)\,aX_1 = 0.$$

Therefore $$X_p \equiv X_1, \quad \text{mod } p,$$

and the congruence $$X_r \equiv X_{r-t(p-1)}, \quad \text{mod } p,$$

holds good for $r = p$, and therefore for all higher values of r.

The remarks in § 3 apply also to the recurring formulæ (ii.) and (iii.).

6. It has thus been shown that, if the X's are defined by any one of the recurring relations

(i.) $(\lambda+1)\,a^nX_n+(n)_1\,a^{n-1}bX_{n-1}+\dots+(n)_{n-1}\,ab^{n-1}X_1+(n)_n\,b^nX_0=c_n,$

(ii.) $(\lambda+1)\,a^{2n}X_{2n}+(2n)_2\,a^{2n-2}b^2X_{2n-2}+\dots$

$$\dots+(2n)_{2n-2}\,a^2b^{2n-2}X_2+(2n)_{2n}\,b^{2n}X_0=c_{2n},$$

(iii.) $(\lambda+1)\,a^{2n+1}X_{2n+1}+(2n+1)_2\,a^{2n-1}b^2X_{2n-1}+\dots$

$$\dots+(2n+1)_{2n-2}\,a^3b^{2n-2}X_3+(2n+1)_{2n}\,ab^{2n}X_1=c_{2n+1},$$

where, in (i.), $\qquad c_n\equiv c_{n-t(p-1)},\quad \text{mod } p,$

in (ii.), $\qquad c_{2n}\equiv c_{2n-t(p-1)},\quad \text{mod } p,$

in (iii.), $\qquad c_{2n+1}\equiv c_{2n+1-t(p-1)},\quad \text{mod } p,$

then $\qquad X_n\equiv X_{n-t(p-1)},\quad \text{mod } p.$

No number which is a divisor of the denominator of any X is an admissible value of p, and there are also the other restrictions mentioned in § 3.

In the recurring formulæ (i.), (ii.), (iii.), the powers of a and b, viz., a^r and b^r, may be replaced by a_r and b_r, where a_r and b_r satisfy congruences similar to that satisfied by c_r in the same relation.

7. The recurring relation connecting the Eulerian numbers, viz.,

$$E_n-(2n)_2\,E_{n-1}+(2n)_4\,E_{n-2}-\dots+(-1)^{n-1}\,(2n)_{2n-2}\,E_1$$
$$+(-1)_n\,(2n)_{2n}\,E_0=0,$$

is a particular case of (ii.), corresponding to

$$\lambda=0,\quad a=1,\quad b=1,\quad c_0=1,\quad c_2,\,c_4,\,\dots=0,$$
$$X_{2n}=(-1)^n\,E_n.$$

In this case, putting $j=\tfrac{1}{2}(p-1)$ as before, the general congruence theorem becomes

$$(-1)^n\,E_n\equiv(-1)^{n-tj}E_{n-tj},\quad \text{mod } p,$$

which is the Sylvester-Stern relation (§ 1). The Eulerian numbers are integers, and therefore all uneven primes are admissible values of p.

8. The recurring equations (i.), (ii.), (iii.) arise respectively from the expansions

$$\text{(i.)}\quad \frac{c_0+c_1\dfrac{x}{a}+\dfrac{c_2}{2!}\dfrac{x^2}{a^2}+\dfrac{c_3}{3!}\dfrac{x^3}{a^3}+\&\text{c.}}{\lambda+e^{\frac{b}{a}x}} = X_0+X_1x+\frac{X_2}{2!}x^2+\frac{X_3}{3!}x^3+\&\text{c.},$$

$$\text{(ii.)}\quad \frac{c_0+\dfrac{c_2}{2!}\dfrac{x^2}{a^2}+\dfrac{c_4}{4!}\dfrac{x^4}{a^4}+\&\text{c.}}{\lambda+\cosh\dfrac{b}{a}x} = X_0+\frac{X_2}{2!}x^2+\frac{X_4}{4!}x^4+\&\text{c.},$$

$$\text{(iii.)}\quad \frac{c_1\dfrac{x}{a}+\dfrac{c_3}{3!}\dfrac{x^3}{a^3}+\dfrac{c_5}{5!}\dfrac{x^5}{a^5}+\&\text{c.}}{\lambda+\cosh\dfrac{b}{a}x} = X_1x+\frac{X_3}{3!}x^3+\frac{X_5}{5!}x^5+\&\text{c.};$$

or, putting ax for x, from the expansions

$$\text{(i.)}\quad \frac{c_0+c_1x+\dfrac{c_2}{2!}x^2+\dfrac{c_3}{3!}x^3+\&\text{c.}}{\lambda+e^{bx}} = X_0+X_1ax+\frac{X_2}{2!}a^2x^2+\frac{X_3}{3!}a^3x^3+\&\text{c.},$$

$$\text{(ii.)}\quad \frac{c_0+\dfrac{c_2}{2!}x^2+\dfrac{c_4}{4!}x^4+\&\text{c.}}{\lambda+\cosh bx} = X_0+\frac{X_2}{2!}a^2x^2+\frac{X_4}{4!}a^4x^4+\&\text{c.},$$

$$\text{(iii.)}\quad \frac{c_1x+\dfrac{c_3}{3!}x^3+\dfrac{c_5}{5!}x^5+\&\text{c.}}{\lambda+\cosh bx} = X_1ax+\frac{X_3}{3!}a^3x^3+\frac{X_5}{5!}a^5x^5+\&\text{c.}$$

9. If we put $\lambda=0$, $a=1$, and replace b^r by b_r, these expansions become

$$\text{(i.)}\quad \frac{\Sigma_0^\infty \dfrac{c_n}{n!}x^n}{\Sigma_0^\infty \dfrac{b_n}{n!}x^n} = \Sigma_0^\infty \frac{X_n}{n!}x^n,$$

$$\text{(ii.)}\quad \frac{\Sigma_0^\infty \dfrac{c_{2n}}{(2n)!}x^{2n}}{\Sigma_0^\infty \dfrac{b_{2n}}{(2n)!}x^{2n}} = \Sigma_0^\infty \frac{X_{2n}}{(2n)!}x^{2n},$$

$$\text{(iii.)}\quad \frac{\Sigma_0^\infty \dfrac{c_{2n+1}}{(2n+1)!}x^{2n+1}}{\Sigma_0^\infty \dfrac{b_{2n}}{(2n)!}x^{2n}} = \Sigma_0^\infty \frac{X_{2n+1}}{(2n+1)!}x^{2n+1}.$$

The sole condition in (1) is that b_n and c_n should satisfy the congruence

$$u_n \equiv u_{n-t(p-1)}, \quad \text{mod } p.$$

If this condition is fulfilled, X_n also satisfies the same congruence. Similarly, in (ii.), if b_{2n} and c_{2n} satisfy the congruence

$$u_{2n} \equiv u_{2n-t(p-1)}, \quad \text{mod } p,$$

then X_{2n} satisfies the same congruence. In (iii.), if

$$c_{2n+1} \equiv c_{2n+1-t(p-1)}, \quad \text{mod } p,$$

and
$$b_{2n} \equiv b_{2n-t(p-1)}, \quad \text{mod } p,$$

then
$$X_{2n+1} \equiv X_{2n+1-t(p-1)}, \quad \text{mod } p.$$

The expansion (i.) shows that, if we have two series of the form

$$\Sigma_0^\infty \frac{a_n}{n!} x^n,$$

in which the coefficient a_n satisfies the congruence

$$u_n \equiv u_{n-t(p-1)}, \quad \text{mod } p,$$

then X_n, the coefficient of $\dfrac{x^n}{n!}$ in their quotient, satisfies the same congruence.

10. The formulæ (i.), (ii.), (iii.) of § 9 include some very general expansions. Thus (i.) includes the expansion of any quantity of the form

$$\frac{Ae^{ax} + Be^{\beta x} + Ce^{\gamma x} + \dots}{A'e^{a'x} + B'e^{\beta' x} + C'e^{\gamma' x} + \dots},$$

where a, β, γ, ..., a', β', γ', ... are integers; (ii.) includes the expansion of any quantity of the form

$$\frac{\Sigma A \cosh ax}{\Sigma A' \cosh a'x};$$

and (iii.) of the form
$$\frac{\Sigma A \sinh ax}{\Sigma A' \cosh a'x}.$$

If in (ii.) and (iii.) we replace the hyperbolic by circular functions, which merely requires the substitution of xi for x on the right-hand

side, we obtain the expansions

$$\text{(ii.)}\quad \frac{\Sigma A \cos ax}{\Sigma A' \cos a'x} = \Sigma_0^\infty \frac{Y_{2n}}{(2n)!} x^{2n},$$

$$\text{(iii.)}\quad \frac{\Sigma A \sin ax}{\Sigma A' \cos a'x} = \Sigma_0^\infty \frac{Y_{2n+1}}{(2n+1)!} x^{2n+1},$$

where, in (ii.), $\qquad Y_{2n} = (-1)^n X_{2n},$

and in (iii.), $\qquad Y_{2n+1} = (-1)^n X_{2n+1}.$

The Y-coefficients therefore satisfy the respective congruences

$$Y_{2n} \equiv (-1)^{t \cdot \frac{1}{2}(p-1)} Y_{2n+t(p-1)}, \qquad \text{mod } p,$$

$$Y_{2n+1} \equiv (-1)^{t \cdot \frac{1}{2}(p-1)} Y_{2n+1-t(p-1)}, \qquad \text{mod } p.$$

11. These expansions include, besides the Eulerian numbers, several similar sets of coefficients which have been considered in some recent papers in the *Quarterly Journal of Mathematics** and *Messenger of Mathematics*.†

The Eulerian numbers may be regarded as defined by the expansion

$$\text{(i.)}\quad \frac{1}{\cos x} = 1 + \frac{E_1}{2!} x^2 + \frac{E_2}{4!} x^4 + \frac{E_3}{6!} x^6 + \&\text{c.},$$

and the other coefficients I_n, H_n, J_n, ... as defined by

$$\text{(ii.)}\quad \frac{1}{2\cos x + 1} = \tfrac{2}{3}\left\{ I_0 + \frac{I_1}{2!} x^2 + \frac{I_2}{4!} x^4 + \frac{I_3}{6!} x^6 + \&\text{c.} \right\},$$

$$\text{(iii.)}\quad \frac{1}{2\cos x - 1} = \tfrac{2}{3}\left\{ H_0 + \frac{H_1}{2!} x^2 + \frac{H_2}{4!} x^4 + \frac{H_3}{6!} x^6 + \&\text{c.} \right\},$$

$$\text{(iv.)}\quad \frac{2\cos x}{2\cos 2x + 1} = \tfrac{1}{3}\left\{ J_0 + \frac{J_1}{2!} x^2 + \frac{J_2}{4!} x^4 + \frac{J_3}{6!} x^6 + \&\text{c.} \right\},$$

$$\text{(v.)}\quad \frac{\cos x}{\cos 2x} = P_0 + \frac{P_1}{2!} x^2 + \frac{P_2}{4!} x^4 + \frac{P_3}{6!} x^6 + \&\text{c.},$$

$$\text{(vi.)}\quad \frac{\sin x}{\cos 2x} = Q_1 x + \frac{Q_2}{2!} x^3 + \frac{Q_3}{5!} x^5 + \&\text{c.},$$

---

* "On the Bernoullian Function," Vol. XXIX., pp. 1–168.

† "On the Definite Integrals connected with the Bernoullian Function," Vol. XXVI., pp. 151–182, and Vol. XXVII., pp. 20–98.

[redacted]

$$\text{(vii.)} \qquad \frac{\cos^2 x}{\cos 3x} = R_0 + \frac{R_1}{2!}x^2 + \frac{R_2}{4!}x^4 + \frac{R_3}{6!}x^6 + \&c.,$$

$$\text{(viii.)} \qquad \frac{\sin x \cos x}{\cos 3x} = T_1 x + \frac{T_2}{3!}x^3 + \frac{T_3}{5!}x^5 + \&c.^*$$

All these coefficients therefore satisfy a congruence of exactly the same form; *ex. gr.*, taking I_n, we have

$$(-1)^n I_n \equiv (-1)^{n-tj} I_{n-tj}, \quad \text{mod } p,$$

that is,

$$I_n \equiv (-1)^{tj} I_{n-tj}, \quad \text{mod } p.$$

The coefficients are all integers, except the I's and J's, and the I's and J's contain only powers of 3 in the denominator (see the next paper). Thus, except in the case of the I's and J's, p may be any uneven prime, and for the I's and J's the value $p = 3$ is alone excluded.

12. The X-coefficients in the expansions of §§ 8 and 9 include the Bernoullian functions $B_n(x)$ and $A_n'(x)$,† which therefore, in general,

---

* The coefficients S_n defined by the equation

$$\frac{\cos 2x}{\cos 3x} = S_0 + \frac{S_1}{2!}x^2 + \frac{S_2}{4!}x^4 + \frac{S_3}{6!}x^6 + \&c.$$

were considered in *Messenger*, Vol. XXVIII., p. 49. The quantities R_n and S are connected by the relation

$$2R_n + E_n = 3S_n.$$

Both R_n and S_n can be expressed in terms of E_n, the formula being

$$R_n = \frac{3^{2n+1}+1}{4} E_n, \quad S_n = \frac{3^{2n}+1}{2} E_n.$$

The quantities H_n and J_n may be expressed in terms of I_n by the formulæ

$$H_n = (2^{2n+1}+1)\,H_n, \quad J_n = 2\,(2^{2n}+1)\,I_n$$

(see § 24 of the next paper).

† The functions $B_n(x)$ and $A_n'(x)$ may be defined as follows:—

$$B_n(x) = \frac{1}{n}\left\{x^n - \frac{n}{2}x^{n-1} + (n)_2 B_1 x^{n-2} - (n)_4 B_2 x^{n-4} + \dots\right\},$$

the series being continued up to the term involving x or x^2, so that the last term is

$$(-1)^{\frac{1}{2}(n-1)}(n)_{n-1} B_{\frac{1}{2}(n-1)}\,x \quad \text{or} \quad (-1)^{\frac{1}{2}n}(n)_{n-2} B_{\frac{1}{2}(n-2)}\,x^2,$$

according as n is uneven or even;

$$A_n'(x) = \frac{1}{n}\left\{\frac{n}{2}x^{n-1} - (n)_2(2^2-1)B_1 x^{n-2} + (n)_4(2^4-1)B_2 x^{n-4} - \dots\right\},$$

the series being continued up to the term involving x or x^0, so that the last term is

$$(-1)^{\frac{1}{2}(n-1)}(n)_{n-1}(2^{n-1}-1)B_{\frac{1}{2}(n-1)}\,x \quad \text{or} \quad (-1)^{\frac{1}{2}n}(2^n-1)B_{\frac{1}{2}n},$$

according as n is uneven or even (*Quarterly Journal*, Vol. XXIX., pp. 7, 94). In these formulæ B_r denotes the r^{th} Bernoullian number.

These definitions have been given at full length, as there are several slightly

satisfy the congruence

$$u_n \equiv u_{n-t(p-1)}, \quad \text{mod } p.$$

It will be shown that the only inadmissible values of p are the divisors of the denominator of x. The modulus p, as in the preceding sections, is restricted to uneven primes.

Taking first the function $A'_n(x)$, we have

$$\frac{e^{bx}}{e^{ax}+1} = A'_1\left(\frac{b}{a}\right) + aA'_2\left(\frac{b}{a}\right)x + a^2A'_3\left(\frac{b}{a}\right)\frac{x^2}{2!} + a^3A'_4\left(\frac{b}{a}\right)\frac{x^3}{3!} + \&c.,^*$$

which is included in formula (i.) of § 8.

It follows therefore that, if a and b be any positive integers, which we may take to be prime to one another, then

$$a^{n-1}A'_n\left(\frac{b}{a}\right) \equiv a^{n-1-t(p-1)}A'_{n-t(p-1)}\left(\frac{b}{a}\right), \quad \text{mod } p.$$

Since $$a^n \equiv a^{n-t(p-1)}, \quad \text{mod } p,$$

we find therefore that, if p is not a divisor of a, then

$$A'_n\left(\frac{b}{a}\right) \equiv A'_{n-t(p-1)}\left(\frac{b}{a}\right), \quad \text{mod } p.$$

It may be remarked that, if a and b are integers, the quantity $a^{n-1}A'_n\left(\frac{b}{a}\right)$ is necessarily an integer, except for a denominator containing powers of 2; for, putting

$$\alpha_{n-1} = a^{n-1}A'_n\left(\frac{b}{a}\right),$$

the expansion is

$$\frac{e^{bx}}{e^{ax}+1} = \alpha_0 + \alpha_1 x + \frac{\alpha_2}{2!}x^2 + \frac{\alpha_3}{2!}x^3 + \&c.,$$

---

differing forms of the Bernoullian function, each of which is specially adapted to some of its applications. Thus for very many purposes it is convenient to use a function $A_n(x)$ in place of $B_n(x)$ as just defined, where $A_{2n+1}(x)$ is the same as $B_{2n+1}(x)$, but

$$A_{2n}(x) = B_{2n}(x) + (-1)^{n-1}\frac{B_n}{2n}.$$

It is also frequently convenient to make a further modification, and use the functions $V_n(x)$ and $U_n(x)$, where

$$V_n(x) = nA_n(x) \quad \text{and} \quad U_n(x) = nA'_n(x)$$

(*Quarterly Journal*, *loc. cit.*, p. 115).

* This formula may be derived from p. 94 of Vol. xxix. of the *Quarterly Journal*.

which gives the recurring relation

$$2\alpha_n+(n)_1 a\alpha_{n-1}+(n)_2 a^2\alpha_{n-2}+\ldots+(n)_n a^n\alpha_0 = b^n,$$

where $\alpha_0=\frac{1}{2}$. Thus α_n must be of the form $\dfrac{\text{integer}}{\text{power of } 2}$.

This recurring relation shows also that, if a be prime to b, the numerator of α_n cannot be divisible by a.

Since the congruence

$$a^{n-1}A_n'\left(\frac{b}{a}\right)\equiv a^{m-1}A_m'\left(\frac{b}{a}\right), \quad \text{mod } p,$$

where $$n-m=t\,(p-1),$$

holds good for all (uneven) values of p, and since $a^nA_n'\left(\dfrac{b}{a}\right)$ contains only a power of 2 in the denominator, and cannot contain a as a factor in the numerator, we see that nothing exceptional occurs when p is a divisor of a, for in this case the congruence does not in general reduce to $0\equiv 0$, mod p.

As an example, take the formula

$$3^{2n}A_{2n+1}'(\tfrac{1}{3})=(-1)^n\frac{H_n}{3},\text{*}$$

where H_n is the same as in the second expansion in § 11.

It follows therefore that $\dfrac{H_n}{3}$ is an integer, except for powers of 2 in the denominator (which, as a fact, do not occur), and that the numerator of $\dfrac{H_n}{3}$ $\left(\text{that is } \dfrac{H_n}{3} \text{ itself}\right)$ is not divisible by 3, and we have, taking $p=3$,

$$\frac{H_n}{3}\equiv(-1)^t\frac{H_{n-t}}{3}, \quad \text{mod } 3.$$

13. The proof just given of the congruence

$$a^{n-1}A_n'\left(\frac{b}{a}\right)\equiv a^{m-1}A_m'\left(\frac{b}{a}\right), \quad \text{mod } p,$$

---

* *Quarterly Journal*, Vol. XXIX., p. 107, or *Messenger*, Vol. XXVI., p. 178.

applies to all values of the suffix n, even or uneven; it is, however, interesting to give the expansion formulæ in which the suffixes are all uneven or all even, and from which the congruence-theorem may be derived separately for uneven and for even suffixes. These expansion formulæ are

$$\tfrac{1}{2}\frac{\cosh(2b-a)x}{\cosh ax} = A_1'\left(\frac{b}{a}\right)+a^2A_3'\left(\frac{b}{a}\right)\frac{(2x)^2}{2!}+a^4A_5'\left(\frac{b}{a}\right)\frac{(2x)^4}{4!}+\&c.,$$

$$\tfrac{1}{2}\frac{\sinh(2b-a)x}{\cosh ax} = aA_2'\left(\frac{b}{a}\right)2x+a^3A_4'\left(\frac{b}{a}\right)\frac{(2x)^3}{3!}$$
$$+a^5A_6'\left(\frac{b}{a}\right)\frac{(2x)^5}{5!}+\&c.,^*$$

which are included respectively in (ii.) and (iii.) of § 8.

14. Passing now to the function $B_n(x)$, we have

$$\frac{e^{bx}-1}{e^{ax}-1} = \frac{b}{a}+aB_2\left(\frac{b}{a}\right)x+a^2B_3\left(\frac{b}{a}\right)\frac{x^2}{2!}+a^3B_4\left(\frac{b}{a}\right)\frac{x^3}{3!}+\&c.,^\dagger$$

and, by dividing both numerator and denominator by e^x-1, the left-hand side becomes

$$\frac{e^{(b-1)x}+e^{(b-2)x}+\ldots+e^x+1}{e^{(a-1)x}+e^{(a-2)x}+\ldots+e^x+1}.$$

This form is included in (i.) of § 9, being a special case of the first form noticed in § 10; so that

$$a^{n-1}B_n\left(\frac{b}{a}\right)\equiv a^{m-1}B_m\left(\frac{b}{a}\right),\quad \text{mod } p,$$

if

$$n-m=t(p-1);$$

and therefore, a and b being prime to each other, and p not being a divisor of a,

$$B_n\left(\frac{b}{a}\right)\equiv B_{n-t(p-1)}\left(\frac{b}{a}\right),\quad \text{mod } p.$$

It is easy to see that $B_n\left(\dfrac{b}{a}\right)$ can contain only powers of a in the denominator, for, putting

$$\mu_r = 1^r+2^r+\ldots+(a-1)^r,$$
$$\nu_r = 1^r+2^r+\ldots+(b-1)^r,$$

* *Quarterly Journal*, Vol. XXIX., p. 107.
† *Ib.*, Vol. XXIX., p. 7.

and $$\beta_{r-1} = a^{r-1} B_r\left(\frac{b}{a}\right),$$

the expansion formula gives

$$\frac{b+\nu_1 x+\dfrac{\nu_2}{2!}x^2+\dfrac{\nu_3}{3!}x^3+\&c.}{a+\mu_1 x+\dfrac{\mu_2}{2!}x^2+\dfrac{\mu_3}{3!}x^3+\&c.} = \frac{b}{a}+\beta_1 x+\frac{\beta_2}{2!}x^2+\frac{\beta_3}{3!}x^3+\&c.,$$

which leads to the recurring relation

$$a\beta_n+(n)_1\mu_1\beta_{n-1}+(n)_2\mu_2\beta_{n-2}+\ldots+(n)_n\mu_n\frac{b}{a}=\nu_n.$$

Since the μ's and ν's are necessarily integers, this relation shows that β_n can contain only powers of a in the denominator.

15. The expansion formulæ in which the suffixes of the B's are all uneven or all even are

$$\tfrac{1}{2}\,\frac{\sinh(2b-a)x}{\sinh ax} = \frac{2b-a}{2a}+a^3B_3\left(\frac{b}{a}\right)\frac{(2x)^2}{2!}+a^4B_5\left(\frac{b}{a}\right)\frac{(2x)^4}{4!}+\&c.,$$

$$\tfrac{1}{2}\,\frac{\cosh(2b-a)x-\cosh ax}{\sinh ax} = aB_2\left(\frac{b}{a}\right)2x+a^3B_4\left(\frac{b}{a}\right)\frac{(2x)^3}{3!}+\&c.^*$$

If r is a positive integer, $\sinh rx$ contains $\sinh x$ as a factor, the other factor being

$$1+2\cosh 2x+2\cosh 4x+\ldots+2\cosh(r-1)x, \quad \text{if } r \text{ is uneven,}$$

and $$2\cosh x+2\cosh 3x+\ldots+2\cosh(r-1)x, \quad \text{if } r \text{ is even.}$$

Thus the left-hand side of the first equation is of the form

$$\frac{\Sigma A\cosh \alpha x}{\Sigma A'\cosh \alpha' x};$$

and, since $\cosh(2b-a)x-\cosh ax = 2\sinh bx\sinh(b-a)x$,

the left-hand side of the second equation is of the form

$$\frac{\Sigma A\sinh \alpha x}{\Sigma A'\sinh \alpha' x}.$$

The two expansion formulæ are therefore included respectively in (ii.) and (iii.) of § 9.

---

* *Quarterly Journal*, Vol. xxix., pp. 5 and 6, or p. 119.

16. In connexion with these formulæ it may be remarked that the expansion

$$\frac{\sinh cx}{\sinh bx} = X_0 + \frac{X_2}{2!}a^2x^2 + \frac{X_4}{4!}a^4x^4 + \frac{X_6}{6!}a^6x^6 + \&c.$$

gives rise to the recurring relation

$$(2n+1)\,a^{2n}bX_{2n} + (2n+1)_3\,a^{2n-2}b^3X_{2n-2} + \ldots + (2n+1)_{2n-1}\,a^2b^{2n-1}X_2$$
$$+ (2n+1)_{2n+1}\,b^{2n+1}X_0 = c^{2n+1}.$$

This relation is of the same kind as (i.), (ii.), (iii.) of § 6, and it is easy to see that the reasoning employed in §§ 2 and 4 holds good also in the case of this formula, and shows that X_{2n} satisfies the congruence

$$X_{2n} \equiv X_{2n-t(p-1)}, \quad \text{mod } p.$$

It would seem, however, that this result could not be of any practical value, since X_{2n}, as calculated from the above recurring relation, might contain in the denominator any uneven numbers up to $2n+1$, so that there might be no admissible value of p; but this is not the case, for the left-hand side, on dividing both numerator and denominator by $\sinh x$, becomes the quotient of expansions which are of the forms $2\Sigma \cosh (2r+1)\,x$ or $1+2\Sigma \cosh 2rx$, and therefore the expansion is included in (ii.) of § 9. We see, too, by forming the recurring equation corresponding to this form of the left-hand side, that $a^{2n}X_{2n}$ can contain only powers of b in the denominator. Thus the only values of p to be excluded are those which are divisors of a and b.

If we put $a = 1$, so that

$$\frac{\sinh cx}{\sinh bx} = X_0 + \frac{X_2}{2!}x^2 + \frac{X_4}{4!}x^4 + \frac{X_6}{6!}x^6 + \&c.,$$

then
$$X_{2n} \equiv X_{2n-t(p-1)}, \quad \text{mod } p,$$

for all uneven values of p that are not divisors of b.

17. We may obtain this result also in another manner; for, from the first formula in § 15, we have

$$\tfrac{1}{2}\frac{\sinh cx}{\sinh bx} = \frac{c}{2b} + b^2B_3\left(\frac{b+c}{2b}\right)\frac{(2x)^2}{2!} + b^4B_5\left(\frac{b+c}{2b}\right)\frac{(2x)^4}{4!} + \&c.$$

Comparing this expansion with that just written, we have

$$X_{2n} = 2^{2n+1} b^{2n} B_{2n+1} \left(\frac{b+c}{2b}\right);$$

and therefore, by § 14, if p is not a divisor of b,

$$X_{2n} \equiv X_{2n-t(p-1)}, \quad \text{mod } p.$$

18. In the preceding sections p has always been supposed to be an uneven prime, and it now remains to consider the case of $p = 2$. The residues of the X-coefficients with respect to modulus 2 may be easily determined, in the case of any of the expansions, by means of the recurring formulæ.

Consider first the recurring formula (i.) of § 6, in which, writing λ' for $\lambda+1$, and putting $n = 1, 2, 3, \ldots$, we have

$$\begin{aligned} &\lambda' a X_1 + b X_0 = c_1, \\ &\lambda' a^2 X_2 + 2_1 ab X_1 + b^2 X_0 = c_2, \\ &\lambda' a^3 X_3 + 3_1 a^2 b X_2 + 3_2 ab^2 X_1 + b^3 X_0 = c_3, \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \end{aligned}$$

Suppose $\lambda' \equiv 1$, mod 2, and let a and b be uneven integers. The assumption $\lambda' \equiv 1$, mod 2, excludes the case of λ' having 2 as a divisor in the denominator. It is supposed that $c_1, c_2, c_3, \ldots$ are all $\equiv 0$ or all $\equiv 1$, mod 2; so that none of them can have 2 as a divisor in the denominator. The quantity X_0 (which is the value of the function expanded, when x is put $= 0$) is also supposed not to have a denominator divisible by 2.*

It will now be shown that $X_1, X_2, X_3, \ldots$ are all $\equiv 0$, mod 2, if X_0, $c_1, c_2, c_3, \ldots$ are all $\equiv 1$, or all $\equiv 0$, mod 2; but that $X_1, X_2, X_3, \ldots$ are all $\equiv 1$, mod 2, if $X_0 \equiv 1$ and $c_1, c_2, c_3, \ldots$ are all $\equiv 0$, mod 2, or if $X_0 \equiv 0$ and $c_1, c_2, c_3, \ldots$ are all $\equiv 1$, mod 2.

I. Let $X_0 \equiv 1$, and $c_1, c_2, c_3, \ldots \equiv 1$, mod 2.

The recurring formulæ give

$$\begin{aligned} &\lambda' a X_1 \equiv 1 - 1 \equiv 0, \text{ mod } 2, \quad \text{so that } X_1 \equiv 0, \text{ mod } 2; \\ &\lambda' a^2 X_2 \equiv 1 - 1 - 0 \equiv 0, \text{ mod } 2, \quad \text{so that } X_2 \equiv 0, \text{ mod } 2; \\ &\lambda' a^3 X_3 \equiv 1 - 1 - 0 - 0 \equiv 0, \text{ mod } 2, \quad \text{so that } X_3 \equiv 0, \text{ mod } 2; \end{aligned}$$

and so on.

* Since $c_0 = \lambda' X_0 \equiv X_0$, mod 2, we may use c_0 in place of X_0 throughout.

II. Let $X_0 \equiv 1$, and $c_1, c_2, c_3, \ldots \equiv 0$, mod 2.

In this case

$\lambda' a X_1 \equiv 0-1 \equiv 1$, mod 2; so that $X_1 \equiv 1$, mod 2.

$\lambda' a^2 X_2 \equiv 0-(b^2+2_1 ab) \equiv a^2-(a+b)^2 \equiv 1$, mod 2;

so that $X_2 \equiv 1$, mod 2,

$\lambda' a^3 X_3 \equiv a^3-(a+b)^3 \equiv 1$, mod 2; so that $X_3 \equiv 1$, mod 2,

and so on; since, a and b being uneven, $a^n-(a+b)^n$ is necessarily uneven.

III. Let $X_0 \equiv 0$, and $c_1, c_2, c_3, \ldots \equiv 1$, mod 2.

In this case

$\lambda' a X_1 \equiv 1-0 \equiv 1$, mod 2; so that $X_1 \equiv 1$, mod 2,

$\lambda' a^2 X_2 \equiv 1-0-2_1 ab \equiv 1+a^2+b^2-(a+b)^2 \equiv 1$, mod 2;

so that $X_2 \equiv 1$, mod 2,

$\lambda' a^3 X_3 \equiv 1+a^3+b^3-(a+b)^3 \equiv 1$, mod 2; so that $X_3 \equiv 1$, mod 2,

and so on.

IV. Let $X_0 \equiv 0$ and $c_1, c_2, c_3, \ldots \equiv 0$, mod 2.

In this case

$\lambda' a X_1 \equiv 0-0 \equiv 0$, mod 2; so that $X_1 \equiv 0$, mod 2,

$\lambda' a^2 X_2 \equiv 0$, mod 2; so that $X_2 \equiv 0$, mod 2,

and so on.

If b is even, the general recurring formula (i.) shows that

$$X_n \equiv c_n, \text{ mod } 2;$$

so that, whether the residue of X_0 be 1 or 0, mod 2, the X's $\equiv$ the c's, mod 2. We may always regard a as uneven, and, in fact, there is no loss of generality in the expansion-formula (§ 8) by putting $a = 1$.

19. Exactly the same reasoning holds good with respect to the recurring formula (ii.) of § 6, viz., we have

$$\lambda' a^2 X_2 + b^2 X_0 = c_2,$$
$$\lambda' a^4 X_4 + 4_2 a^2 b^2 X_2 + b^4 X_0 = c_4,$$
$$\lambda' a^6 X_6 + 6_2 a^4 b^2 X_4 + 6_4 a^2 b^4 X_2 + b^6 X_0 = c_6,$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots,$$

and, under the same conditions as those expressed at the beginning of the preceding section, viz., $\lambda' \equiv 1$, mod 2, a and b uneven, and

$c_2, c_4, c_6, \ldots$ all $\equiv 1$ or all $\equiv 0$, mod 2, we find that X_2, X_4, X_6, are all $\equiv 0$, mod 2, if X_0, c_2, c_4, c_6, are all $\equiv 1$, or all $\equiv 0$, mod 2, but that X_2, X_4, X_6, ... are all $\equiv 1$, mod 2, if $X_0 \equiv 1$, and $c_2, c_4, c_6, \ldots$ are all $\equiv 0$, mod 2, or if $X_0 \equiv 0$, and $c_2, c_4, c_6, \ldots$ are all $\equiv 1$, mod 2.

In proving these results we have, in Case II.,

$$\lambda' a^{2n} X_{2n} \equiv a^{2n} + b^{2n} - \frac{(a+b)^{2n} + (a-b)^{2n}}{2}, \quad \text{mod } 2,$$

the right-hand side of which is necessarily even.

If b is even, we have evidently

$$X_{2n} \equiv c_{2n}, \quad \text{mod } 2.$$

20. In the case of the recurring formula (iii.) of § 6, we have

$$\lambda' a^3 X_3 + 3_2 ab^2 X_1 = c_3,$$

$$\lambda' a^5 X_5 + 5_2 a^3 b^2 X_3 + 5_4 ab^4 X_1 = c_5,$$

$$\lambda' a^7 X_7 + 7_2 a^5 b^2 X_5 + 7_4 a^3 b^4 X_3 + 7_6 ab^6 X_1 = c_7.$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

and supposing $\lambda' \equiv 1$, mod 2, a and b being uneven integers, and $c_3, c_5, c_7, \ldots$ all $\equiv 1$ or $\equiv 0$, mod 2, and separating the four cases

I. *$X_1 \equiv 1$, and $c_3, c_5, c_7, \ldots$ all $\equiv 1$, mod 2;

II. $X_1 \equiv 1$, and $c_3, c_5, c_7, \ldots$ all $\equiv 0$, mod 2;

III. $X_1 \equiv 0$, and $c_3, c_5, c_7, \ldots$ all $\equiv 1$, mod 2;

IV. $X_1 \equiv 0$, and $c_3, c_5, c_7, \ldots$ all $\equiv 0$, mod 2;

we find that X_3, X_5, X_7, ... are all $\equiv 0$, mod 2, if $X_1, c_3, c_5, c_7, \ldots$ are all $\equiv 1$, or all $\equiv 0$, mod 2, but that $X_3, X_5, X_7, \ldots$ are all $\equiv 1$, mod 2, if $X_1 \equiv 1$ and $c_3, c_5, c_7, \ldots$ are all $\equiv 0$, mod 2, or if $X_1 \equiv 0$ and $c_3, c_5, c_7, \ldots$ are all $\equiv 1$, mod 2.

In proving these results, we have, in Case II.,

$$\lambda' a^{2n+1} X_{2n+1} \equiv \frac{(a+b)^{2n+1} + (a-b)^{2n+1}}{2} - a^{2n+1}, \quad \text{mod } 2,$$

the right-hand side of which is necessarily even; in Case III. the extra term 1 occurs, but the term in X_1 which $\equiv 1$, mod 2, is omitted, so that in this case also

$$X_{2n+1} \equiv 1, \qquad \text{mod } 2.$$

If b is even, we have $\qquad X_{2n+1} \equiv c^{2n+1}, \quad \text{mod } 2.$

* Since $ac = \lambda' X_1$, we have $c_1 \equiv X_1$, mod 2, and we may therefore use c_1 in place of X_1 throughout.

21. In the eight expansions of § 11, we have respectively

(i.) $\lambda' = 1$. $a = 1$: $b = 1$: $c_2, c_4, \ldots = 0$; $X_0 = 1$ $[X_{2n} = (-1)^n E_n]$.

(ii.) $\lambda' = \frac{3}{2}$: $a = 1$: $b = 1$; $c_2, c_4, \ldots = 0$; $X_0 = \frac{1}{2}$ $[X_{2n} = (-1)^n I_n]$.

(iii.) $\lambda' = \frac{1}{2}$: $a = 1$; $b = 1$; $c_2, c_4, \ldots = 0$; $X_0 = \frac{3}{2}$ $[X_{2n} = (-1)^n H_n]$.

(iv.) $\lambda' = \frac{3}{2}$; $a = 1$: $b = 2$; $c_2, c_4, \ldots = 1$; $X_0 = 2$ $[X_{2n} = (-1)^n J_n]$.

(v.) $\lambda' = 1$; $a = 1$: $b = 2$; $c_2, c_4, \ldots = 1$; $X_0 = 1$ $[X_{2n} = (-1)^n P_n]$.

(vi.) $\lambda' = 1$: $a = 1$; $b = 2$; $c_3, c_5, \ldots = 1$; $X_1 = 1$

$[X_{2n-1} = (-1)^{n-1} Q_n]$.

(vii.) $\lambda' = 1$; $a = 1$: $b = 3$; $c_2, c_4, \ldots \equiv 0, \text{ mod } 2$; $X_0 = 1$

$[X_{2n} = (-1)^n R_n]$.

(viii.) $\lambda' = 1$: $a = 1$; $b = 3$; $c_3, c_5, \ldots \equiv 0, \text{ mod } 2$; $X_1 = 1$

$[X_{2n-1} = (-1)^{n-1} T_n]$.

These expansions are included in (i.) of § 8, except the sixth and eighth, which belong to (iii.) of § 8.

The theorems of the three preceding sections are applicable to the first, and to the last four, of these expansions, and show that in these cases the coefficients $\equiv 1$, mod 2. Since these coefficients are all integers, we thus see that E_n, P_n, Q_n, R_n, T_n are all uneven numbers.

22. When λ' and X_0 contain powers of 2 in the denominator, as in (ii.), (iii.), (iv.), it is easy to determine in each case the residues of the X's with respect to modulus 2 by means of the recurring relation. For example, in (ii.), where $\lambda' = \frac{3}{2}$, $X_0 = \frac{1}{2}$, and $c_2, c_4, \ldots = 0$, we see at once, from the recurring relation, that

$$3X_{2n}+1 \equiv 0, \text{ mod } 2, \quad \text{so that } X_{2n} \equiv 1, \text{ mod } 2;$$

and similarly, in (iii.),

$$X_{2n}+3 \equiv 0, \text{ mod } 2, \quad \text{so that } X_{2n} \equiv 1, \text{ mod } 2;$$

in (iv.), $$3X_{2n} \equiv 0, \text{ mod } 2, \quad \text{so that } X_{2n} \equiv 0, \text{ mod } 2.$$

23. The fact that in each of the expansions the coefficients when integral (such as the E's, the P's, &c.) end in one or other of two

digits, or all end in the same digit,* is explained by the consideration of their residues with respect to the modulus 2 and the modulus 5; for, if all the coefficients have the same residue to modulus 2, and if all the alternate coefficients have the same residue to modulus 5, it is evident that all the alternate coefficients must have the same residue to modulus 10; so that the difference between any coefficient and the coefficient next but one to it must, if integral, be a multiple of 10.

Putting $p = 5$ in the general congruence formulæ of § 6, we have,

when X_n is defined by (i.) of § 8, $X_n \equiv X_{n-4t}$, mod 5,

" X_{2n} " " (ii.) " $X_{2n} \equiv X_{2n-4t}$, "

" X_{2n+1} " " (iii.) " $X_{2n+1} \equiv X_{2n+1-4t}$, "

Therefore, putting $X_{2n} = (-1)^n Z_n$ in (ii.), so that the expansion is

$$\text{(ii.)} \qquad \frac{c_0 - \frac{c_2}{2!}x^2 + \frac{c_4}{4!}x^4 - \&c.}{\lambda + \cos bx} = Z_0 + \frac{Z_1}{2!}x^2 + \frac{Z_2}{4!}x^4 + \&c.,$$

we have $$Z_n \equiv Z_{n-2t}, \quad \text{mod } 5;$$

and, putting $X_{2n-1} = (-1)^{n-1} Z_n$ in (iii.), so that the expansion is

$$\text{(iii.)} \qquad \frac{c_1 x - \frac{c_3}{3!}x^3 + \frac{c_5}{5!}x^5 - \&c.}{\lambda + \cos bx} = Z_1 x + \frac{Z_2}{3!}x^3 + \frac{Z_3}{5!}x^5 + \&c.,$$

we have $$Z_n \equiv Z_{n-2t}, \quad \text{mod } 5.$$

Thus, in both (ii.) and (iii.), the difference between two alternate Z's $\equiv 0$, mod 5, and when the Z's are all congruent to one another, mod 2, this difference must be $\equiv 0$, mod 10.

It will be noticed that, by putting $p = 3$, we see that the sum of any two consecutive Z's, both in (ii.) and (iii.), $\equiv 0$, mod 3, except, of course, when 3 occurs as a factor in the denominator of any of the Z's.

---

* The Eulerian numbers end in 1 and 5 alternately, the H's all end in 3, the P's end in 3 and 7 alternately, the Q's all end in 1, the R's end in 7 and 5 alternately, and the T's in 1 and 3 alternately (*Quarterly Journal*, Vol. XXIX., pp. 63, 66, 71, 76; or *Messenger*, Vol. XXVIII., p. 51). The endings of the I's and J's are considered in the next paper (§§ 17, 18, and 30). A table of all the coefficients up to $n = 5$ was given in the *Messenger*, Vol. XXVIII., p. 51. More extensive tables of I_n, H_n, J_n (up to $n = 13$) are contained in the next paper (§§ 14, 25).

24. In Vol. XXVIII., pp. 75, 76, of the *Messenger*, it was shown by means of recurring formula that, if $2n-1$ is prime,

$$\left.\begin{array}{r} E_n \equiv Q_n \equiv T_n \equiv (-1)^{n-1} \\ H_n \equiv P_n \equiv (-1)^{n-1} 3 \end{array}\right\}, \quad \bmod 2n-1,$$

with similar congruences relating to other coefficients, and it was stated that these results might be extended to the modulus $2n-3$, if prime, to the modulus $2n-5$, if prime, &c., and indeed to any prime modulus. It is this extension which has formed the subject of the present paper.

25. It may be remarked that, by putting $n = p-1$, where p is any uneven prime, in the recurring equation (i.) of § 6, we obtain a congruence connecting $X_0, X_1, X_2, \ldots, X_{p-1}$, mod p, viz., we have

$$(\lambda+1)\, a^{p-1} X_{p-1} + (p-1)_1\, a^{p-2} b X_{p-2} + (p-1)_2 a^{p-3} b^2 X_{p-3} + \ldots$$
$$\ldots + (p-1)_{p-1} b^{p-1} X_0 = c_{p-1},$$

giving

$$(\lambda+1)\, a^{p-1} X_{p-1} - a^{p-2} b X_{p-2} + a^{p-3} b^2 X_{p-3} - \ldots + b^{p-1} X_0 \equiv c_{p-1}, \quad \bmod p.$$

Similarly, by putting $2n = p-1$ in the relation (ii.) of § 6, we find

$$(\lambda+1)\, a^{p-1} X_{p-1} + a^{p-3} b^2 X_{p-3} + a^{p-5} b^4 X_{p-5} + \ldots\ b^{p-1} X_0 \equiv c_{p-1}, \quad \bmod p.$$

In the case of the Eulerian numbers this congruence gives, since $E_0 = E_1$,

$$E_{\frac{1}{2}(p-1)} - E_{\frac{1}{2}(p-3)} + E_{\frac{1}{2}(p-5)} - \ldots + (-1)^{\frac{1}{2}(p-5)} E_2 \equiv 0, \quad \bmod p\,;$$

for the I-numbers, since $\qquad I_0 - I_1 = \frac{1}{6}$,

it gives

$$\tfrac{3}{2} I_{\frac{1}{2}(p-1)} - I_{\frac{1}{2}(p-3)} + I_{\frac{1}{2}(p-5)} - \ldots + (-1)^{\frac{1}{2}(p-5)} I_2 \equiv (-1)^{\frac{1}{2}(p-3)} \tfrac{1}{6}, \quad \bmod p,$$

and so on.

26. In the formula (iii.) of § 6 we cannot put $2n+1 = p-1$, but, by putting $2n+1 = p-2$, we find

$$(\lambda+1)\, a^{p-2} X_{p-2} + 3a^{p-4} b^2 X_{p-4} + 5a^{p-6} b^4 X_{p-6} + \ldots$$
$$\ldots + (p-2)\, a b^{p-3} X_1 \equiv c_{p-2}, \bmod p.$$

By putting $n = p-2$ in (i.), we obtain a formula of the same character connecting $X_0, X_1, X_2, \ldots, X_{p-2}$, viz.,

$$(\lambda+1)\, a^{p-2} X_{p-2} - 2a^{p-3} b X_{p-3} + 3a^{p-4} b^2 X_{p-4} - \ldots$$
$$\ldots - (p-1)\, b^{p-2} X_0 \equiv c_{p-2}, \quad \bmod p.$$

Other relations may be derived from the recurring formulæ in the same manner by putting n or $2n = p-2$, &c., but the numerical coefficients are less simple.

[27. Since this paper was communicated to the Society, my attention has been called to a paper by Lucas in Vol. VI.* of the *Bulletin de la Société Mathématique de France*, in which the use of a recurring series to prove the congruence property of the Eulerian numbers is indicated. Denoting the r^{th} Eulerian number by $(-1)^r E_{2r}$, Lucas shows, by putting $2n = p-1$ in the recurring relation connecting the first n Eulerian numbers, that

$$E_{p-1} + E_{p-3} + E_{p-5} + \ldots + E_2 + E_0 \equiv 0, \quad \text{mod } p,$$

which is the formula obtained in § 25, and, by putting $2n = p+1$, $p+3$, $p+5$, ..., he shows that $E_{p+1} \equiv E_2$, $E_{p+3} \equiv E_4$, $E_{p+5} \equiv E_6$, ..., mod p, whence it is inferred that generally $E_{2n} \equiv E_{2n+k(p-1)}$, mod p. The extension, however, to the general theorem seems to me to require a definite investigation of the same kind as that given in §§ 2–4 of the present paper. Lucas points out that a similar congruence property would also hold good with respect to the coefficients in the expansion of $\left(\dfrac{2}{e^x+e^{-x}}\right)^a$, and, with restrictions, to any function of e^x. These forms are included in the general expressions of § 10.

I may mention that, since writing this paper, I have proved by means of the theorem in § 5 that, B_n denoting the n^{th} Bernoullian number,

$$\frac{B_n}{n} \equiv (-1)^{tj} \frac{B_{n-tj}}{n-tj}, \quad \text{mod } p,$$

where, as in § 1, $j = \frac{1}{2}(p-1)$, and p is any uneven prime, such that $p-1$ is not a divisor of $2n$. This theorem and its consequences have been considered in two papers in the *Messenger*† and one in the *Quarterly Journal*.‡ In this last paper the theorem in § 5 is proved separately in detail.]

* "Sur les congruences des nombres eulériens et des coefficients différentiels des fonctions trigonométriques, suivant un module premier," pp. 49–54.

† "Fundamental Theorems relating to the Bernoullian Numbers," Vol. XXIX., p. 49 and p. 128.

‡ "A Congruence Theorem relating to the Bernoullian Numbers," Vol. XXXI., p. 253.

On a Set of Coefficients analogous to the Eulerian Numbers. **By J. W. L. Glaisher. Communicated June 8th, 1899. Received August 30th, 1899.**

1. The object of the present paper is to investigate the properties of a system of numbers I_n which are very similar in character to the Eulerian numbers E_n. As they are included in the class of coefficients considered in the preceding paper, they necessarily satisfy the same general congruence theorems, and they also possess special congruence properties in the case when the modulus is a power of 3. Unlike the Eulerian numbers, some of the I-numbers have denominators, which, however, consist only of powers of 3.

2. The I-numbers, which have been referred to in § 11 of the preceding paper, may be defined by the expansion

$$\frac{1}{e^x+e^{-x}+1} = \tfrac{2}{3}\left\{ I_0 - \frac{I_1}{2!}x^2 + \frac{I_2}{4!}x^4 - \frac{I_3}{6!}x^6 + \&c. \right\},*$$

corresponding to the definition

$$\frac{1}{e^x+e^{-x}} = \tfrac{1}{2}\left\{ E_0 - \frac{E_1}{2!}x^2 + \frac{E_2}{4!}x^4 - \frac{E_3}{6!}x^6 + \&c. \right\}$$

of the Eulerian numbers.

From these equations we obtain the fundamental recurring formulæ

$$\tfrac{3}{2} I_n - (2n)_2 I_{n-1} + (2n)_4 I_{n-2} - \ldots + (-1)^{n-1}(2n)_{2n-2} I_1 + (-1)^n I_0 = 0,$$

$$E_n - (2n)_2 E_{n-1} + (2n)_4 E_{n-2} - \ldots + (-1)^{n-1}(2n)_{2n-2} E_1 + (-1)^n E_0 = 0,$$

where $I_0 = \frac{1}{2}$, $E_0 = 1$, and, as in the preceding paper, $(n)_r$ denotes the number of combinations of n things taken r together.

---

* *Quarterly Journal*, Vol. XXVIII., p. 157, or Vol. XXIX., p. 35; *Messenger*, Vol. XXVI., p. 165.

The recurring relation

$$(2n+1)\,I_n-(2n+1)_3\,3^2I_{n-1}+\dots+(-1)^{n-1}\,(2n+1)_{2n-1}\,3^{2n-2}I_1$$
$$+(-1)^n\,3^{2n}\,I_0=(-1)^n\,(2^{2n}-\tfrac{1}{2})^*$$

will also be used.

3. The first recurring equation

$$\tfrac{3}{2}I_n-(2n)_2\,I_{n-1}+(2n)_4\,I_{n-2}-\dots+(-1)^{n-1}\,(2n)_{2n-2}\,I_1+(-1)^n\,I_0=0$$

shows that the I-numbers can contain only powers of 3 in the denominator. It will now be shown that the power of 3 which forms the denominator of I_n is the highest power of 3 by which $2n+1$ is divisible.

4. Putting $$G_n=(2n+1)\,I_n,$$

the second recurring I-formula may be written

$$G_n=\frac{(2n+1)\,2n}{2.3}\,3^2G_{n-1}-\frac{(2n+1)\,2n\,(2n-1)(2n-2)}{2.3.4.5}\,3^4G_{n-2}+\dots$$
$$\dots+(-1)^n\,\frac{(2n+1)\,2n\,\dots\,4}{2.3\,\dots\,(2n-1)}\,3^{2n-2}G_1+(-1)^{n-1}\left\{\frac{3^{2n}+1}{2}-2^{2n}\right\}.$$

In this equation all the terms must be integers, except for powers of 3, which may occur as denominators through the I's. In considering therefore whether the terms are integral, we need only consider the possibility of the occurrence of 3 or powers of 3 in the denominators of the terms.

If we multiply the coefficients

$$\frac{(2n+1)\,2n}{2.3},\quad\frac{(2n+1)\,2n\,(2n-1)(2n-2)}{2.3.4.5},\quad\dots$$

---

* This relation was given in Vol. XXIX., p. 56, of the *Quarterly Journal*. It can be obtained by equating the coefficients of x^{2n+1} in

$$\sinh 3x\times\frac{1}{1+2\cosh x}=\sinh 2x-\sinh x.$$

Other recurring relations are given in the *Quarterly Journal*, *loc. cit.*, pp. 38, 39, 43, 56.

[redacted]

by $2n+2$, we obtain the quantities $(2n+2)_3$, $(2n+2)_5$, ..., which are necessarily integral. If, therefore, $2n+2$ is not divisible by 3, that is, if n is of the form $3h$ or $3h+1$, the coefficients $\frac{(2n+1)\,2n}{2.3}$, ... cannot have a 3 or a power of 3 in the denominator; but, if $2n+2$ is divisible by 3, that is, if n is of the form $3h+2$, the denominators may contain powers of 3 up to 3^s, where 3^s is the highest power of 3 by which $n+1$ is divisible.

By putting $n = 3h+2$ in the coefficients, we see that the first coefficient necessarily has 3 in the denominator, that the second and third coefficients cannot have 3 in the denominator, that the fourth may have 3^2 in the denominator, and so on. By putting $n = 3h+1$, we see that in this case the second coefficient must contain 3 as a factor of the numerator. We obtain no additional results of this kind by putting $n = 3h$.

Now suppose that $G_1, G_2, \ldots, G_{n-1}$ are all integers; then, putting, for the moment,

$$Q_n = \frac{3^{2n}+1}{2} - 2^{2n},$$

we have, if $n = 3h$ or $3h+1$,

$$G_n = \text{multiple of } 3^2 + (-1)^{n-1}\, Q_n,$$

and, if $n = 3h+2$,

$$G_n = \text{multiple of } 3 + (-1)^{n-1}\, Q_n.$$

The quantity Q_n is an integer which $\equiv 1$, mod 3, so that, for all values of n, we have

$$G_n = 3m + (-1)^{n-1},$$

where m is an integer; therefore G_n is an integer, and, moreover, an integer which is not divisible by 3.

Since $G_1 = 1$, $G_2 = 5$, ..., we have therefore proved that all the G's are integers not divisible by 3.

Now let $2n+1$ contain 3^t as a factor, this being the highest power of 3 by which $2n+1$ is divisible; then, since G_n, $= (2n+1)\, I_n$, is an integer not divisible by 3, we see that 3^t must be a factor of the denominator of I_n, and, since the denominator of I_n can only consist of a power of 3, it follows that 3^t must be the denominator of I_n.

Therefore, if n is of the form $3h$ or $3h+2$, I_n is an integer; but, if n is of the form $3h+1$, I_n is of the form

$$\frac{\text{integer not divisible by } 3}{3^t},$$

where 3^t is the highest power of 3 by which $2n+1$ is divisible. Thus, when n is given, we can always assign the denominator (if any) of I_n.

5. Taking account of some of the results obtained in the preceding section, the recurring formula (§ 4) shows that

(i.) if $n = 3h$, $\quad G_n = (6h+1)\,h\,.\,3^2G_{n-1} + \mu\,.\,3^4 + (-1)^{n-1}Q_n$,

(ii.) if $n = 3h+1$, $\quad G_n = (2h+1)(3h+1)\,.\,3^2G_{n-1} + \mu\,.\,3^5 + (-1)^{n-1}Q_n$,

(iii.) if $n = 3h+2$, $\quad G_n = (6h+5)(3h+2)\,.\,3G_{n-1} + \mu\,.\,3^4 + (-1)^{n-1}Q_n$,

where $\mu\,.\,a$ is used to denote a multiple of a.

6. Now consider the residues of Q_n to mods $3, 3^2, \ldots$. We have

$$Q_n = \frac{3^{2n}+1}{2} - 2^{2n} = \frac{3^{2n}-1}{3-1} + 1 + (3-1)^{2n}$$

$$= 1+3+3^2+\ldots+3^{2n-1} + (2n)_1 3 - (2n)_2 3^2 + (2n)_3 3^3 - \ldots + (2n)_{2n} 3^{2n}.$$

Therefore

$$\begin{aligned} Q_n &\equiv 1, \quad \text{mod } 3, \\ &\equiv 1+3+(2n)_1 3, \quad \text{mod } 3^2, \\ &\equiv 1+3+3^2+(2n)_1 3-(2n)_2 3^2, \quad \text{mod } 3^3, \\ &\equiv 1+3+3^2+3^3+(2n)_1 3-(2n)_2 3^2+(2n)_3 3^3, \quad \text{mod } 3^4, \end{aligned}$$

and so on.

7. Thus, taking the modulus to be 3^2, we have

(i.) $n = 3h$, $\quad G_n \equiv (-1)^{n-1}Q_n \equiv (-1)^{n-1}\{4+6h\,.\,3\}$
$\qquad\qquad \equiv (-1)^{n-1}4, \quad \text{mod } 3^2$,

(ii.) $n = 3h+1$, $\quad G_n \equiv (-1)^{n-1}Q_n \equiv (-1)^{n-1}\{4+(6h+2)\,3\}$
$\qquad\qquad \equiv (-1)^{n-1}, \quad \text{mod } 3^2$,

(iii.) $n = 3h+2$, $\quad G_n \equiv 10\,.\,3\,\{(-1)^n + \mu\,.\,3^2\}$
$\qquad\qquad + (-1)^{n-1}\{4+(6h+4)\,3\}, \quad \text{mod } 3^2$,
$\qquad\qquad \equiv (-1)^{n-1}4, \quad \text{mod } 3^2$.

8. For modulus 3^3, we have

(i.) $n = 3h$,

$$G_n \equiv (6h+1)\,h.3^2\{(-1)^n+\mu.3\}$$
$$+(-1)^{n-1}\{13+6h.3-3h(6h-1)\,3^2\}, \quad \text{mod } 3^3,$$

(ii.) $n = 3h+1$,

$$G_n \equiv (2h+1)(3h+1)\,3^2\{(-1)^n+\mu.3\}$$
$$+(-1)^{n-1}\{13+(6h+2)\,3-(3h+1)(6h+1)\,3^2\}, \quad \text{mod } 3^3,$$

(iii.) $n = 3h+2$,

$$G_n \equiv (6h+5)(3h+2)\,3\{(-1)^n+\mu.3^2\}$$
$$+(-1)^{n-1}\{13+(6h+4)\,3-(3h+2)(6h+3)\,3^2\}, \quad \text{mod } 3^3,$$

giving

(i.) $n = 3h$, $\quad G_n \equiv (-1)^{n-1}(9h+13) \equiv (-1)^{n-1}(3n+13)$, mod 3^3.

(ii.) $n = 3h+1$, $\quad G_n \equiv (-1)^{n-1}$, mod 3^3,

(iii.) $n = 3h+2$, $\quad G_n \equiv (-1)^{n-1}(18h+22) \equiv (-1)^{n-1}(6n+10)$, mod 3^3.

9. The results of §§ 7 and 8 show that

(i.) $n = 3h$, $\quad G_{n-1} = (-1)^n\,4+\mu.3^2$,

(ii.) $n = 3h+1$, $\quad G_{n-1} = (-1)^n\,4+\mu.3^2$,

(iii.) $n = 3h+2$, $\quad G_{n-1} = (-1)^n+\mu.3^3$,

whence, considering the residues of G_n to mod 3^4, we have

(i.) $n = 3h$,

$$G_n \equiv (6h+1)\,h.3^3\,(-1)^n\,4$$
$$+(-1)^{n-1}\{40+6h.3-3h(6h-1)\,3^2+h\,(6h-1)(6h-2)\,3^3\}, \quad \text{mod } 3^4,$$

(ii.) $n = 3h+1$,

$$G_n \equiv (2h+1)(3h+1)\,3^2\,(-1)^n\,4$$
$$+(-1)^{n-1}\{40+(6h+2)\,3-(3h+1)\,(6h+1)\,3^2$$
$$+(6h+2)(6h+1)\,h.3^3\}, \quad \text{mod } 3^4,$$

(iii.) $n = 3h+2$,

$$G_n \equiv (6h+5)\,(3h+2)\,3\,(-1)^n$$
$$+(-1)^{n-1}\{40+(6h+4)\,3-(3h+2)\,(6h+3)\,3^2$$
$$+(3h+2)(2h+1)(6h+2)\,3^3\}, \quad \text{mod } 3^4,$$

giving

(i.) $n = 3h$,

$$G_n \equiv (-1)^{n-1}(27h^2+63h+40) \equiv (-1)^{n-1}(3n^2+21n+40), \quad \text{mod } 3^4,$$

(ii.) $n = 3h+1$,

$$G_n \equiv (-1)^{n-1}(27h^2+54h+1) \equiv (-1)^{n-1}(3n^2+12n+67), \quad \text{mod } 3^4,$$

(iii.) $n = 3h+2$,

$$G_n \equiv (-1)^{n-1}(27h^2+45h+1) \equiv (-1)^{n-1}(3n^2+3n+58), \quad \text{mod } 3^4.$$

It will be noticed that the residue in (ii.) may be written

$$(-1)^{n-1}\{27h(h+2)+1\},$$

so that it $\equiv (-1)^{n-1}$, mod 3^4, if h or $h+2$ is a multiple of 3. Thus, if n is of the form $9h+1$ or $9h+4$,

$$G_n \equiv (-1)^{n-1}, \quad \text{mod } 3^4,$$

but, if n is of the form $9h+7$,

$$G_n \equiv (-1)^{n-1}55, \quad \text{mod } 3^4.$$

10. The results obtained in the preceding four sections are contained in the following table, which gives (irrespective of sign) the value of the residue of G_n to mods 3, 3^2, 3^3, 3^4 for the forms $3h$, $3h+1$, $3h+2$ of n. In all cases the value given in the table is to be multiplied by $(-1)^{n-1}$.

	mod 3	mod 3^2	mod 3^3	mod 3^4
$n = 3h$	1	4	$3n+13$	$3n^2+21n+40$
$n = 3h+1$	1	1	1	$3n^2+12n+67$
$n = 3h+2$	1	4	$6n+10$	$3n^2+3n+58$

11. We may also exhibit this table in the following form, in which n is classified according to its residue to mod 9.

	mod 3	mod 3^2	mod 3^3	mod 3^4
$n = 9h$	1	4	13	$3n+40$
$n = 9h+1$	1	1	1	1
$n = 9h+2$	1	4	22	$6n+64$
$n = 9h+3$	1	4	22	$3n+40$
$n = 9h+4$	1	1	1	1
$n = 9h+5$	1	4	13	$6n+37$
$n = 9h+6$	1	4	4	$3n+15$
$n = 9h+7$	1	1	1	55
$n = 9h+8$	1	4	4	$6n+64$

As before, the residues of G_n are derived from the values in the table by multiplying by $(-1)^{n-1}$.

12. From the residues of G_n to mods 3, 3^2, ..., we may derive those of I_n if integral, and of I'_n, the numerator of I_n, if I_n is fractional. In the former case $2n+1$ is not divisible by 3, and in the latter case we obtain the residue of aI'_n, where a is the residual factor of $2n+1$ when the highest power of 3 by which $2n+1$ is divisible has been thrown out; so that a is not divisible by 3.

Taking mod 3, we have

$$\begin{array}{lll} \text{(i.)} & n = 3h, & I_n \equiv (-1)^{n-1}, \quad \text{mod } 3, \\ \text{(ii.)} & n = 3h+1, & aI'_n \equiv (-1)^{n-1}, \quad \text{mod } 3, \\ \text{(iii.)} & n = 3h+2, & I_n \equiv (-1)^{n-1}, \quad \text{mod } 3. \end{array}$$

Therefore, when I_n is integral,

$$I_n \equiv 1 \text{ or } 2, \quad \text{mod } 3,$$

according as $n \equiv 2$ and 3 or 5 and 0, mod 6, *i.e.*, I_{6h+2} and I_{6h+3} are of the form $3r+1$ and I_{6h+5} and I_{6h} are of the form $3r+2$.

Thus the residues of the integral I's, viz., $I_2, I_3, I_5, I_6, I_8, I_9, \ldots$ to mod 3 are 1, 1, 2, 2, 1, 1, ... in this order, viz., the 1's and 2's occur in pairs.

For fractional I's, we have

$$I'_n \equiv a, \quad \text{mod } 3, \quad \text{if } n = 6h+1,$$

$$I'_n \equiv -a, \quad \text{mod } 3, \quad \text{if } n = 6h+4,$$

where a is the residual factor of $2n+1$ when powers of 3 have been thrown out.

13. Taking mod 3^2, we find that, for the integral I's,

$$I_n \equiv (-1)^n,\ (-1)^n 2,\ (-1)^{n-1} 2,\ (-1)^{n-1},\ (-1)^n 4,\ (-1)^{n-1} 4, \quad \text{mod } 9,$$

according as $n \equiv 2, 3, 5, 6, 8, 0$, mod 9; and that, when $n \equiv 1$, mod 3, so that I_n is fractional,

$$aI' \equiv (-1)^{n-1}, \quad \text{mod } 9,$$

a being, as before, the residual factor of $2n+1$ when powers of 3 have been thrown out.

Thus, taking the integral values of I_n, the residues of I_n to mod 9 occur in the cycle 1, 7, 2, 8, 4, 4, 8, 2, 7, 1, 5, 5, corresponding to

$$n \equiv 2, 3, 5, 6, 8, 9, 11, 12, 14, 15, 17, 18, \quad \text{mod } 18.$$

Similar results for mods 3^3 and 3^4 may be deduced from the table in § 11.

14. The following table, giving the values of I_n as far as I_{13}, was calculated by means of the recurring formula

$$\tfrac{3}{2} I_n - (2n)_2 I_{n-1} + (2n)_4 I_{n-2} - \ldots + (-1)^{n-1} (2n)_{2n-2} I_1 + (-1)^n \tfrac{1}{2} = 0 \quad (\S 2).$$

The second column, containing the values of G_n was deduced from the first by multiplication by $2n+1$.

n	I_n	$G_n = (2n+1) I_n$.
1	$\frac{1}{3}$	1
2	1	5
3	7	49
4	$\frac{809}{9}$	809
5	1847	20317
6	55601	722813
7	$\frac{6921461}{3}$	34607305
8	126235201	2145998417
9	8806171927	167317266613
10	$\frac{2288629046003}{3}$	16020403322021
11	80348736972167	1848020950359841
12	10111159088668001	252778977216700025
13	$\frac{40453941942593304589}{27}$	40453941942593304589

I have verified that the residues of the G's in the table to mods 3, 3^2, 3^3, 3^4 are the same as those given by the formulæ in §§ 10 and 11.

15. The integral I-numbers are all uneven, and the numerators of the fractional I-numbers are also uneven. This is evident from the recurring formula in the preceding section, which gives

$$3I_n \equiv (-1)^{n-1}, \quad \text{mod } 2,$$

so that, for all values of n, $\quad I_n \equiv 1, \quad \text{mod } 2.$

When I_n is fractional, since the denominator is a power of 3, and therefore uneven, it follows that the numerator I'_n must be uneven.

16. The I-numbers are included in the general expansion formula (ii.) of § 8 in the preceding paper, and they therefore satisfy the

congruence

$$I_n \equiv (-1)^{\varpi} I_{n-\varpi}, \quad \text{mod } p,^*$$

where p is any prime except 3.

In the next five sections some of the results obtained by giving special values to p will be considered.

17. Taking $p = 5$, we have, for all values of n,

$$I_n \equiv I_{n-2}, \quad \text{mod } 5.$$

Considering first the integral I's, viz., I_2, I_3, I_5, I_6, I_8, I_9, ..., that is, all the I's in which the suffix is not of the form $3h+1$, this congruence shows that all of them in which the suffix is even are congruent to one another, mod 5, and that all in which the suffix is uneven are also congruent to one another, mod 5. Since the integral I's are all uneven numbers, it follows that the difference between any two of them with even suffixes is divisible by 10, and that the difference between any two of them with uneven suffixes is also divisible by 10. Thus, all the integral I-numbers in which the suffix is even must end in the same digit, which is 1, since I_2 is 1, and all the integral I-numbers in which the suffix is uneven must end in the same digit, which is 7, since $I_3 = 7$. The series of integral I-numbers beginning with I_2 must end therefore in the digits 1, 7, 7, 1, 1, 7, 7, 1, 1, ... in this order, *i.e.*, after the first, in 7's and 1's in pairs; or, in other words, I_{6h} and I_{6h+2} end in 1, I_{6h+3} and I_{6h+5} end in 7.

18. Now consider the I's which have denominators. In these the suffix is of the form $3h+1$. The congruence

$$I_n \equiv I_{n-2}, \quad \text{mod } 5,$$

shows that

$$\text{any fractional } I \equiv \text{any other fractional } I, \quad \text{mod } 5,$$

if both have even, or both have uneven, suffixes.

Thus
$$I_{6h+1} \equiv I_1 \equiv \tfrac{1}{3}, \qquad \text{mod } 5,$$
$$I_{6h+4} \equiv I_4 \equiv \tfrac{809}{9}, \quad \text{mod } 5.$$

* The series defining the I-numbers is (ii.) of § 11 (p. 202); this formula is given in the same section.

All the I's $\equiv 1$, mod 2 (§ 15), and therefore

$$I_{6h+1} \equiv \tfrac{1}{3}, \qquad \text{mod } 10,$$

$$I_{6h+4} \equiv \tfrac{209}{9}, \quad \text{mod } 10.$$

The second of these congruences shows that the numerator and denominator of I_{6h+4} must end in the same digit, and the first shows that the numerator of I_{6h+1} when multiplied by 3 must end in the same digit as its denominator. Putting, as before,

$$I_n = \frac{I'_n}{3^t},$$

so that I'_n is the numerator of I_n and 3^t its denominator, we see that, when $n = 6h+1$, I'_n ends in the same digit as 3^{t-1}, and, when $n = 6h+4$, I'_n ends in the same digit as 3^t. The powers of 3 end in the digits 1, 3, 9, 7, so that, when the suffix of a fractional I is uneven, the numerator ends in 7, 1, 9, 3, according as the denominator ends in 1, 3, 7, 9; and, when the suffix is even, the numerator and denominator end in the same digit.

We may express these results as follows. If t be the exponent of 3 in the denominator of I_n, then,

if $n = 6h+1$, I'_n ends in 1, 3, 9, 7 according as $t \equiv 1, 2, 3, 0$, mod 4,
" $n = 6h+4$, " " 3, 9, 7, 1 " " " " .

By using the congruence $\quad I_n \equiv I_{n-2}$, mod 5,

to connect integral and fractional I's we merely reproduce these results.

19. Now take $p = 7$. The congruence is

$$I_n \equiv (-1)^u I_{n-3u}, \quad \text{mod } 7;$$

so that in this case only integral I's are connected together, and only fractional I's.

Taking the integral I's first, we have

$$I_{3h+3} \equiv (-1)^h I_3 \equiv (-1)^h 7 \equiv 0, \quad \text{mod } 7,$$

$$I_{6h+2} \equiv I_2 \equiv 1, \quad \text{mod } 7,$$

$$I_{6h+5} \equiv I_5 \equiv 1847 \equiv 6, \quad \text{mod } 7.$$

Therefore I_{3h} is always divisible by 7, *i.e.*, is of the form $7r$; and I_{6h+2} and I_{6h+5} are respectively of the forms $7r+1$ and $7r-1$.

20. Considering now the fractional I's, we have

$$I_{3h+1} \equiv (-1)^h I_1 \equiv (-1)^h \tfrac{1}{3}, \quad \text{mod } 7;$$

so that, 3^t being the denominator of I_{3h+1},

$$I'_{3h+1} \equiv (-1)^h 3^{t-1}, \quad \text{mod } 7.$$

Now $\qquad 3^{t-1} \equiv 1, 3, 2, 6, 4, 5, \quad \text{mod } 7,$

according as $\qquad t \equiv 1, 2, 3, 4, 5, 0, \quad \text{mod } 6;$

and therefore, if $n = 6h+1$,

$$I'_n \equiv 1, 3, 2, 6, 4, 5, \quad \text{mod } 7$$

according as $\qquad t \equiv 1, 2, 3, 4, 5, 0, \quad \text{mod } 6;$

and, if $n = 6h+4$,

$$I'_n \equiv 6, 4, 5, 1, 3, 2, \quad \text{mod } 7,$$

according as $\qquad t \equiv 1, 2, 3, 4, 5, 0, \quad \text{mod } 6.$

21. The most interesting values of p are those for which $\frac{p-1}{2}$ is a multiple of 3, for then, as in the case of $p = 7$, the integral I's and the fractional I's are kept distinct. Passing over therefore $p = 11$, let $p = 13$, for which the general congruence is

$$I_n \equiv I_{n-6t}, \quad \text{mod } 13.$$

Considering the integral I's first, we have

$$I_{6h+2} \equiv I_2 \equiv 1, \quad \text{mod } 13,$$
$$I_{6h+3} \equiv I_3 \equiv 7, \quad \text{mod } 13,$$
$$I_{6h+5} \equiv I_5 \equiv 1847 \equiv 1, \quad \text{mod } 13,$$
$$I_{6h} \equiv I_6 \equiv 55601 \equiv 0, \quad \text{mod } 13.$$

Thus I_{6h} is always divisible by 13, *i.e.*, is of the form $13r$; I_{3h+2} is of the form $13r+1$; and I_{6h+3} of the form $13r+7$.

22. For the fractional I's, we have, 3^t being the denominator,

$$I_{6h+1} \equiv I_1 \equiv \tfrac{1}{3}, \quad \text{mod } 13,$$
$$I_{6h+4} \equiv I_4 \equiv \tfrac{802}{9}, \quad \text{mod } 13;$$

so that $\qquad I'_{6h+1} \equiv 3^{t-1}, \quad I'_{6h+4} \equiv \tfrac{802}{9} \times 3^t \equiv 3^{t-1}, \quad \text{mod } 13.$

Thus, generally, $\qquad I'_{3h+1} \equiv 3^{t-1}, \quad \text{mod } 13,$

Now $$3^{t-1} \equiv 1,\ 3,\ 9, \quad \text{mod } 13,$$
according as $$t \equiv 1,\ 2,\ 0, \quad \text{mod } 3,$$
and therefore $$I'_{3h+1} \equiv 1,\ 3,\ 9, \quad \text{mod } 13,$$
according as $$t \equiv 1,\ 2,\ 0. \quad \text{mod } 3.$$

23. Combining the results obtained in the cases of mods 7 and 13, we find, for the integral I's,

$$I_{6h+2} \equiv 1, \quad \text{mod } 91,$$
$$I_{6h+3} \equiv 7, \quad \text{mod } 91,$$
$$I_{6h+5} \equiv 27, \quad \text{mod } 91,$$
$$I_{6h} \equiv 0, \quad \text{mod } 91;$$

and for the fractional I's,

if $n = 6h+1$, $$I'_n \equiv 1,\ 3,\ 9,\ 27,\ 81,\ 61, \quad \text{mod } 91,$$
according as $$t \equiv 1,\ 2,\ 3,\ 4,\ 5,\ 0, \quad \text{mod } 6,$$
if $n = 6h+5$, $$I'_n \equiv 27,\ 81,\ 61,\ 1,\ 3,\ 9, \quad \text{mod } 91,$$
according as $$t \equiv 1,\ 2,\ 3,\ 4,\ 5,\ 0. \quad \text{mod } 6.$$

I have verified all the results given by the formulæ for mods 7 and 13 as far as the extent of the table in § 14 permits.

The next value of p for which $\frac{p-1}{2}$ is a multiple of 3 is $p = 19$, but the results in this case do not seem to be of sufficient interest to be worth giving in detail.

24. The coefficients H_n and J_n which occur in the expansions

$$\frac{1}{2\cos x - 1} = \tfrac{2}{3}\left\{H_0 + \frac{H_1}{2!}x^2 + \frac{H_2}{4!}x^4 + \frac{H_3}{6!}x^6 + \&c.\right\},$$

$$\frac{2\cos x}{2\cos 2x + 1} = \tfrac{1}{3}\left\{J_0 + \frac{J_1}{2!}x^2 + \frac{J_2}{4!}x^4 + \frac{J_3}{6!}x^6 + \&c.\right\},*$$

---

* These series are (iii.) and (iv.) of § 11 of the preceding paper (p. 202). The quantities I_n, H_n, J_n have been considered in the *Quarterly Journal*, Vol. XXIX., pp. 35–59, 131–148, and the *Messenger*, Vol. XXVI., pp. 165–170, and Vol. XXVIII., pp. 48–51.

where $H_0 = \frac{3}{2}$, $J_0 = 2$, are connected with the I-numbers by the relations

$$H_n = (2^{2n+1}+1)\, I_n, \quad J_n = (2^{2n+1}+2)\, I_n;$$

so that

$$J_n = H_n + I_n.$$

25. The following table gives the values of H_n and J_n up to $n = 13$. The values of H_n were deduced from those of I_n (§ 14) by multiplication by $2^{2n+1}+1$, and those of J_n were obtained by adding H_n and I_n.

n	H_n	J_n
1	3	$\frac{10}{3}$
2	33	34
3	903	910
4	46113	$\frac{415826}{9}$
5	3784503	3786350
6	455538993	455594594
7	75603118503	$\frac{226816276970}{3}$
8	16546026500673	16546152735874
9	4616979073434903	4616987879606830
10	1599868423237443153	$\frac{4709607558341375462}{3}$
11	674014138103352845703	674014218452089817870
12	339274210193051498798433	3392742220304210587466434
13	2010976376530637671311142903	$\frac{5420636257086663655134162970}{27}$

16. The simplest of the recurring relations connecting the H's and the J's are respectively

$$\tfrac{3}{2}H_n - (2n)_2\, H_{n-1} + (2n)_4\, H_{n-2} - \ldots + (-1)^{n-1}\,(2n)_{2n-2}\, H_1 = (-1)^{n-1}\, \tfrac{3}{2},$$

$$\tfrac{3}{2}J_n - (2n)_2\, 2^2 J_{n-1} + (2n)_4\, 2^4 J_{n-2} - \ldots + (-1)^{n-1}\,(2n)_{2n-2}\, 2^{2n-2}\, J_1$$
$$= (-1)^{n-1}\,(2^{2n+1}-3).*$$

* *Quarterly Journal*, Vol. XXIX., pp. 49, 45. Other recurring relations are given on pp. 44, 49, 50, 55, 57 of the same volume.

The first of these relations shows at sight that H_n is always an integer; and, by putting $n = 1, 2, 3, \ldots$ we see that $H_1, H_2, H_3, \ldots$ are all divisible by 3. The second relation shows that J_n can only have a power of 3 as denominator.

27. These results may also be derived from the properties of the I-numbers. For, if $2n+1 = a.3^t$, where a is prime to 3 and t may be zero, then $2^{2n+1}+1$ is divisible by 3^{t+1}. This is easily seen by writing $2^{2n+1}+1$ in the form $1-(1-3)^{a.3^t}$ and expanding by the binomial theorem, each term in the expansion being evidently divisible by 3^{t+1}. Since $H_n = (2^{2n+1}+1) I_n$, and since the denominator of I_n is 3^t, it follows that H_n must be an integer divisible by 3. Similarly, since $J_n = 2(2^{2n}+1) I_n$, and since $2^{2n}+1$ cannot be divisible by 3, we see that J_n must have the same denominator as I_n.

28. The quantities H_n and J_n satisfy the general congruence theorems

$$H_n \equiv (-1)^{tj} H_{n-tj}, \quad \text{mod } p,$$

$$J_n \equiv (-1)^{tj} J_{n-tj}, \quad \text{mod } p,$$

where p is any uneven prime except 3 (§ 11 of the preceding paper) and $j = \frac{1}{2}(p-1)$.* In the case of the former congruence, we merely obtain $0 \equiv 0$ by putting $p = 3$, but we know from § 12 of the preceding paper that

$$\frac{H_n}{3} \equiv (-1)^t \frac{H_{n-t}}{3}, \quad \text{mod } 3.$$

29. The recurring formula for H_n shows that all the H's are uneven, and, since

$$H_n \equiv H_{n-2}, \quad \text{mod } 5,$$

we have therefore
$$H_n \equiv H_{n-2}, \quad \text{mod } 10.$$

Thus the H's can end in only one or other of two digits, and, since H_1 and H_2 both end in 3, all the H's must end in 3.

30. Since
$$J_n = H_n + I_n,$$
and since all the H's end in 3, it follows from § 17 that the integral J's, viz., $J_2, J_3, J_5, J_6, J_8, J_9, \ldots$, end in the digits 4, 0, 0, 4, 4, 0, ... in

* These results may also be derived from the corresponding I-congruences (§ 16 of the present paper) since

$$2^{2n} \equiv 2^{2n-t(p-1)}, \quad \text{mod } p.$$

this order; or, in other words, J_{6h} and J_{6h+2} end in 4, and J_{6h+3} and J_{6h+5} end in 0.

Similarly, from the I-results of § 18 we may deduce that, if J'_n be the numerator of J_n, so that

$$J_n = \frac{J'_n}{3^t},$$

then, if $n = 6h+1$, J'_n always ends in 0, and, if $n = 6h+4$, J'_n ends in 2, 6, 8, 4 according as the denominator ends in 3, 9, 7, 1, that is, according as $t \equiv 1, 2, 3, 0$, mod 4.

These J-results may also be easily obtained independently by the methods of §§ 17 and 18.

31. It was shown in Vol. XXVIII., p. 75, of the *Messenger* that, for all values of n,

$$(2n+1)\left\{I_n+(-1)^{n+1}\right\} \text{ is a multiple of 4}$$

and $$(2n+1)\left\{J_n+(-1)^{n+1}2\right\} \text{ is a multiple of 16.}$$

The first of these formulæ shows that, when I_n is integral, it is of the form $4r+(-1)^n$; viz., that I_{6h} and I_{6h+2} are of the form $4r+1$, and that I_{6h+3} and I_{6h+5} are of the form $4r+3$. For the fractional I's, the formula shows that,

if $n = 6h+1$, the numerator + the denominator $\equiv 0$, mod 4;

if $n = 6h+4$, the numerator − the denominator $\equiv 0$, mod 4.

Combining these results for the integral I's with the corresponding results in § 12, we see that I_{6h+2} is of the form $12r+1$, I_{6h+3} of the form $12r+7$, I_{6h+5} of the form $12r+11$, and I_{6h} of the form $12r+5$.

32. The second formula shows that, when J_n is integral, it is of the form $16r+(-1)^n 2$; viz., that J_{6h} and J_{6h+2} are of the form $16r+2$ and that J_{6h+3} and J_{6h+5} are of the form $16r-2$. For the fractional J's, we have,

if $n = 6h+1$, the numerator + twice the denominator $\equiv 0$, mod 16,

if $n = 6h+4$, the numerator − twice the denominator $\equiv 0$, mod

33. It was also shown (*Messenger*, *loc. cit.*) that

$$H_n \equiv (-1)^n, \quad \text{mod } 4;$$

so that H_{2h} is of the form $4r+1$ and H_{2h+1} of the form $4r+3$.

34. Since all the H-coefficients are divisible by 3, it would, perhaps, be preferable to use H'_n instead of H_n, where

$$H'_n = \tfrac{1}{3} H_n.$$

The expansion formula (§ 24) then becomes

$$\frac{1}{2\cos x - 1} = 2\left\{ H'_0 + \frac{H'_1}{2!}x^2 + \frac{H'_2}{4!}x^4 + \frac{H'_3}{6!}x^6 + \&c. \right\},$$

with $H'_0 = \tfrac{1}{2}$; and the recurring formula (§ 27) is

$$\tfrac{1}{2}H'_n - (2n)_2 H'_{n-1} + (2n)_4 H'_{n-2} - \dots + (-1)^{n-1}(2n)_{2n-2} H'_1 = (-1)^{n-1}\tfrac{1}{2}.$$

The H's all end in 1; for the difference between any two H's is a multiple of 10, and therefore, dividing by 3, the difference between any two H's is a multiple of 10; so that the H's must all end in the same digit, which is 1, since $H'_1 = 1$.

35. The following is a table of the values of H'_n up to H'_{13} derived from the table of H_n in § 23 by division by 3 :—

	H'_n
1	1
2	11
3	301
4	15371
5	1261501
6	151846331
7	25201039501
8	5515342166891
9	1538993024478301
10	533289474412481051
11	224671379367784281901
12	113091403397683832932811
13	67032545884354589043714301

36. The relation $$H'_n \equiv (-1)^t H'_{n-t}, \quad \text{mod } 3 \quad (\S 28),$$

shows that $$H'_n \equiv (-1)^{n-1} H'_1 \equiv (-1)^{n-1}, \quad \text{mod } 3;$$

so that H'_{2h+1} is of the form $3r+1$ and H'_{2h} of the form $3r+2$.

It follows from § 33 that H'_{2h+1} is of the form $4r+1$ and H'_{2h} of the form $4r+3$.

Combining these results, we see that H'_{2h+1} is of the form $12r+1$ and H'_{2h} of the form $12r-1$.

37. By taking $p = 7$ and $p = 13$, we find, as in §§ 19 and 21, that

$$\left.\begin{array}{ll} H'_{6h+1} \equiv 1, & H'_{6h+2} \equiv 4, \\ H'_{6h+4} \equiv 6, & H'_{6h+5} \equiv 3, \end{array} \quad H'_{3h} \equiv 0 \right\}, \quad \text{mod } 7,$$

and

$$\left.\begin{array}{lll} H'_{6h+1} \equiv 1, & H'_{6h+2} \equiv 11, & H'_{6h+3} \equiv 2 \\ H'_{6h+4} \equiv 5, & H'_{6h+5} \equiv 7, & H'_{6h} \equiv 0 \end{array}\right\}, \quad \text{mod } 13.$$

Combining these results, we see that

$$H'_n \equiv 1, 11, 28, 83, 59, 0, \quad \text{mod } 91,$$

according as $$n \equiv 1, 2, 3, 4, 5, 0, \quad \text{mod } 6.$$

38. The residues of H'_n for mods $3^2, 3^3, \ldots$ may be easily deduced from those of G_n (§ 10). For,

$$G_n = (2n+1) I_n,$$

and

$$H'_n = \frac{2^{2n+1}+1}{3} I_n = \frac{1-(1-3)^{2n+1}}{3} I_n$$

$$= \tfrac{1}{3}\left\{(2n+1)\,3 - \frac{(2n+1)\,2n}{2!}\,3^2 + \frac{(2n+1)\,2n\,(2n-1)}{3!}\,3^3 - \ldots\right\} I_n$$

$$= \{1-3n+3n\,(2n-1) - \text{terms in } 3^2\}\,(2n+1)\,I_n.$$

Thus, taking mod 9, $\quad H'_n \equiv (6n^2-6n+1)\,G_n, \quad \text{mod } 9;$

and therefore,

$$\begin{array}{llll} \text{if} & n = 3h, & H'_n \equiv \ G \ \equiv (-1)^{n-1}4, & \text{mod } 9; \\ \text{,,} & n = 3h+1, & H'_n \equiv \ G_n \equiv (-1)^{n-1}, & \text{,,} \\ \text{,,} & n = 3h+2, & H'_n \equiv 4G_n \equiv (-1)^{n-1}\,7, & \text{,,} \end{array}$$

Thus $$H'_n \equiv 1, 2, 4, 8, 7, 5, \quad \text{mod } 9,$$
according as $$n \equiv 1, 2, 3, 4, 5, 0, \quad \text{mod } 6.$$

39. With respect to the applications of the numbers I_n, H_n, J_n, it may be mentioned that, besides being simple cases* of the Bernoullian function, and therefore sharing in its general applications, they are analogous to the Eulerian numbers in being rational factors which occur in the summation of certain series of reciprocals by means of powers of π. For, putting

$$u_n = 1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} - \&c.,$$

$$g_n = 1 - \frac{1}{2^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{7^n} - \&c.,$$

$$j_n = 1 + \frac{1}{2^n} - \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{7^n} + \&c.,$$

$$h_n = 1 - \frac{1}{5^n} + \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} - \&c.,$$

where, in u_n the positive terms contain numbers $\equiv 1$, mod 4, and the negative terms numbers $\equiv 3$, mod 4; in g_n the positive terms contain numbers $\equiv 1$, mod 3, and the negative terms numbers $\equiv 2$, mod 3; in j_n the positive terms contain numbers $\equiv 1$ and 2, mod 6, and the negative terms numbers $\equiv 4$ and 5, mod 6; and in h_n the positive terms contain numbers $\equiv 1$, mod 6, and the negative terms numbers $\equiv 5$, mod 6; we have

$$u_{2n+1} = \tfrac{1}{2} \frac{E_n}{(2n)!} \left(\frac{\pi}{2}\right)^{2n+1},$$

$$g_{2n+1} = \frac{1}{\sqrt{3}} \frac{I_n}{(2n)!} \left(\frac{2\pi}{3}\right)^{2n+1},$$

$$j_{2n+1} = \frac{1}{\sqrt{3}} \frac{J_n}{(2n)!} \left(\frac{\pi}{3}\right)^{2n+1},$$

$$h_{2n+1} = \frac{1}{\sqrt{3}} \frac{H_n}{(2n)!} \left(\frac{\pi}{3}\right)^{2n+1}.\dagger$$

* *Quarterly Journal*, Vol. XXIX., pp. 107, 131, or *Messenger*, Vol. XXVI., pp. 178, 179.

† *Quarterly Journal*, Vol. XXIX., pp. 50–52.

40. The numbers E_n, I_n, J_n, H_n also serve to express the values of the following definite integrals :—

$$\int_0^\infty \frac{t^{2n}\,dt}{\cosh t} = E_n \left(\frac{\pi}{2}\right)^{2n+1},$$

$$\int_0^\infty t^{2n} \frac{\sinh \frac{1}{2}t}{\sinh \frac{3}{2}t}\,dt = \int_0^\infty \frac{t^{2n}\,dt}{2\cosh t + 1} = \frac{J_n}{\surd 3}\left(\frac{2\pi}{3}\right)^{2n+1},$$

$$\int_0^\infty t^{2n} \frac{\cosh \frac{1}{2}t}{\cosh \frac{3}{2}t}\,dt = \int_0^\infty \frac{t^{2n}\,dt}{2\cosh t - 1} = \frac{J_n}{\surd 3}\left(\frac{\pi}{3}\right)^{2n+1},$$

$$\int_0^\infty t^{2n} \frac{\sinh 2t}{\sinh 3t}\,dt = \int_0^\infty \frac{2t^{2n}\cosh t\,dt}{2\cosh 2t + 1} = \frac{H_n}{\surd 3}\left(\frac{\pi}{3}\right)^{2n+1}.*$$

41. In the papers in the *Quarterly Journal* and the *Messenger* which have been already referred to certain other quantities P_n, Q_n, R_n, S_n, T_n have been considered; the expansions in which they occur as coefficients were given in § 11 of the preceding paper (pp. 202, 203). These quantities are all integers, and their congruence properties are therefore similar to those of the Eulerian numbers.

On the Theory of Simultaneous Partial Differential Equations. *By* J. E. CAMPBELL. Read December 8th, 1898. Received, in revised form, May 24th, 1899.

The necessary and sufficient condition that any number of partial differential equations, of any orders whatever, in one dependent and n independent variables may be consistent is that by repeated differentiations of the equations and eliminations it should not be possible to deduce any relation between the independent variables.

Such a consistent system of differential equations is said to be integrable (Goursat, *Equations aux dérivées partielles du second ordre*, Tome II., p. 41). If p is the order of the lowest differential equation which can be deduced by mere algebra from the system, and if by successive differentiations and eliminations no equation algebraically

---

* *Messenger*, Vol. XXVI., pp. 174, 175.

independent of the given equations of the system, and of order equal to or less than p, can be deduced, then the system is said to be completely integrable.

When an equation system is integrable, it is not to be expected that the most general solution of the system is a general solution of any one of the equations which make up the system; in fact, in the most ordinary case of an integrable system, the solution involves no arbitrary functions, but only a finite number of arbitrary constants. Thus, if we write down two partial differential equations $f_1 = 0$, $f_2 = 0$ at random, they will not be consistent; if $f_1 = 0$ and $f_2 = 0$ are consistent, it must be owing to a relation between the forms of f_1 and f_2; if we consider the form of one of these equations, say $f_1 = 0$, as known, then the form of the second f_2 considered as a function of the variables and the differential coefficients it contains must satisfy certain differential equations. Now, when f_2 satisfies these equations, $f_1 = 0$ and $f_2 = 0$ will be consistent; but the common solutions of this system will ordinarily involve only a finite number of constants.

It is here that we notice an essential difference between the theory of partial differential equations of the first order and those of the second and higher orders: given any partial differential equation of the first order $f_1 = 0$, then a second equation $f_2 = 0$, also of the first order, always exists such that $f_1 = 0$ and $f_2 = 0$ have common solutions involving $n-1$ arbitrary functions; on the other hand, if f_1 is of the second order, then it is not generally true that any other equation $f_2 = 0$ exists having in common with $f_1 = 0$ solutions involving any arbitrary function. If f_1 is of a special form, then $f_1 = 0$ may be an equation which belongs to a system having solutions involving an infinity of constants. Such systems have been called by Lie "systems of Darboux" (Goursat, *ibid.*, p. 41) or "systems in involution."

If we have any integrable system, by repeated differentiations and eliminations we can add new equations till, after a finite number of operations, we have a completely integrable system. If such a system is not in involution, it has the property that differential coefficients above a certain order can be expressed in terms of coefficients of lower order; the complete theory of such a system is given in Lie-Engel, *Transformations-Gruppen*, I., Kap. 10. The object of the present paper is to develop certain formulæ analogous to the Jacobian series of combinants, by aid of which it may be decided whether or no a system is integrable.

Before the results arrived at can be stated, certain preliminary

explanations and definitions must be given. If f is a partial differential expression of order p, and, if we write for $\dfrac{\partial^{a_1+\ldots+a_n} z}{\partial x_1^{a_1} \ldots \partial x_n^{a_n}}$ $z_{a_1 \ldots a_n}$, then the quantic in the set of auxiliary variables $\xi_1 \ldots \xi_n$,

$$\Sigma\, \xi_1^{a_1} \ldots \xi_n^{a_n} \frac{\partial f}{\partial z_{a_1 \ldots a_n}}$$

(the summation being for all zero and positive integral values of $a_1 \ldots a_n$ such that $a_1 + \ldots + a_n = p$, and $\dfrac{\partial f}{\partial z_{a_1 \ldots a_n}}$ denoting the partial differential coefficient of f with respect to $z_{a_1 \ldots a_n}$), is said to correspond to the differential expression f.*

---

* The following geometrical interpretation may be given to this quantic. Cauchy's existence theorem may be thus stated: "If a differential equation of order p contains the derivative $\dfrac{\partial^p z}{\partial x_1^p}$, then a definite number of solutions of the equation can be found which are of the form $z = F(x_1 \ldots x_n)$; F is a holomorphic function of $x_1 \ldots x_n$ which can be so chosen that the locus $z = F$ passes through the $n-1$-dimensional locus $\begin{cases} z = \phi(x_2 \ldots x_n) \\ x_1 = 0 \end{cases}$, where ϕ is arbitrarily assigned, and has contact of $(p-1)^{\text{st}}$ order at all points on $\begin{cases} z = \phi \\ x_1 = 0 \end{cases}$ with any arbitrarily assigned n-dimensional locus which passes through $\begin{cases} z = \phi. \\ x_1 = 0 \end{cases}$" Apply now to the differential equation the point-transformation

$$\begin{aligned} z' &= z, \\ x_1' &= \psi(x_1 \ldots x_n), \\ x_2' &= x_2, \\ &\vdots \\ x_n' &= x_n; \end{aligned}$$

then the transformed equation will contain $\dfrac{\partial^p z'}{\partial x_1'^p}$ if, and only if,

$$\Sigma \left(\frac{\partial\psi}{\partial x_1}\right)^{a_1} \left(\frac{\partial\psi}{\partial x_2}\right)^{a_2} \ldots \left(\frac{\partial\psi}{\partial x_n}\right)^{a_n} \frac{\partial f}{\partial z_{a_1 \ldots a_n}} \neq 0;$$

so that Cauchy's theorem may be stated as follows:—"A definite number of solutions of the differential equation $f = 0$ can be found which are of the form $z = F(x_1 \ldots x_n)$. F is a holomorphic function of $x_1 \ldots x_n$ which can be so chosen that the locus $z = F$ passes through $\begin{cases} z = \phi(x_2 \ldots x_n) \\ \psi(x_1 \ldots x_n) = 0 \end{cases}$, and has contact of $(p-1)^{\text{st}}$ order at all points of $\begin{cases} z = \phi \\ \psi = 0 \end{cases}$ with any arbitrarily assigned n-dimensional

If we have any quantic $\Sigma a_{a_1 \dots a_n} \xi_1^{a_1} \dots \xi_n^{a_n}$, then $\Sigma a_{a_1 \dots a_n} \dfrac{d^{a_1+\dots+a_n}}{dx_1^{a_1} \dots dx_n^{a_n}}$ * is said to be the operation which corresponds to the quantic. It follows that the operation which corresponds to the quantic which corresponds to f is

$$\Sigma \frac{\partial f}{\partial z_{\dots a_n}} \frac{d^{a_1+\dots+a_n}}{dx_1^{a_1} \dots dx_n^{a_n}}.$$

We may speak of this as the operation which corresponds to the differential expression f.

Let $w_1 \dots w_s$ be the s quantics which correspond to the s differential expressions $f_1 \dots f_s$ which are respectively of orders $p_1 \dots p_s$; let

$$\left.\begin{array}{l} v_{11},\ v_{21},\ \dots,\ v_{s1} \\ v_{12},\ v_{22},\ \dots,\ v_{s2} \\ \vdots \\ v_{1r},\ v_{2r},\ \dots,\ v_{sr} \end{array}\right\} \tag{1}$$

be r sets of quantics such that for all values of κ from 1 up to r inclusive

$$v_{1\kappa} w_1 + v_{2\kappa} w_2 + \dots + v_{s\kappa} w_s \equiv 0\ ; \tag{2}$$

then, if $\lambda_1 \dots \lambda_r$ are any other arbitrary quantics, the identity

$$\sum_{\substack{\kappa=1 \\ h=1}}^{\substack{\kappa=r \\ h=s}} \lambda_\kappa v_{h\kappa} w_h \equiv 0$$

is merely an algebraic consequence of the identities (2), and is said to be reducible.

An identity of the form (2) which is not a consequence of identities of the same form and of lower degree is said to be simple.

---

locus which also passes through $\begin{cases} z = \phi \\ \psi = 0 \end{cases}$; the only limitations placed on the arbitrarily assigned holomorphic functions ϕ and ψ are that the direction cosines $\xi_1 \dots \xi_n$ of the normals to $\psi = 0$ must not satisfy the equation of the quantic which corresponds to f."

* The symbol $\dfrac{d}{dx_r}$ is used to denote total differentiation with respect to x_r, thus,

$$\frac{d}{dx_r} = \frac{\partial}{\partial x_r} + z_r \frac{\partial}{\partial z} + z_{1r} \frac{\partial}{\partial z_1} + z_{2r} \frac{\partial}{\partial z_2} + \dots .$$

It will be proved (§ 1) that, given the quantics $w_1 \dots w_n$, there are only a finite number of simple identities.

Let the r sets of quantics (1) generate simple identities, and let

$$\left.\begin{array}{l} \phi_{11},\ \phi_{21},\ \dots,\ \phi_{s1} \\ \phi_{12},\ \phi_{22},\ \dots,\ \phi_{s2} \\ \ \vdots \\ \phi_{1r},\ \phi_{2r},\ \dots,\ \phi_{sr} \end{array}\right\} \tag{3}$$

be the set of operations which correspond to them; then

$$\phi_{1\kappa} f_1 + \phi_{2\kappa} f_2 + \dots + \phi_{s\kappa} f_s \tag{4}$$

is said to be a combinant of the differential expressions $f_1 \dots f_s$.

From the definition here given of a combinant, and from the fact that there are only a finite number of simple identities, it at once follows that there are only a finite number of such combinants. If a combinant vanishes in consequence of the vanishing of $f_1 \dots f_s$ and the total differential coefficients of these expressions which do not contain differential coefficients of z of order higher than appear in the combinant, then the combinant is said to be satisfied.

The first theorem, then, to be stated is: "If all the combinants are satisfied, the differential equations $f_1 = 0 \dots f_s = 0$ will be integrable (§ 3).

If the combinants are not all satisfied, then we take those which are not satisfied as new equations, additional to $f_1 = 0 \dots f_s = 0$.

It may now happen that we have more equations than are algebraically sufficient to determine all the differential coefficients involved in them; in this case we see that the equation system is inconsistent; if not, we proceed as before with this increased system, and find its combinants. If these are satisfied, then the new equation system, and as a consequence the original one, is consistent; if not, we proceed further till we finally reach a satisfied system of combinants, or obtain more equations than are sufficient to determine the coefficients involved: in the former case the original system is consistent, in the latter it is not. That this question will be decided in every case by a finite number of operations is, I believe, true, but I have not yet succeeded in finding a general proof of its truth.

When all the differential equations are of the first order, combinants, as I have defined them, are easily seen to coincide with the Jacobian series of combinants and are all of the first order; the

operations in this case then obviously form a closed series; so that the method of this paper will prove that Jacobi's conditions are necessary and sufficient, and may therefore be considered an extension of those conditions to the case of equations of higher order.

It may, perhaps, be worth while to point out that the question whether or no given equations are consistent is of interest quite apart from any light it may throw on the solution of equations; thus the question might be asked: "Is there any infinitesimal transformation $\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}$, which leaves the equation

$$x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial y^2} = 0$$

unaltered?" The answer to this would depend on the possibility of certain differential equations having common solutions, and it may be proved that these equations are inconsistent; so that

$$x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial y^2} = 0$$

does not admit any infinitesimal transformation.

It will be proved (§ 2) that, if we have s quantics $w_1 \ldots w_s$, where $s \leqq n$, then, unless a special relation exists between their coefficients, the only simple identities are the obvious ones

$$w_h w_\kappa - w_\kappa w_h \equiv 0.$$

It follows then from the definition of a combinant that, if we have s differential equations $f_1 = 0 \ldots f_s = 0$, where $s \leqq n$, then, unless the quantics which correspond to them are of special form, the only combinants are of the form

$$\Sigma \frac{\partial f_h}{\partial z_{\alpha_1 \ldots \alpha_n}} \frac{d^{\alpha_1 + \ldots + \alpha_n} f_\kappa}{dx_1^{\alpha_1} \ldots dx_n^{\alpha_n}} - \Sigma \frac{\partial f_\kappa}{\partial z_{\beta_1 \ldots \beta_n}} \frac{d^{\beta_1 + \ldots + \beta_n} f_h}{dx_1^{\beta_1} \ldots dx_n^{\beta_n}},$$

the summation in first Σ being for all zero and positive integral values of $\alpha_1 \ldots \alpha_n$ such that $\alpha_1 + \ldots + \alpha_n = p_h$, and in the second for such values of $\beta_1 \ldots \beta_n$ as make $\beta_1 + \ldots + \beta_n = p_\kappa$.

As an example of the application of the methods discussed in this paper, it is proved (§ 4) that, if

$$F(x, y, z) = 0$$

is any minimum surface, then

$$F\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) = 0$$

and
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

are consistent, and their common solutions involve two arbitrary functions; and no other equation of the first order of the form

$$\phi\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) = 0$$

has a solution satisfying $\nabla^2 u = 0$ and involving two arbitrary functions. It is shown how, given any minimum surface, a solution of $\nabla^2 u = 0$ can be made to depend on the solution of a partial differential equation of the second order in two independent variables. These results were suggested by Prof. Forsyth's paper in the *Messenger*, as was also the second example discussed. The second example (§ 5) proves that not only are the equations

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} + \frac{\partial^2 u}{\partial x_4^2} = 0$$

and
$$\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2 + \left(\frac{\partial u}{\partial x_3}\right)^2 + \left(\frac{\partial u}{\partial x_4}\right)^2 = 0$$

consistent (as the common solutions obtained in the *Messenger* show), but that they form with one other equation of the second order a completely integrable system whose common solutions involve four arbitrary functions of one argument.

1. If we take any number of *given* quantics $w_1 \dots w_s$ in any number of variables $x_1 \dots x_m$, and of any degrees, the question arises as to the form of s quantics $v_1 \dots v_s$ such that

$$v_1 w_1 + \dots + v_s w_s \equiv 0.$$

It must first be proved that there are only a finite number of simple identities of the above form—that is to say, there are only a limited number r of sets of quantics

$$\begin{array}{l} v_{11},\ v_{21},\ \dots,\ v_{s1}, \\ v_{12},\ v_{22},\ \dots,\ v_{s2}, \\ \vdots \\ v_{1r},\ v_{2r},\ \dots,\ v_{sr}, \end{array}$$

such that $\qquad v_{1\kappa}w_1+v_{2\kappa}w_2+\ldots+v_{s\kappa}w_s=0 \quad (\kappa=1, 2, \ldots, r),$

and that every other set $v_1 \ldots v_s$, such that

$$v_1w_1+\ldots+v_sw_s\equiv 0$$

is given by $\qquad v_\rho=\lambda_1v_{\rho 1}+\lambda_2v_{\rho 2}+\ldots+\lambda_rv_{\rho s} \quad (\rho=1, 2, \ldots, s).$

This theorem is almost an immediate consequence of Hilbert's very general theorem: "If S denotes *any* system of forms in n variables $x_1, x_2 \ldots x_n$, there can be so selected from S a finite number of forms $F_1, F_2 \ldots F_\mu$ that every form F of S can be expressed in the form

$$F\equiv A_1F_1+A_2F_2+\ldots+A_\mu F_\mu,$$

where $A_1, A_2 \ldots A_\mu$ are forms in the variables $x_1, x_2 \ldots x_n$" (Weber's *Algebra*, first edition, Vol. II., pp. 165–168). Now take for the system S that of forms which can be expressed in both the shapes

$$v_1w_1+v_2w_2+\ldots+v_{s-1}w_{s-1} \quad \text{and} \quad -v_sw_s.$$

The theorem which I wish to prove follows, except as regards sets of $v_1, v_2 \ldots v_s$ in which $v_s=0$. As to these, take for S the system of forms which can be expressed in both the shapes

$$v_1w_1+\ldots+v_{s-2}w_{s-2} \quad \text{and} \quad -v_{s-1}w_{s-1}.$$

It follows as to sets in which $v_s=0$, but $v_{s-1}\neq 0$. Continue in like manner. After at most $s-1$ repetitions, the theorem follows in its generality.*

An example on the calculation of simple identities having an interesting application to the differential equation

$$\frac{d^2V}{dx_1^2}+\frac{d^2V}{dx_2^2}+\frac{d^2V}{dx_3^2}+\frac{d^2V}{dx_4^2}=0$$

had perhaps best be given here.

Let $$w_1\equiv\left(\frac{y}{b}-\frac{z}{c}\right)^2\equiv X^2,$$

$$w_2\equiv\left(\frac{z}{c}-\frac{x}{a}\right)^2\equiv Y^2,$$

$$w_3\equiv\left(\frac{x}{a}-\frac{y}{b}\right)^2.$$

* [I owe this reference and proof to the kindness of Prof. Elliott, who has also given me very much valued help in other parts of the paper. I desire to express to him and to both of the referees my thanks for the great trouble which they have taken in considering this paper.]

If, now, $$v_1 w_1 + v_2 w_2 + v_3 w_3 \equiv 0,$$

then $$v_1 X^2 + v_2 Y^2 + v_3 (X+Y)^2 \equiv 0,$$

or $$(v_1+v_3) X^2 + (v_2+v_3) Y^2 + 2v_3 XY \equiv 0;$$

therefore v_1+v_3 must be divisible by Y, so that

$$v_1+v_3 \equiv 2YP,$$

where P is some function of x, y, and z. Similarly,

$$v_2+v_3 = 2XQ;$$

consequently $$2YPX^2 + 2XQY^2 + 2XYv_3 \equiv 0;$$

and therefore $$2v_3 + 2PX + 2QY \equiv 0,$$

that is, $$v_1 \equiv (2Y+X)P + YQ,$$

$$v_2 \equiv (2X+Y)Q + XP,$$

$$v_3 \equiv -XP - YQ.$$

The simple identities are then given by

$$\left.\begin{array}{llll} & v_1 = \dfrac{y}{b} + \dfrac{z}{c} - \dfrac{2x}{a}, & v_2 = \dfrac{y}{b} - \dfrac{z}{c}, & v_3 = \dfrac{z}{c} - \dfrac{y}{b} \\ \text{and} & v_1 = -\dfrac{z}{c} + \dfrac{x}{a}, & v_2 = \dfrac{x}{a} + \dfrac{z}{c} - \dfrac{2y}{b}, & v_3 = \dfrac{z}{c} - \dfrac{x}{a} \end{array}\right\}. \quad (10)'$$

2. It is now to be proved that when $s \leqq n$, unless the quantics $w_1 \dots w_s$ are such that their coefficients are connected by certain relations, there are no simple identities except those of the type

$$w_h w_\kappa - w_\kappa w_h \equiv 0.$$

Any term $x_1^{a_1} \dots x_n^{a_n}$ is said to be derived if $a_1 \geqq p_1$, or $a_2 \geqq p_2 \dots$ or $a_s \geqq p_s$, where s is a given integer $\not> n$ and $p_1 \dots p_s$ any s given integers; a term which is not derived is called arbitrary; Σa is the order of the above term.

It is easily seen that the number of arbitrary terms of order r is the coefficient of x^r in

$$(1-x^{p_1}) \dots (1-x^{p_s})(1-x)^{-n} = a_0+a_1x+\dots+a_rx^r+\dots;$$

$a_0, a_1, \dots$ are, of course, positive integers; if $s<n$, the series is infinite; if $s=n$, the series is finite. From the series

$$(1-x)^{-n} = (a_0+a_1x+\dots)(1-x^{p_1})^{-1} \dots (1-x^{p_s})^{-1},$$

it follows that, if H_r denotes the coefficient of x^r in $(1-x)^{-n}$,

$$H_r = a_r + \sum_{\kappa=1}^{\kappa=s} a_{r-p_\kappa} + \sum_{\substack{\kappa=1\\t=1}}^{\substack{\kappa=s\\t=s}} a_{r-p_\kappa-p_t} + \dots,$$

the summation on the right being continued so long as the suffixes are non-negative.

It should be noticed that $H_r \nless a_r$, and that, if $s=n$, there are no arbitrary terms of order higher than $(p_1-1)(p_2-1) \dots (p_n-1)$.

If we have s quantics of order r in the n variables $x_1 \dots x_n$, we can form a matrix; thus the first row consists of the coefficients of the first r^{ic} in any assigned order, the second of the corresponding coefficients of the second in the same assigned order, and so on; *i.e.*, the coefficients of the same term in each r^{ic} form a column of the matrix.

From any quantic w of order p, we can form H_{r-p} derived quantics $x_1^{a_1} \dots x_n^{a_n} w$, by taking all positive integral and zero values of a, such that $\Sigma a = r-p$. Let us therefore form the matrix of the $\sum\limits_{\kappa=1}^{\kappa=s} H_{r-p_\kappa}$ derived r^{ics} of $w_1 \dots w_s$.

Now it must be shown that in general not every H_r-a_r-rowed determinant of this matrix will vanish. To prove this it will be sufficient to take

$$w_1 \equiv x_1^{p_1} \dots w_s \equiv x_s^{p_s}.$$

Then we can choose as H_r-a_r derived quantics the H_r-a_r derived terms, that is, the distinct terms which contain x_1^p or $x_2^p \dots$ or x_s^p. In this case, we see that in each row there is one, and only one, term which is not a zero coefficient; and no column can contain two non-zero coefficients, so that the matrix will contain one determinant of order H_r-a_r which does not vanish.

Unless, then, the quantics $w_1 \dots w_s$ are of "special form," not all

$H_r - a_r$-rowed determinants of the matrix will vanish; and we may assume without any real loss of generality that, in particular, the determinant of the derived terms will not vanish.

It will now be proved that all $H_r - a_r + 1$-rowed determinants of the matrix do vanish.

We may assume that, b_r denoting zero or some positive integer, not all $H_r - a_r + b_r$-rowed determinants vanish, but that all $H_r - a_r + b_r + 1$-rowed determinants do vanish.

It follows that we can express all derived terms, and a certain b_r arbitrary terms, in terms of $a_r - b_r$ remaining arbitrary terms and the derived quantics of $w_1 \dots w_s$.

Every r^{ic} can therefore be expressed in the form

$$w_1 v_1 + \dots + w_s v_s + P_r,$$

where $v_1 \dots v_s$ are respectively quantics of degree $r - p_1 \dots r - p_s$, and P_r is an r^{ic} which only contains the above $a_r - b_r$ arbitrary terms.

Treating v_1 by the same method, we see that it can be expressed in the form

$$w_1 v_{11} + \dots + w_s v_{s1} + P_{r-p_1},$$

where $v_{\kappa 1}$ is of degree $r - p_\kappa - p_1$, and P_{r-p_1} only contains $a_{r-p_1} - b_{r-p_1}$ arbitrary terms: proceeding thus, it is clear that every r^{ic} can be expressed in the form

$$P_r + w_1 P_{r-p_1} + \dots + w_s P_{r-p_s} + w_1^2 P_{r-2p_1} + w_1 w_2 P_{r-p_1-p_2} + \dots, \qquad \text{(I.)}$$

where the term P_{r-p_κ}, for instance, represents a quantic of degree $r - p_\kappa$, which only contains $a_{r-p_\kappa} - b_{r-p_\kappa}$ arbitrary terms and no derived terms.

The number of arbitrary coefficients in the above form cannot then exceed

$$a_r - b_r + \sum_{\kappa=1}^{\kappa=s} (a_{r-p_\kappa} - b_{r-p_\kappa}) + \sum_{\substack{\kappa=1 \\ t=1}}^{\substack{\kappa=s \\ t=s}} (a_{r-p_\kappa-p_t} - b_{r-p_\kappa-p_t}) + \dots .$$

It is said that the number of arbitrary coefficients cannot exceed the above limit, rather than that it is equal to it, because of possible identities of the form (I.).

Now the number of effective arbitrary constants in any r^{ic} is H_r, so that

$$H_r \leqq a_r - b_r + \sum_{\kappa=1}^{\kappa=s} (a_{r-p_\kappa} - b_{r-p_\kappa}) + \dots ;$$

but
$$H_r = a_r + \sum_{\kappa=1}^{\kappa=s} a_{r-p_\kappa} + \dots;$$

therefore
$$b_r + \sum_{\kappa=1}^{\kappa=s} b_{r-p_\kappa} + \dots \leqq 0,$$

an inequality which (since b_κ is a positive integer or zero) can only hold when
$$b_\kappa = 0.$$

The conclusions that we draw are, firstly, that every $H_r - a_r + 1$-rowed determinant of the matrix of s non-special quantics does vanish, and that therefore the derived terms, and no others, can be expressed in terms of the arbitrary terms and the derived quantics; and, secondly, that every r^{ic} can be expressed in one definite way only in the form
$$P_r + \sum_{\kappa=1}^{\kappa=s} w_\kappa P_{r-p_\kappa} + \dots, \qquad \text{(II.)}$$

where $P_r \dots$ denote quantics, of degree equal to their suffix, and only containing arbitrary terms; and consequently there can be no identity of this form. When an r^{ic} is so expressed, it is said to be in "standard form."

It is now required to investigate the form of s quantics $v_1 \dots v_s$, such that
$$v_1 w_1 + \dots + v_s w_s \equiv 0. \qquad \text{(III.)}$$

Remembering that $v_1 \dots v_s$ can each be thrown into standard form, and that there can be no identical relation between $w_1 \dots w_s$ and arbitrary terms of the form (II.), we conclude that the coefficients of each arbitrary in the above identity must be zero. The problem is therefore really reduced to finding the forms of s rational integral functions of $w_1 \dots w_s$, such that
$$v_1 w_1 + \dots + v_s w_s \equiv 0.$$

Now any rational integral function of $w_1 \dots w_s$ may be written in the form
$$v_1 \equiv P_1 + w_2 P_{12} + \dots + w_s P_{1s} + \sum_{\substack{\kappa=1 \\ t=1}}^{\substack{\kappa=s \\ t=s}} w_\kappa w_t P_{1\kappa t} + \dots + w_2 w_3 \dots w_s P_{12\dots s},$$

where $P_{1\kappa t}$, for instance, denotes a rational integral function of w_1, w_κ, and w_t only

Expressing $v_2 \dots v_s$ also in similar forms, we deduce from the equation (III.)

$$P_m = P_h = P_t = \dots = 0,$$

$$P_{hm} + P_{mh} = 0, \quad P_{hmt} + P_{mht} + P_{tmh} = 0,$$

$$P_{qmht} + P_{mqht} + P_{tmhq} + P_{hmqt} = 0.$$

It is clear that in the P functions all the suffixes except the first may be interchanged without altering the form of the functions.

In case of equal suffixes the equations deduced differ slightly; thus, if $q = m$, he equations last written would be replaced by

$$P_{mmht} + P_{tmhm} + P_{hmmt} = 0;$$

and, if $q = m = t$, by $\quad P_{mmhm} + P_{hmmm} = 0;$

if, finally, $h = m$, by $\quad P_{mmmm} = 0.$

It is not difficult to see that consequently $v_1 \dots v_s$ may be written in the form

$$\left.\begin{array}{l} v_1 = Q_{12} w_2 + Q_{13} w_3 + \dots + Q_{1s} w_s \\ v_2 = Q_{21} w_1 + Q_{23} w_3 + \dots + Q_{2s} w_s \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ v_s = Q_{s1} w_1 + Q_{2s} w_2 + \dots + Q_{s,s-1} w_{s-1} \end{array}\right\}, \tag{IV.}$$

where $\quad Q_{hm} + Q_{mh} \equiv 0,$

but except for this restriction the Q's are any functions whatever of $w_1 \dots w_s$.

It follows that the only simple identities are of the form

$$w_h w_\kappa - w_\kappa w_h \equiv 0.$$

In case all the quantics are linear forms in $x_1 \dots x_n$, it is obvious that the system is non-special; in fact, we lose no essential generality in taking $w_1 \equiv x_1 \dots w_s \equiv w_s$, in which it has been shown that not all $H_r - a_r$-rowed determinants of the matrix disappear.

We can now write down the combinants of the differential expressions $f_1 \dots f_s$ for the case here considered, viz., when $n \not< s$, and the quantics $w_1 \dots w_s$ which correspond to $f_1 \dots f_s$ are non-special. Since the only quantics which now generate simple identities are $w_1 \dots w_s$, we see that in (1) $v_{mh} = w_\kappa$, where κ is some integer $\not> s$ and $\neq h$, and $v_{m\kappa} = -w_h$, and all other quantics in the row which contains v_{mh} and $v_{m\kappa}$ are zero.

It follows that the operations ϕ_{mh} and $\phi_{m\kappa}$ in (3) which correspond to these are

$$\Sigma \frac{\partial f_\kappa}{\partial z_{\beta_1 \dots \beta_n}} \frac{d^{\beta_1 + \dots + \beta_n}}{dx_1^{\beta_1} \dots dx_n^{\beta_n}},$$

the summation being for all non-negative integral values of $\beta_1 \dots \beta_n$, such that

$$\beta_1 + \dots + \beta_n = p_\kappa,$$

and
$$-\Sigma \frac{\partial f_h}{\partial z_{\alpha_1 \dots \alpha_n}} \frac{d^{\alpha_1 + \dots + \alpha_n}}{dx_1^{\alpha_1} \dots dx_n^{\alpha_n}},$$

the summation being for such values of $\alpha_1 \dots \alpha_n$ that

$$\alpha_1 + \dots + \alpha_n = p_h;$$

and therefore we get the typical form of combinant for non-special cases to be

$$\Sigma \frac{\partial f_h}{\partial z_{\alpha_1 \dots \alpha_n}} \frac{d^{\alpha_1 + \dots + \alpha_n} f_\kappa}{dx_1^{\alpha_1} \dots dx_n^{\alpha_n}} - \Sigma \frac{\partial f_\kappa}{\partial z_{\beta_1 \dots \beta_n}} \frac{d^{\beta_1 + \dots + \beta_n} f_h}{dx_1^{\beta_1} \dots dx_n^{\beta_n}}.$$

We can easily verify the fundamental property of this combinant, that all partial derivatives of order $p_\kappa + p_h$ disappear from it, for the derivatives $z_{\alpha_1+\beta_1 \dots \alpha_n+\beta_n}$ appear under each summation with the coefficient

$$\frac{\partial f_h}{\partial z_{\alpha_1 \dots \alpha_n}} \frac{\partial f_\kappa}{\partial z_{\beta_1 \dots \beta_n}},$$

and consequently the terms cancel.

3. From the s differential equations $f_1 = 0 \dots f_s = 0$ respectively of orders $p_1 \dots p_s$, we obtain, to determine the differential coefficients of the r^{th} order, the system of equations

$$\frac{d^{\alpha_1 + \dots + \alpha_n} f_\kappa}{dx_1^{\alpha_1} \dots dx_n^{\alpha_n}} = 0,$$

where all zero and positive integral values of $\alpha_1 \dots \alpha_n$ are to be taken, such that

$$\Sigma \alpha = r - p_\kappa,$$

and κ is to have any value from 1 up to s inclusive.

After r attains a certain value there will be more equations of this system than there are differential coefficients of the r^{th} order; so that

we can eliminate the coefficients of the r^{th} order, and obtain a reduced system of equations not containing any coefficients of order higher than $r-1$. The system of the r^{th} order may then be divided into two parts: the first will not contain more equations than are sufficient to determine the coefficients of the r^{th} order in terms of coefficients of lower order (and it may not contain so many)—it will be convenient to speak of these equations as the effective ones of the r^{th} order; the second part will consist of reduced equations not containing coefficients of the r^{th} order. The system of the r^{th} order will then contain effective and reduced equations; the reduced equations of the r^{th} order may of course be effective in determining coefficients of the $(r-1)^{\text{st}}$ orders. It is now necessary to examine the forms of these reduced equations.

The highest differential coefficients which occur in $\dfrac{d^{a_1+a_2+\ldots+a_n} f_\kappa}{dx_1^{a_1} \ldots dx_n^{a_n}}$ occur in the part

$$\Sigma z_{a_1+l_1,\, a_2+l_2 \ldots a_n+l_n} \frac{\partial f_\kappa}{\partial z_{l_1 l_2 \ldots l_n}},$$

where the summation is to be taken for all positive integral and zero values of $l_1, l_2 \ldots l_n$, such that

$$l_1+l_2+\ldots+l_n = p_\kappa,$$

p_κ being the order of the highest derivative in f_κ. They occur linearly.

For
$$\sum^{\kappa} \sum^{a_1 \ldots a_n} \lambda_{\kappa\, a_1 a_2 \ldots a_n} \frac{d^{a_1+a_2+\ldots+a_n} f_\kappa}{dx_1^{a_1} \ldots dx_n^{a_n}} = 0, \tag{11}$$

where $\lambda_{\kappa a_1 \ldots a_n}$ is some function of $x_1 x_2 \ldots x_n z$, and differential coefficients of order not exceeding p_κ, and where the summations cover all non-negative integral values of $a_1 \ldots a_n$, for which $\Sigma a = r - p_\kappa$, and all integral values of κ from 1 to s inclusive, to be free from differential coefficients of order exceeding $r-1$, and so to be an equation of the reduced system, it is then necessary and sufficient that

$$\sum^{\kappa} \sum^{a_1 a_2 \ldots a_n} \lambda_{\kappa a_1 a_2 \ldots a_n} \sum^{l_1 l_2 \ldots l_n} z_{a_1+l_1,\, a_2+l_2 \ldots a_n+l_n} \frac{\partial f_\kappa}{\partial z_{l_1 l_2 \ldots l_n}} \tag{12}$$

vanish identically, where the summations for $l_1 l_2 \ldots l_n$, for $a_1 a_2 \ldots a_n$, and for κ are as explained above. And this sum will vanish identically if, and only if, the sum

$$\sum^{\kappa} \sum^{a_1 a_2 \ldots a_n} \lambda_{\kappa a_1 a_2 \ldots a_n} \sum^{l_1 l_2 \ldots l_n} \xi_1^{a_1+l_1} \xi_2^{a_2+l_2} \ldots \xi^{a_n+l_n} \frac{\partial f_\kappa}{\partial z_{l_1 l_2 \ldots l_n}},$$

where $\xi_1, \xi_2 \dots \xi_n$ are any distinct quantities or symbols, vanishes identically; *i.e.*, if

$$\sum^{\kappa} \sum^{a_1 \dots a_n} \lambda_{\kappa a_1 \dots a_n} \xi_1^{a_1} \xi_2^{a_2} \dots \xi_n^{a} \sum^{l_1 l_2 \dots l_n} \xi_1^{l_1} \xi_2^{l_2} \dots \xi_n^{l_n} \frac{\partial f_\kappa}{\partial z_{l_1 l_2 \dots l_n}} \qquad (13)$$

vanishes identically.

Now here, for any κ, $\sum^{a_1 \dots a_n} \lambda_{\kappa a_1 a_2 \dots a_n} \xi_1^{a_1} \xi_2^{a_2} \dots \xi_n^{a_n}$ is what we have earlier defined as the quantic which corresponds to the operation

$$\sum^{a_1 \dots a_n} \lambda_{\kappa a_1 \dots a_n} \frac{d^{a_1+a_2+\dots+a_n}}{dx_1^{a_1} \dots dx_n^{a_n}},$$

and
$$\sum^{l_1 l_2 \dots l_n} \xi_1^{l_1} \xi_2^{l_2} \dots \xi_n^{l_n} \frac{\partial f_\kappa}{\partial z_{l_1 l_2 \dots l_n}}$$

is what we have defined as the quantic which corresponds to f_κ.

We have, then, established that, if $v_1, v_2 \dots v_s$ be the quantics which correspond to the s operations

$$\sum^{a_1 a_2 \dots a_n} \lambda_{\kappa a_1 \dots a_n} \frac{d^{a_1+a_2+\dots+a_n}}{dx_1^{a_1} dx_2^{a_2} \dots dx_n^{a_n}} \quad (\kappa = 1, 2 \dots s),$$

and if $w_1, w_2 \dots w_s$ be the quantics which correspond to $f_1, f_2 \dots f_s$ respectively, what is necessary and sufficient that (11) may be an equation of the reduced system is that

$$v_1 w_1 + v_2 w_2 + \dots + v_s w_s \equiv 0.$$

Here $w_1, w_2 \dots w_s$ are definitely given quantics. We apply then the conclusion of § 1, and are enabled to state that every v_κ must be of the form

$$\lambda_1 v_{\kappa 1} + \lambda_2 v_{\kappa 2} + \dots + \lambda_m v_{\kappa m} \quad (\kappa = 1, 2 \dots s),$$

where

$$\begin{array}{l} v_{11},\ v_{21}\ \dots\ v_{s1}, \\ v_{12},\ v_{22}\ \dots\ v_{s2}, \\ \dots\ \ \dots\ \ \dots \\ v_{1m},\ v_{2m}\ \dots\ v_{sm} \end{array}$$

are the m sets of quantics which occur in the simple identities of $w_1, w_2 \dots w_s$; and where $\lambda_1, \lambda_2 \dots \lambda_m$ are of orders $r-p_1, r-p_2 \dots r-p_m$.

We proceed to apply this conclusion to the supposed reduced equation (11).

If we write it
$$\phi_1 f_1+\phi_2 f_2+\ldots+\phi_s f_s=0,$$

then ϕ_κ $(\kappa=1, 2 \ldots s)$ is the operation to which corresponds the quantic v_κ, *i.e.*, is the result of replacing $\xi_1, \xi_2 \ldots \xi_n$ by $\frac{d}{dx_1}, \frac{d}{dx_2} \ldots \frac{d}{dx_n}$ in that quantic. Now let $\phi_{\kappa\nu}$ $(\kappa=1, 2 \ldots s;\ \nu=1, 2 \ldots m)$ be the operations to which correspond in like manner the quantics $v_{\kappa\nu}$, and let $\mu_1, \mu_2 \ldots \mu_m$ be the operations to which correspond the quantics $\lambda_1 \ldots \lambda_m$. What we have learned is that ϕ_κ is the result of omitting from
$$\mu_1\phi_{\kappa 1}+\mu_2\phi_{\kappa 2}+\ldots+\mu_m\phi_{\kappa m}$$

all operations of differentiation of lower order than the highest which occur, *i.e.*, than order $r-p_\kappa$.*

Consequently (11) differs from
$$\mu_1\sum_{\kappa=1}^{\kappa=s}\phi_{\kappa 1}f_\kappa+\mu_2\sum_{\kappa=1}^{\kappa=s}\phi_{\kappa 2}f_\kappa+\ldots+\mu_m\sum_{\kappa=1}^{\kappa=s}\phi_{\kappa m}f_\kappa=0$$

only by terms which have for factors derivatives
$$\frac{d^{a_1+\ldots+a_n}}{dx_1^{a_1}\ldots dx_n^{a_n}} \quad (\kappa=1, 2 \ldots s),$$

for which $a_1+a_2+\ldots+a_n < r-p_\kappa$, *i.e.*, by terms whose vanishing is a result of the vanishing of total derivatives of $f_1, f_2 \ldots f_s$ which are not of high enough order to involve partial derivatives of z with regard to $x_1, x_2 \ldots x_n$ of order exceeding $r-1$; that is, all reduced equations of the r^{th} order [since (11) was the general form of such equations] can be obtained by differentiation of the combinants, and by the addition of total derivatives of $f_1, f_2 \ldots f_s$ which are not of high enough order to involve partial derivatives of z with regard to $x_1, x_2 \ldots x_n$ of order exceeding $r-1$.

Now suppose that all the combinants are satisfied; then there will be no reduced equations of the r^{th} order; and, proceeding similarly with the equations of the $(r-1)^{\text{st}}$ and lower orders, we see that there are none except the effective ones. The number of effective equations is never greater than is sufficient to determine the coefficients,

---

* In the actual v_κ, $\lambda_1 v_{\kappa 1}$, $\lambda_2 v_{\kappa 2}$, &c. are mere algebraical products; whereas the operator $\mu_1\phi_{\kappa 1}$ is the sum of such an algebraic product and other parts resulting from operations of μ_1 on the coefficients of symbols of differentiation in $\phi_{\kappa 1}$.

so that in this case the system must be integrable; and by subtracting the number of effective equations of any order from the number of differential coefficients of that order we measure the generality of the common solution possible.

4. It is required to find the form of the most general differential equation

$$F\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) = 0,$$

such that

$$F = 0$$

and

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

may have common solutions involving two arbitrary functions.

We have wherewith to determine the derivatives of the second order

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0,$$

$$\frac{dF}{dx} = 0, \quad \frac{dF}{dy} = 0, \quad \frac{dF}{dz} = 0;$$

F must therefore be of such form that it is not possible to deduce any equation of the second order independent, algebraically, of these four; it follows that the first combinant of F and $\nabla^2 u$ must vanish identically by aid of $F = 0$ and these four equations. It will lighten the labour of determining F if we use the following notation

$$\frac{\partial u}{\partial x} = \lambda_1, \quad \frac{\partial u}{\partial y} = \lambda_2, \quad \frac{\partial u}{\partial z} = \lambda_3, \quad \frac{\partial^2 u}{\partial y \partial z} = \lambda_{23},$$

with similar expressions for $\dfrac{\partial^2 u}{\partial x^2}, \dfrac{\partial^2 u}{\partial x \partial z}, \ldots,$

$$\frac{\partial F}{\partial \lambda_\kappa} \equiv F_\kappa, \quad \frac{\partial^2 F}{\partial \lambda_\kappa \partial \lambda_h} \equiv F_{\kappa h}.$$

Since there are only two equations $F = 0$ and $\nabla^2 u = 0$, and the quantics which correspond to these are respectively a line, and $\xi_1^2 + \xi_2^2 + \xi_3^2$, and the latter does not break into factors, we see that the only combinant is

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right) F - \left(F_1 \frac{d}{dx} + F_2 \frac{d}{dy} + F_3 \frac{d}{dz}\right) \nabla^2 u.$$

Now
$$\frac{dF}{dx_\kappa} = \lambda_{1\kappa} F_1 + \lambda_{2\kappa} \dot{F}_2 + \lambda_{3\kappa} F_3,$$

so that, remembering that derivatives higher than the second disappear identically from the combinant, we obtain without much labour that the combinant is

$$F_{11}(\lambda_{11}^2 + \lambda_{12}^2 + \lambda_{13}^2) + \ldots + 2F_{23}\{\lambda_{12}\lambda_{13} + \lambda_{23}(\lambda_{22} + \lambda_{33})\};$$

then, from the fact that

$$\frac{dF}{dx_1} = \frac{dF}{dx_2} = \frac{dF}{dx_3} = 0,$$

we have
$$\left.\begin{array}{l} \lambda_{11} F_1 + \lambda_{12} F_2 + \lambda_{13} F_3 = 0 \\ \lambda_{12} F_1 + \lambda_{22} F_2 + \lambda_{23} F_3 = 0 \\ \lambda_{13} F_1 + \lambda_{23} F_2 + \lambda_{33} F_3 = 0 \end{array}\right\}. \tag{14}$$

If now we write
$$\lambda_{11} = a, \quad \lambda_{22} = b, \quad \lambda_{33} = c,$$
$$\lambda_{12} = h, \quad \lambda_{23} = f, \quad \lambda_{31} = g,$$

and employ the notation usual in the theory of conics, we have (since $a+b+c=0$) as combinant

$$-(B+C)F_{11} - (C+A)F_{22} - (A+B)F_{33} + 2GHF_{23} + 2FHF_{31} + 2FGF_{12}.$$

From (14) we deduce

$$\frac{F_1}{GH} = \frac{F_2}{FH} = \frac{F_3}{FG},$$

since the discriminant

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0,$$

and
$$B + C = \frac{FH}{G} + \frac{GH}{F}$$

for the same reason, so that, finally, the combinant takes the simple form

$$(F_2^2 + F_3^2)F_{11} + (F_3^2 + F_1^2)F_{22} + (F_1^2 + F_2^2)F_{33}$$
$$= 2F_2F_3F_{23} + 2F_3F_1F_{13} + 2F_1F_2F_{12},$$

that is, when $F = 0$ is looked on as a surface in space whose coordinates are $\lambda_1, \lambda_2, \lambda_3$, it has the sum of its principal curvatures everywhere zero, that is, is a minimum surface.

We may verify this result, and at the same time see how to obtain particular classes of solutions of the equation

$$\nabla^2 u = 0$$

in the following method.

Let $$z = f(x, y)$$

be any solution whatever of the equation

$$(1+p^2)\,t+(1+q^2)\,r-2pqs = 0,$$

that is, any minimum surface. It is well known that

$$u = ax+by+f\,(a, b)\,z+\phi\,(a, b),$$

where we consider x, y, and z as independent variables, and $\phi\,(a, b)$ is any arbitrary function of a and b, and a and b are given by

$$\frac{\partial u}{\partial a} = 0, \quad \frac{\partial u}{\partial b} = 0,$$

is the general integral of

$$\frac{du}{dz} = f\left(\frac{du}{dx}, \frac{du}{dy}\right).$$

We wish to find the form of ϕ in order that this may be an integral of

$$\nabla^2 u = 0.$$

Let us write $$\frac{\partial^2 f}{\partial a^2} = A', \quad \frac{\partial^2 f}{\partial a \partial b} = H', \quad \frac{\partial^2 f}{\partial b^2} = B',$$

$$\frac{\partial^2 \phi}{\partial a^2} = A, \quad \frac{\partial^2 \phi}{\partial a \partial b} = H, \quad \frac{\partial^2 \phi}{\partial b^2} = B,$$

$$\frac{\partial f}{\partial a} = P', \quad \frac{\partial f}{\partial b} = Q', \quad \frac{\partial \phi}{\partial a} = P, \quad \frac{\partial \phi}{\partial b} = Q,$$

$\dfrac{\partial u}{\partial a} = 0$ is then $$x+P'z+P = 0,$$

$\dfrac{\partial u}{\partial b} = 0$ is $$y+Q'z+Q = 0.$$

Differentiating these two equations with respect to x, y, z, we get

$$1+(zA'+A)\frac{\partial a}{\partial x}+(zH'+H)\frac{\partial b}{\partial x}=0,$$

$$(zA'+A)\frac{\partial a}{\partial y}+(zH'+H)\frac{\partial b}{\partial y}=0,$$

$$P'+(zA'+A)\frac{\partial a}{\partial z}+(zH'+H)\frac{\partial b}{\partial z}=0,$$

$$1+(zB'+B)\frac{\partial b}{\partial y}+(zH'+H)\frac{\partial a}{\partial y}=0,$$

$$(zB'+B)\frac{\partial b}{\partial x}+(zH'+H)\frac{\partial a}{\partial x}=0,$$

$$Q'+(zB'+B)\frac{\partial b}{\partial z}+(zH'+H)\frac{\partial a}{\partial z}=0.$$

Now $$\frac{\partial u}{\partial x}=a,\quad \frac{\partial u}{\partial y}=b,\quad \frac{\partial u}{\partial z}=f(a,\,b),$$

so that $\nabla^2 u = 0$, if, and only if,

$$\frac{\partial a}{\partial x}+\frac{\partial b}{\partial y}+P'\frac{\partial a}{\partial z}+Q'\frac{\partial b}{\partial z}=0.$$

Solving the first six equations, we obtain

$$\frac{\partial a}{\partial x}=-\frac{zB'+B}{D},$$

$$\frac{\partial b}{\partial y}=-\frac{zA'+A}{D},$$

$$\frac{\partial a}{\partial z}=-\frac{P'(zB'+B)-Q'(zH'+H)}{D},$$

$$\frac{\partial b}{\partial z}=-\frac{Q'(zA'+A)-P'(zH'+H)}{D},$$

where $D=z^2(A'B'-H'^2)+z(AB'+BA'-2HH')+AB-H^2$,

and we at once deduce

$$z\left\{A'(1+Q'^2)+B'(1+P'^2)-2H'P'Q'\right\}$$
$$+A(1+Q'^2)+B(1+P'^2)-2HP'Q'=0;$$

but the coefficient of z vanishes, from the definition of f, and we see that ϕ must satisfy the equation

$$\frac{\partial^2\phi}{\partial a^2}\left\{1+\left(\frac{\partial f}{\partial b}\right)^2\right\}+\frac{\partial^2\phi}{\partial b^2}\left\{1+\left(\frac{\partial f}{\partial a}\right)^2\right\}-2\frac{\partial^2\phi}{\partial a\,\partial b}\,\frac{\partial f}{\partial a}\,\frac{\partial f}{\partial b}=0.$$

Knowing now the form of f, and choosing ϕ so as to satisfy the above equation, we see that

$$u = ax+by+f(a,\,b)\,z+\phi\,(a,\,b)$$

will be a solution of $\nabla^2 u = 0$, provided that we choose a and b so as to satisfy

$$\frac{\partial u}{\partial a}=0,\quad \frac{\partial u}{\partial b}=0.$$

5. In the *Messenger of Mathematics* (November, 1897, p. 100), Prof. Forsyth proves, amongst other theorems, that, if p_1, p_2, p_3, p_4 denote four arbitrary functions of u subject to the single condition

$$p_1^2+p_2^2+p_3^2+p_4^2=0,$$

and if u be determined as a function of x_1, x_2, x_3, x_4 by the equation

$$au = x_1p_1+x_2p_2+x_3p_3+x_4p_4,$$

where a is a constant, then, if v denote any arbitrary function of u, it satisfies the equation

$$\frac{\partial^2 v}{\partial x_1^2}+\frac{\partial^2 v}{\partial x_2^2}+\frac{\partial^2 v}{\partial x_3^2}+\frac{\partial^2 v}{\partial x_4^2}=0,$$

and also the equation

$$\left(\frac{\partial u}{\partial x_1}\right)^2+\left(\frac{\partial v}{\partial x_2}\right)^2+\left(\frac{\partial v}{\partial x_3}\right)^2+\left(\frac{\partial v}{\partial x_4}\right)^2=0.$$

Now, it is very easily verified that u not only satisfies the above two equations, but also the equation

$$\frac{\partial^2 u}{\partial x_1^2}\left(\frac{\partial u}{\partial x_2}\right)^2+\frac{\partial^2 u}{\partial x_2^2}\left(\frac{\partial u}{\partial x_1}\right)^2-2\frac{\partial^2 u}{\partial x_1\partial x_2}\left(\frac{\partial u}{\partial x_1}\right)\left(\frac{\partial u}{\partial x_2}\right)=0$$

and five others of the same type. The question is thus suggested whether these six are mere consequences of the first two; it will be found that, though consistent with them, as of course they must be, they are not necessary consequences. The system

$$\left(\frac{\partial u}{\partial x_1}\right)^2+\left(\frac{\partial u}{\partial x_2}\right)^2+\left(\frac{\partial u}{\partial x_3}\right)^2+\left(\frac{\partial u}{\partial x_4}\right)^2=0,$$

$\nabla^2 u = 0$, and the combinant of these two will, however, be proved to form a complete system whose common solutions involve four arbitrary functions of one argument.

Let us write

$$\frac{\partial u}{\partial x_1} = \lambda_1 \ldots \frac{\partial^2 u}{\partial x_1^2} = \lambda_1^2 a_{11}, \quad \frac{\partial^2 u}{\partial x_1 \partial x_2} = \lambda_1 \lambda_2 a_{12};$$

the equations which we have to consider are

$$f_1 \equiv \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 = 0,$$

$$f_2 \equiv a_{11}\lambda_1^2 + a_{22}\lambda_2^2 + a_{33}\lambda_3^2 + a_{44}\lambda_4^2 = 0.$$

Forming the combinant (here there is obviously only one)

$$(f_1, f_2) \equiv \left(\frac{d^2}{dx_1^2} + \ldots + \frac{d^2}{dx_4^2}\right)(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2)$$
$$-2\left(\lambda_1 \frac{d}{dx_1} + \ldots + \lambda_4 \frac{d}{dx_4}\right)(a_{11}\lambda_1^2 + \ldots + a_{44}\lambda_4^2),$$

we get

$$\begin{aligned} &\lambda_1^2(\lambda_1^2 a_{11}^2 + \lambda_2^2 a_{12}^2 + \lambda_3^2 a_{13}^2 + \lambda_4^2 a_{14}^2) \\ &+\lambda_2^2(\lambda_1^2 a_{21}^2 + \lambda_2^2 a_{22}^2 + \lambda_3^2 a_{23}^2 + \lambda_4^2 a_{24}^2) \\ &+\lambda_3^2(\lambda_1^2 a_{31}^2 + \lambda_2^2 a_{32}^2 + \lambda_3^2 a_{33}^2 + \lambda_4^2 a_{34}^2) \\ &+\lambda_4^2(\lambda_1^2 a_{41}^2 + \lambda_2^2 a_{42}^2 + \lambda_3^2 a_{43}^2 + \lambda_4^2 a_{44}) = 0. \end{aligned}$$

Differentiating $f_1 = 0$ with respect to $x_1 \ldots x_4$, we get

$$\left.\begin{aligned} a_{11}\lambda_1^2 + a_{12}\lambda_2^2 + a_{13}\lambda_3^2 + a_{14}\lambda_4^2 = 0 \\ a_{21}\lambda_1^2 + a_{22}\lambda_2^2 + a_{23}\lambda_3^2 + a_{24}\lambda_4^2 = 0 \\ a_{31}\lambda_1^2 + a_{32}\lambda_2^2 + a_{33}\lambda_3^2 + a_{34}\lambda_4^2 = 0 \\ a_{41}\lambda_1^2 + a_{42}\lambda_2^2 + a_{43}\lambda_3^2 + a_{44}\lambda_4^2 = 0 \end{aligned}\right\}. \qquad (15)$$

If now we write

$$\begin{aligned} a_{11} + a_{22} - 2a_{12} &= b_{12}, \\ a_{11} + a_{33} - 2a_{13} &= b_{13}, \\ a_{22} + a_{33} - 2a_{23} &= b_{23}, \\ a_{11} + a_{44} - 2a_{14} &= b_{14}, \\ a_{22} + a_{44} - 2a_{24} &= b_{24}, \\ a_{33} + a_{44} - 2a_{34} &= b_{34}. \end{aligned}$$

(Notice Prof. Forsyth's solutions require all the b's to vanish.)

The above four equations take the simpler form (by aid of $f_1 = 0$ and $f_2 = 0$)

$$\left.\begin{aligned} b_{12}\lambda_2^2+b_{13}\lambda_3^2+b_{14}\lambda_4^2 &= 0\\ b_{21}\lambda_1^2+b_{23}\lambda_3^2+b_{24}\lambda_4^2 &= 0\\ b_{31}\lambda_1^2+b_{32}\lambda_2^2+b_{34}\lambda_4^2 &= 0\\ b_{14}\lambda_1^2+b_{24}\lambda_2^2+b_{34}\lambda_3^2 &= 0 \end{aligned}\right\}, \tag{16}$$

which equations may also be written in the form

$$\left.\begin{aligned} &\lambda_2^2\lambda_3^2 b_{23}+\lambda_3^2\lambda_1^2 b_{13}+\lambda_1^2\lambda_2^2 b_{12} = 0\\ &\lambda_2^2\lambda_3^2 b_{23} = \lambda_1^2\lambda_4^2 b_{14},\quad \lambda_1^2\lambda_3^2 b_{13} = \lambda_2^2\lambda_4^2 b_{24},\quad \lambda_1^2\lambda_2^2 b_{12} = \lambda_3^2\lambda_4^2 b_{34} \end{aligned}\right\}. \tag{17}$$

Expressing all such terms as $2a_{12}$ in the equivalent form $a_{11}+a_{22}-b_{12}$, we see [by aid of $f_1 = 0$, $f_2 = 0$ and (16)] that the combinant which is

$$\begin{aligned} 2\Sigma\lambda_1^4 a_{11}^2+\Sigma\lambda_1^2\lambda_2^2(a_{11}+a_{22}-b_{12})^2 &= \Sigma\lambda_1^4 a_{11}^2+\Sigma a_{11}^2\lambda_1^2(\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2)+2\Sigma\lambda_1^2\lambda_2^2 a_{11}a_{22}\\ &\quad -2\Sigma a_{11}\lambda_1^2(\lambda_2^2 b_{12}+\lambda_3^2 b_{13}+\lambda_4^2 b_{14})+\Sigma\lambda_1^2\lambda_2^2 b_{12}^2 \end{aligned}$$

may be written $$\Sigma\lambda_1^2\lambda_2^2 b_{12}^2. \tag{18}$$

The combinant can also be thrown into the form

$$\begin{aligned} (\lambda_2^2\lambda_3^2+\lambda_1^2\lambda_4^2)(b_{12}+b_{34}-b_{13}-b_{24})^2+(\lambda_3^2\lambda_1^2+\lambda_2^2\lambda_4^2)(b_{12}+b_{34}-b_{23}-b_{14})^2&\\ +(\lambda_1^2\lambda_2^2+\lambda_3^2\lambda_4^2)(b_{23}+b_{14}-b_{13}-b_{24})^2.& \quad (18)' \end{aligned}$$

To prove the identity of these two expressions (18) and (18)′ write

$$x = b_{23},\quad y = b_{31},\quad z = b_{12},$$

and let us write for sake of brevity

$$a = \lambda_2^2\lambda_3+\lambda_1^2\lambda_4^2,\quad b = \lambda_1^2\lambda_3^2+\lambda_2^2\lambda_4^2,\quad c = \lambda_1^2\lambda_2^2+\lambda_3^2\lambda_4^2;$$

then the expression (18)′ is equal to

$$\Sigma a\left(\frac{cz}{\lambda_3^2\lambda_4^2}-\frac{by}{\lambda_2^2\lambda_4^2}\right)^2,$$

by aid of (17); but $$\left(\frac{x}{\lambda_1^2}+\frac{y}{\lambda_2^2}+\frac{z}{\lambda_3^2}\right)^2 = 0,$$

by the first of equations (17); therefore (18)′ may be written in the form

$$\frac{(bc+ca+ab)}{\lambda_4^4\lambda_1^4\lambda_2^4\lambda_3^4}\,(ax^2\lambda_2^4\lambda_3^4+by^2\lambda_1^4\lambda_3^4+cz^2\lambda_1^4\lambda_2^4), \tag{18}''$$

and this, by (17),

$$\equiv \frac{(bc+ca+ab)}{\lambda_1^2\lambda_2^2\lambda_3^2\lambda_4^2}\,(\lambda_2^2\lambda_3^2b_{23}^2+\lambda_1^2\lambda_4^2b_{14}^2+\lambda_3^2\lambda_1^2b_{13}^2+\lambda_2^2\lambda_4^2b_{24}^2+\lambda_1^2\lambda_2^2b_{12}^2+\lambda_3^2\lambda_4^2b_{34}^2),$$

that is, the equations obtained by equating (18) and (18)′ to zero are equivalent.

We must now prove that the system $f_1=0$, $f_2=0$, and $(f_1f_2)=0$ is complete; and first we shall prove that the combinant $[f_1(f_1f_2)]$ is satisfied.

Notice that

$$\frac{1}{\lambda_\kappa}\frac{d}{dx_\kappa}a_{12}=a_{12\kappa}-a_{12}(a_{1\kappa}+a_{2\kappa}), \tag{19}$$

where

$$a_{12\kappa}\equiv\frac{\partial^3u}{\partial x_1\partial x_2\partial x_\kappa}\div\lambda_1\lambda_2\lambda_\kappa;$$

therefore

$$\left(\lambda_1\frac{d}{dx_1}+\lambda_2\frac{d}{dx_2}+\lambda_3\frac{d}{dx_3}+\lambda_4\frac{d}{dx_4}\right)a_{h\kappa}$$
$$=\lambda_1^2a_{h\kappa1}+\lambda_2^2a_{h\kappa2}+\lambda_3^2a_{h\kappa3}+\lambda_4^2a_{h\kappa4},$$

the other terms disappearing by (15).

Forming the combinant of $a_{h\kappa}$ and f_1, we get

$$\frac{1}{\lambda_h\lambda_\kappa}\frac{d^2}{dx_hdx_\kappa}(\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2)-2\left(\lambda_1\frac{d}{dx_1}+\ldots+\lambda_4\frac{d}{dx_4}\right)a_{h\kappa}$$
$$=2\,(\lambda_1^2a_{1\kappa}a_{1h}+\lambda_2^2a_{2\kappa}a_{2h}+\lambda_3^2a_{3\kappa}a_{3h}+\lambda_4^2a_{4\kappa}a_{4h}).$$

If, then, we form the combinant of $a_{12}+a_{34}-a_{13}-a_{24}$ with f_1, we get

$$\sum_{\kappa=1}^{\kappa=4}\lambda_\kappa\,(a_{\kappa1}a_{\kappa2}+a_{\kappa3}a_{\kappa4}-a_{\kappa1}a_{\kappa3}-a_{\kappa2}a_{\kappa4}).$$

Expressing every term $2a_{12}$ in this in its equivalent form $a_{11}+a_{22}-b_{12}$ as before, and using (17) to reduce this expression into terms in b_{12}, b_{13}, b_{23} only, we see that it vanishes identically; but

$$a_{12}+a_{34}-a_{13}-a_{24}\equiv b_{12}+b_{34}-b_{13}-b_{24};$$

therefore the combinant of the latter with f_1 is satisfied.

[redacted]

It follows that the combinant of $(b_{12}+b_{34}-b_{13}-b_{24})^2$ with f_1 is also satisfied.

Notice now that $\sum_{\kappa=1}^{\kappa=4} \lambda_\kappa \frac{d}{dx_\kappa}$ annihilates any function of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ only. since, in operating on such a function, the coefficient of $\frac{\partial}{\partial \lambda_\kappa}$ is

$$\lambda_1^2 a_{1\kappa}+\lambda_2^2 a_{2\kappa}+\lambda_3^2 a_{3\kappa}+\lambda_4^2 a_{4\kappa},$$

which is zero, by (15). Using these results, we see that the combinant of f_1 with $(f_1 f_2)$ is satisfied.

We must now prove that the combinant $[f_2\,(f_1 f_2)]$ is satisfied; when this is done, we can say that $f_1 = 0$, $f_2 = 0$ and $(f_1 f_2) = 0$ form a complete system.

Instead of directly proving this, it will be sufficient to prove that the combinant of $(f_1 f_2)$ with ϕ_2 is satisfied, where $\phi_2 = 0$ is any expression of the form

$$\mu_2 f_2 + \sum_{\kappa=1}^{\kappa=4} \mu_{1\kappa} \frac{\partial f_1}{\partial x_\kappa} + \mu f_1 = 0,$$

$\mu_{11} \dots \mu_{14}$ being any functions which do not contain derivatives of the second or higher orders, μ a function not containing derivatives of the first or higher orders, and μ_2 a function which does not vanish identically nor contain derivatives higher than the first order.

If, then, we prove that

$$(f_1 f_2) = 0 \quad \text{and} \quad \lambda_2^2\lambda_3^2 b_{23}+\lambda_3^2\lambda_1^2 b_{13}+\lambda_1^2\lambda_2^2 b_{12} = 0,$$

or any two equations algebraically equivalent with these, are complete in themselves, that is, if all their combinants are satisfied, we may conclude that $f_1 = 0, f_2 = 0$, and $(f_1 f_2) = 0$ form a complete system.

Now, from $(f_1 f_2) = 0$, in its form (18)″, and

$$\lambda_2^2\lambda_3^2 b_{23}+\lambda_3\lambda_1^2 b_{13}+\lambda_1^2\lambda_2^2 b_{12} = 0,$$

we deduce by algebraical solution

$$\frac{b_{23}}{\lambda_1^2\,(\lambda_1\lambda_2 \pm \lambda_3\lambda_4)^2} = \frac{b_{31}}{\lambda_2^2\,(\lambda_1^2+\lambda_3^2)^2} = \frac{b_{12}}{\lambda_3^2\,(\lambda_2\lambda_3 \mp \lambda_1\lambda_4)^2}.$$

Writing $\qquad x = b_{23}, \quad y = b_{13}, \quad z = b_{12},$

we have now to test two equations of the forms

$$x = p^2 z,$$

$$y = q^2 z,$$

p and q being homogeneous functions of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ of zero degree, and connected by the relation

$$p+q+1 \equiv 0.$$

We shall now prove that any such pair of equations is complete.

The quantics which correspond respectively to x, y, and z are w_1, w_2, w_3, where

$$w_1 \equiv \left(\frac{\xi_2}{\lambda_2} - \frac{\xi_3}{\lambda_3}\right)^2, \quad w_2 \equiv \left(\frac{\xi_3}{\lambda_3} - \frac{\xi_1}{\lambda_1}\right)^2, \quad w_3 = \left(\frac{\xi_1}{\lambda_1} - \frac{\xi_2}{\lambda_2}\right)^2,$$

and we proved (p. 243) that there were two simple identities in this case (10)

$$v_1 \equiv \frac{\xi_3}{\lambda_3} + \frac{\xi_2}{\lambda_2} - \frac{2\xi_1}{\lambda_1}, \quad v_2 \equiv \frac{\xi_2}{\lambda_2} - \frac{\xi_3}{\lambda_3}, \qquad v_3 \equiv \frac{\xi_3}{\lambda_3} - \frac{\xi_2}{\lambda_2},$$

$$\text{and } v_1 \equiv \frac{\xi_1}{\lambda_1} - \frac{\xi_3}{\lambda_3}, \qquad v_2 \equiv \frac{\xi_1}{\lambda_1} + \frac{\xi_3}{\lambda_3} - \frac{2\xi_2}{\lambda_2} \quad v_3 \equiv \frac{\xi_3}{\lambda_3} - \frac{\xi_1}{\lambda_1}.$$

There are therefore two combinants,

$$\left(\frac{1}{\lambda_2}\frac{d}{dx_2} - \frac{1}{\lambda_3}\frac{d}{dx_3}\right)(b_{13}+b_{23}-b_{12}) + 2\left(\frac{1}{\lambda_3}\frac{d}{dx_3} - \frac{1}{\lambda_1}\frac{d}{dx_1}\right) b_{23} \tag{20}$$

and

$$\left(\frac{1}{\lambda_3}\frac{d}{dx_3} - \frac{1}{\lambda_1}\frac{d}{dx_1}\right)(b_{13}+b_{23}-b_{12}) + 2\left(\frac{1}{\lambda_2}\frac{d}{dx_2} - \frac{1}{\lambda_3}\frac{d}{dx_3}\right) b_{13}. \tag{21}$$

Using (19), we see that

$$\left(\frac{1}{\lambda_2}\frac{d}{dx_2} - \frac{1}{\lambda_3}\frac{d}{dx_3}\right) a_{h\kappa} = a_{h\kappa 2} - a_{h\kappa 3} - a_{h\kappa}\,(a_{h2} + a_{\kappa 2} - a_{h3} - a_{\kappa 3}).$$

Expressing all terms b which are to be operated upon in terms of $a_{h\kappa} \ldots$, we verify that the combinants (as we expected) do not contain derivatives above the second order, and are, in fact, the first

$$(a_{13}-a_{12})(b_{13}+b_{32}-b_{12}) - 2\,(a_{23}-a_{12})\, b_{23}, \tag{22}$$

and the second

$$-(a_{23}-a_{12})(b_{13}+b_{23}-b_{12}) + 2\,(a_{13}-a_{12})\, b_{13}. \tag{23}$$

If then we write $\quad x'$ for $\dfrac{1}{\lambda_2}\dfrac{d}{dx_2} - \dfrac{1}{\lambda_3}\dfrac{d}{dx_3}$,

y' for $\dfrac{1}{\lambda_3}\dfrac{d}{dx_3} - \dfrac{1}{\lambda_1}\dfrac{d}{dx_1}$,

we may express these results by the formulæ

$$x'(y+x-z)+2y'x \equiv \quad (a_{13}-a_{12})(y+x-z)-2(a_{23}-a_{12})\,x. \qquad \textbf{(24)}$$

$$y'(x+y-z)+2x'y \equiv -(a_{23}-a_{12})(y+x-z)+2(a_{13}-a_{33})\,y. \qquad \textbf{(25)}$$

Now $\quad \{(1-q)\,y'-qx'\}\,(x-p^2z)+\{(1-p)\,x'-py'\}\,(y-q^2z)$

is easily seen to be the only combinant of $x-p^2z$ and $y-q^2z$, and, expanding it, we get

$$(1-q)\,y'x-(1-q)\,p^2y'z-qx'x+qp^2x'z+(1-p)\,x'y$$
$$-(1-p)\,q^2x'z-py'y+pq^2y'z+z\{-(1-q)\,y'+qx'\}\,p^2$$
$$+z\,\{py'-(1-p)\,x'\}\,q^2.$$

The last two terms taken together are

$$z\,\{q\,(x'+y')\,p^2+p\,(x'+y')\,q^2-y'p^2-x'q^2\}$$
$$=z\,\{2qp\,(x'+y')(p+q)-2py'p-2qx'q\},$$

which becomes (since $p+q=-1$. a constant and therefore annihilated by $x'+y'$)

$$-2z\,(py'p+qx'q);$$

the other terms reduce (using the fact $p+q=1$) to

$$-p\,\{y'(x+y-z)+2x'y\}-q\,\{x'(x+y-z)+2y'x\}.$$

Using (24) and (25) and remembering that

$$x=p^2z, \quad y=q^2z,$$

we see that these terms all disappear.

We have now only to prove that $py'p+qx'q$ or $(py'-qx')\,p$ vanishes. First, we shall prove that

$$(py'-qx')\,\frac{\lambda_1}{\lambda_3}=0;$$

this is to prove that

$$\left(\frac{1}{\lambda_3}\,\frac{d}{dx_3}+\frac{p}{\lambda_1}\,\frac{d}{dx_1}+\frac{q}{\lambda_2}\,\frac{d}{dx_2}\right)\frac{\lambda_1}{\lambda_3}=0:$$

that is to prove that

$$-a_{13}+a_{33}-p\,(a_{11}-a_{13})-q\,(a_{12}-a_{23})=0.$$

The expression on the left may be written

$$-p\,(a_{11}-2a_{13}+a_{33})-q\,(a_{12}+a_{33}-a_{23}-a_{13})=-py+\frac{q}{2}\,(z-x-y).$$

which vanishes when we put

$$x = p^2 z, \quad y = q^2 z.$$

Similarly, we may prove that

$$(py' - qx') \frac{\lambda_2}{\lambda_3} = 0,$$

and therefore $py' - qx'$ annihilates $\dfrac{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}{\lambda_3^2}$, and therefore $\dfrac{\lambda_4}{\lambda_3}$; that is, $py' - qx'$ annihilates any homogeneous function of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ which is of zero degree, so that

$$(py' - qx')\, p = 0.$$

We have seen that

$$\{(1-q)\, y' - qx'\}\, (x - p^2 z) + \{(1-p)\, x' - py'\}\, (y - q^2 z)$$

is the combinant of $x - p^2 z$ and $y - q^2 z$, and we have now proved that it is satisfied.

The system $f_1 = 0$, $f_2 = 0$, and $(f_1 f_2) = 0$ is now proved to be complete, and, as we have two equations of the second order and one of the first in four independent variables, the formula

$$(1 - x^2)^2 (1 - x)^1 (1 - x)^{-4} = (1 + x)^2 (1 - x)^{-1} = 1 + 3x + 4x^2 + 4x^3 + \ldots$$

shows us that four derivatives of any order above the first are arbitrary; we could take these to be

$$\frac{\partial^r u}{\partial x_1^r}, \quad \frac{\partial^r u}{\partial x_2 \partial x_1^{r-1}}, \quad \frac{\partial^r u}{\partial x_3 \partial x_1^{r-1}}, \quad \text{and} \quad \frac{\partial^r u}{\partial x_4 \partial x_1^{r-1}},$$

so that the most general common solution could be taken to be

$$u = a_0 + a_2 x_2 + a_3 x_3 + a_4 x_4 + \ldots,$$

a series in powers of x_2, x_3, x_4, the coefficients being functions of x_1, the first four arbitrary and the remaining ones given in terms of these.

Zeroes of the Spherical Harmonic $P_n^m(\mu)$ ***considered as a Function of*** n. *By* **H. M. Macdonald. Read May 11th, 1899.** **Received, with additional note, December 1st, 1899.**

For the solution of potential and other allied problems for a space bounded by two concentric spheres, or by two surfaces differing but slightly from them, the solutions of Laplace's equation which are required are

$$f_n(r)\, P_n^{-m}(\mu) \begin{matrix}\cos\\ \sin\end{matrix} m\phi,$$

where m and n are integers, and μ is real, such that $1 > \mu > -1$. When the space is that bounded by two coaxal cones, the solutions of Laplace's equation which are required are

$$r^{\pm(n+\frac{1}{2})-\frac{1}{2}} \left\{A_n P_n^{-m}(\mu) + B_n P_n^{m}(\mu)\right\} \begin{matrix}\cos\\ \sin\end{matrix} m\phi,$$

where $A_n P_n^{-m}(\mu) + B_n P_n^{m}(\mu)$ vanishes for each of the two values of μ which belong to the surfaces of the cones;* to find all the solutions which can occur, it is necessary to find all the values of n for which the above conditions are satisfied. In particular, when the space is that bounded by one cone, the solutions required are

$$P_n^{-m}(\mu) \begin{matrix}\cos\\ \sin\end{matrix} m\phi \,.\, r^{\pm(n+\frac{1}{2})-\frac{1}{2}},$$

and all the values of n have to be found which make $P_n^{-m}(\mu)$ vanish, when $\mu = \mu_0$. Among other physical applications of the zeroes of such a function may be mentioned the case of the free vibrations of a spherical layer of air bounded by a small circle on the sphere,† and the case of a spherical condenser bounded in the same way.‡ In the following the values of n for which $P_n^m(\mu)$ vanishes, when $\mu = \mu_0$, where μ_0 is a real quantity and $1 > \mu_0 > -1$, m being real, are discussed; the method could be easily extended to the discussion of

* Thomson and Tait, *Natural Philosophy*, Vol. I., Part I., p. 196.
† Rayleigh, *Theory of Sound*, Vol. II., p. 259.
‡ Larmor, *Proc. Lond. Math. Soc.*, Vol. XXV., p. 133.

the values of n for which

$$A_n P_n^m(\mu) + B_n P_n^{-m}(\mu) \quad \text{or} \quad A_n P_n^m(\mu) + B_n Q_n^m(\mu)$$

vanishes for two given real values of μ lying between -1 and 1, but its applications are of minor importance compared with those of the case discussed. The notation used throughout is that of Hobson, *Phil. Trans.*, 1896.

In § 1 the reality of the zeroes of $P_n^{-m}(\mu)$, m being a real positive quantity, is demonstrated. A formula suitable for calculating the zeroes when $\cos^{-1}\mu$ is not near to 0 or π is given in § 2, and a suitable formula when $\cos^{-1}\mu$ is near to 0 is given in § 3, and when $\cos^{-1}\mu$ is near to π, m not being an integer, in § 4. The form of $Q_n^m(\mu)$ when m is an integer is discussed in § 5, and a formula for the zeroes of $P_n^{-m}(\mu)$ when $\cos^{-1}\mu$ is near to π, m being an integer, is given in § 6. It is shown in § 7 that the zeroes diminish as $\cos^{-1}\mu$ increases from 0 to π. A short discussion of the zeroes of $P_n^m(\mu)$ is given in § 8.

1. *All the Zeroes of $P_n^{-m}(\mu_0)$ are Real when m and μ_0 are Real and m is Positive.*

The function $P_n^{-m}(\mu)$ is that solution of the differential equation

$$(1-\mu^2)\frac{d^2y}{d\mu^2} - 2\mu\frac{dy}{d\mu} + \left\{n(n+1) - \frac{m^2}{1-\mu^2}\right\} y = 0 \tag{1}$$

which vanishes when $\mu = 1$, m being a real positive quantity.

Let $P_{n'}^{-m}(\mu)$ be the corresponding solution of

$$(1-\mu^2)\frac{d^2y'}{d\mu^2} - 2\mu\frac{dy'}{d\mu} + \left\{n'(n'+1) - \frac{m^2}{1-\mu^2}\right\} y' = 0; \tag{2}$$

then $$\frac{d}{d\mu}\left[(1-\mu^2)\left(y'\frac{dy}{d\mu} - y\frac{dy'}{d\mu}\right)\right] + (n-n')(n+n'+1)\,yy' = 0;$$

therefore

$$(n'-n)(n+n'+1)\int_{\mu_0}^{1} P_n^{-m}(\mu)\,P_{n'}^{-m}(\mu)\,d\mu$$
$$= (1-\mu_0^2)\left(P_n^{-m}\frac{dP_{n'}^{-m}}{d\mu} - P_{n'}^{-m}\frac{dP_n^{-m}}{d\mu}\right)_{\mu=\mu_0};$$

hence, if n and n' are two different values of n for which $P_n^{-m}(\mu_0)$ vanishes,

$$\int_{\mu_0}^{1} P_n^{-m}(\mu)\,P_{n'}^{-m}(\mu)\,d\mu = 0,$$

unless $n+n'+1$ vanishes, which possibility can be excluded by the consideration that only zeroes with real part positive need be taken into account, as

$$P_n^{-m}(\mu) = P_{-n-1}^{-m}(\mu).$$

Now, if $P_n^{-m}(\mu_0)$ has a complex zero n, it has a conjugate complex zero n', for $P_n^{-m}(\mu)$ is expressible as a converging power series in $(1-\mu)$, with real coefficients so long as $|1-\mu| < 2$; hence, n and n' being conjugate complex zeroes of $P_n^{-m}(\mu_0)$, $P_n^{-m}(\mu)\,P_{n'}^{-m}(\mu)$ must be a real positive quantity for all real values of μ such that $1>\mu>-1$; and therefore

$$\int_{\mu_0}^{1} P_n^{-m}(\mu)\,P_{n'}^{-m}(\mu)\,d\mu$$

cannot vanish; hence $P_n^{-m}(\mu_0)$ can have no complex zeroes, that is, all its zeroes are real. The above argument cannot be applied to the function $P_n^{m}(\mu)$, inasmuch as it does not converge necessarily when $\mu=-1$. The argument applied in a previous paper (*Proceedings*, Vol. XXIX.) can be used in the present case, and shows that $P_n^{-m}(\mu)$ has all its zeroes real as above, while $P_n^{m}(\mu)$ has an infinite number of real zeroes, and, in addition, at most $2k$ complex zeroes, where k is the greatest integer contained in m.

2. *Investigation of a Formula for the Zeroes of* $P_n^{-m}(\mu_0)$.

The expression for $P_n^{-m}(\cos\theta)$ convenient for use when n is large is

$$
\begin{aligned}
&{}^{*}P_n^{-m}(\cos\theta) \\
&= \frac{2}{\sqrt{\pi}}\,\frac{\Pi(n-m)}{\Pi(n+\frac{1}{2})}\left[\frac{\cos\left\{(n+\frac{1}{2})\theta-\frac{\pi}{4}-\frac{m\pi}{2}\right\}}{(2\sin\theta)^{\frac{1}{2}}}\right. \\
&\qquad + \frac{1^2-4m^2}{2\,.\,2n+3}\,\frac{\cos\left\{(n+\frac{3}{2})\theta-\frac{3\pi}{4}-\frac{m\pi}{2}\right\}}{(2\sin\theta)^{\frac{3}{2}}} \\
&\qquad \left. + \frac{1^2-4m^2\,.\,3^2-4m^2}{2\,.\,4\,.\,2n+3\,.\,2n+5}\,\frac{\cos\left\{(n+\frac{5}{2})\theta-\frac{5\pi}{4}-\frac{m\pi}{2}\right\}}{(2\sin\theta)^{\frac{5}{2}}}+\dots\right].
\end{aligned}
$$

* Hobson, *Phil. Trans.*, 1896, p. 486.

The large values of n for which $P_n^{-m}(\cos\theta)$ vanishes are given by

$$(n+\tfrac{1}{2})\,\theta-\frac{\pi}{4}-\frac{m\pi}{2}=(2k+1)\,\frac{\pi}{2},$$

where k is an integer, that is, by

$$n+\tfrac{1}{2}=\frac{\pi}{2\theta}(2k+m+\tfrac{3}{2}).$$

Put $$n+\tfrac{1}{2}=x,\quad \frac{\pi}{2\theta}(2k+m+\tfrac{3}{2})=x_0\,;$$

then, for all positive values of n which make $P_n^{-m}(\cos\theta)$ vanish,

$$(x-x_0)\,\theta=\psi,$$

where ψ is to be determined from the relation

$$\tan\psi$$

$$=\frac{\dfrac{1-4m^2}{2^2(1+x)}\,\dfrac{\sin\left(\frac{\pi}{2}-\theta\right)}{2\sin\theta}+\dfrac{(1-4m^2)(3^2-4m^2)}{2^4(1+x)(2+x)\,2\,!}\,\dfrac{\sin(\pi-2\theta)}{(2\sin\theta)^2}+\dots}{1+\dfrac{1-4m^2}{2^2(1+x)}\,\dfrac{\cos\left(\frac{\pi}{2}-\theta\right)}{2\sin\theta}+\dfrac{(1-4m^2)(3^2-4m^2)}{2^4(1+x)(2+x)\,2\,!}\,\dfrac{\cos(\pi-2\theta)}{(2\sin\theta)^2}+\dots}.$$

This may be written

$$\tan\psi=\frac{c_1}{1+x}+\frac{c_2}{(1+x)^2}+\frac{d_2}{(1+x)(2+x)}+\frac{c_3}{(1+x)^3}+\frac{d_3}{(1+x)^2(2+x)}$$
$$+\frac{e_3}{(1+x)(2+x)(3+x)}+\dots,$$

where $c_1=b_1,\;\; c_2=-a_1b_1,\;\; d_2=b_2,\;\; c_3=a_1^2b_1,\;\; d_3=-a_2b_1-a_1b_2,$

$e_3=b_3,\;\; c_4=-a_1^3b_1,\;\; d_4=2a_1a_2b_1+a_1^2b_2,\;\; e_4=-a_2b_2,$

$f_4=-a_1b_3-a_3b_1,\qquad g_4=b_4,\quad$ &c.,

and $$a_1=\frac{1-4m^2}{2^2}\,\frac{\cos\left(\frac{\pi}{2}-\theta\right)}{2\sin\theta},\quad b_1=\frac{1-4m^2}{2^2}\,\frac{\sin\left(\frac{\pi}{2}-\theta\right)}{2\sin\theta},$$

$$a_2=\frac{(1-4m^2)(3^2-4m^2)}{2^4\,(2\sin\theta)^2\,2!}\cos(\pi-2\theta),\;\text{ \&c.};$$

whence $$\tan (x-x_0)\,\theta = a_1+a_2+a_3+a_4+\ldots,$$

where $$a_1 = \frac{c_1}{1+x},\quad a_2 = \frac{c_2}{(1+x)^2}+\frac{d_2}{(1+x)(2+x)},\ \text{\&c.}$$

Expanding by Lagrange's theorem and neglecting terms of the order $\dfrac{1}{(1+x_0)^5}$,

$$x = x_0+\frac{1}{\theta}\left(a_1+a_2+a_3+a_4-\frac{a_1^3}{3}-a_1^2a_2\right)+\frac{1}{\theta^2}\left(a_1a_1'+a_1a_2'+a_1'a_2\right)+\ldots,$$

where x_0 is written for x in $a_1, a_2, \ldots$, and a_1' denotes $\dfrac{da_1}{dx_0}$. Substituting for a_1, a_2 their values

$$\begin{aligned} x = x_0 &+ \frac{b_1}{\theta(1+x_0)}+\frac{b_2}{\theta(1+x_0)(2+x_0)}-\frac{a_1b_1}{\theta(1+x_0)^2}+\frac{3a_1^2b_1-b_1^3}{3\theta(1+x_0)^3} \\ &-\frac{a_2b_1+a_1b_2}{\theta(1+x_0)^2(2+x_0)}+\frac{b_3}{\theta(1+x_0)(2+x_0)(3+x_0)}-\frac{b_1^2}{\theta^2(1+x_0)^3} \\ &+\frac{a_1b_1^3-a_1^3b_1}{\theta(1+x_0)^4}+\frac{2a_1a_2b_1+a_1^2b_2-b_1^2b_2}{\theta(1+x_0)^3(2+x_0)}-\frac{a_2b_2}{\theta(1+x_0)^2(2+x_0)^2} \\ &-\frac{a_1b_3+a_3b_1}{\theta(1+x_0)^2(2+x_0)(3+x_0)}+\frac{b_4}{\theta(1+x_0)(2+x_0)(3+x_0)(4+x_0)} \\ &+\frac{3a_1b_1^2}{\theta^2(1+x_0)^4}-\frac{2b_1b_2}{\theta^2(1+x_0)^3(2+x_0)}+\frac{b_1b_2}{\theta^2(1+x_0)^2(2+x_0)^2}+\ldots. \end{aligned}$$

The negative values of n for which $P_n^{-m}(\mu_0)$ vanishes are obtained from the above by changing the sign of the right-hand side, as

$$P_n^{-m}(\mu) = P_{-n-1}^{-m}(\mu).$$

When θ lies between $\pi/4$ and $3\pi/4$ it will be found that the omission of terms of order $\dfrac{1}{(1+x_0)^4}$ and higher orders at most affects the fifth decimal place, and the series can be extended so as to approximate more closely. As θ diminishes to 0 or increases to π, a greater number of terms must be taken to obtain a good approximation, and when θ is near to 0 or π the series is unsuitable; series suitable for these cases are given below.

[redacted]

3. *Zeroes of $P_n^{-m}(\cos\theta)$ when θ is a small quantity.*

From the expression for $P_n^{-m}(\mu)$,

$$P_n^{-m}(\mu) = \left(\frac{1-\mu}{1+\mu}\right)^{\frac{1}{2}m} \sum_0^\infty \frac{\Pi(n+r)\cos r\pi}{\Pi(n-r)\,\Pi(m+r)\,\Pi(r)} \left(\frac{1-\mu}{2}\right)^r,$$

it may be shown, by expanding $\dfrac{\Pi(n+r)}{\Pi(n-r)}$ in powers of n and arranging the result for $P_n^{-m}(\mu)$ in powers of $2n\sin\dfrac{\theta}{2}$, that

$$P_n^{-m}(\cos\theta)$$

$$= \frac{1}{\left(n\cos\frac{\theta}{2}\right)^m}\left[J_m(x) - \sin\frac{\theta}{2}J_{m+1}(x) - \sin^2\frac{\theta}{2}\left\{\tfrac{1}{2}J_{m+2}(x) - \frac{x}{6}J_{m+3}(x)\right\}\right.$$

$$-\sin^3\frac{\theta}{2}\left\{\frac{2}{x}J_{m+2}(x) - \tfrac{3}{2}J_{m+3}(x) + \frac{x}{6}J_{m+4}(x)\right\}$$

$$\left.+\sin^4\frac{\theta}{2}\left\{\frac{x^2}{72}J_{m+6}(x) - \frac{17x}{60}J_{m+5}(x) + \tfrac{11}{8}J_{m+4}(x) - \frac{4}{3x}J_{m+3}(x)\right\} - \&c.\right],$$

where $$x = 2n\sin\frac{\theta}{2}.$$

*[For $\dfrac{\Pi(n+r)}{\Pi(n-r)} = (n^2+rn)(n^{2r-2} + a_1^{(r-1)}n^{2r-4} + a^{(r-1)}n^{2r-6} + \dots),$

where $(n^2-1^2)(n^2-2^2)\dots\{n^2-(r-1)^2\} = n^{2r-2} + a^{(r-1)}n^{2r-4} + \dots;$

whence $$a_1^{(r-1)} = -\sum^{r-1} s^2,$$

$$a_2^{(r-1)} = -\sum^{r-1} s^2 a_1^{(s-1)}, \ \&c.,$$

that is,

$$a^{(r-1)} = -\tfrac{1}{3}r(r-1)(r-2) - \tfrac{1}{2}r(r-1),$$

$$a_2^{(r-1)} = \tfrac{1}{18}\frac{\Pi(r)}{\Pi(r-6)} + \tfrac{17}{30}\frac{\Pi(r)}{\Pi(r-5)} + \tfrac{11}{8}\frac{\Pi(r)}{\Pi(r-4)} + \tfrac{2}{3}\frac{\Pi(r)}{\Pi(r-3)}, \ \&c.$$

* Added December 1st, 1899.

Hence

$$P_n^{-m}(\mu) = \left(\frac{1-\mu}{1+\mu}\right)^{\frac{1}{2}m} \sum_0^\infty \frac{\cos r\pi}{\Pi(m+r)\,\Pi(r)} \left(\frac{1-\mu}{2}\right)^r$$

$$\times \left[n^{2r} + rn^{2r-1} - \left\{ \tfrac{1}{3}\frac{\Pi(r)}{\Pi(r-3)} + \tfrac{1}{2}\frac{\Pi(r)}{\Pi(r-2)} \right\} n^{2r-2} \right.$$

$$- \left\{ \tfrac{1}{3}\frac{\Pi(r)}{\Pi(r-4)} + \tfrac{3}{2}\frac{\Pi(r)}{\Pi(r-3)} + \frac{\Pi(r)}{\Pi(r-2)} \right\} n^{2r-3}$$

$$\left. + \left\{ \tfrac{1}{18}\frac{\Pi(r)}{\Pi(r-6)} + \tfrac{17}{30}\frac{\Pi(r)}{\Pi(r-5)} + \tfrac{11}{8}\frac{\Pi(r)}{\Pi(r-4)} + \tfrac{2}{3}\frac{\Pi(r)}{\Pi(r-3)} \right\} n^{2r-4} + \&c. \right],$$

that is

$$P_n^{-m}(\mu)$$

$$= \left(\frac{x}{2n\cos\frac{\theta}{2}}\right)^m \sum_0^\infty \left[\frac{1}{\Pi(m+r)\,\Pi(r)} \left(-\frac{x^2}{4}\right)^r - n\sin^2\frac{\theta}{2}\,\frac{1}{\Pi(m+r)\,\Pi(r-1)} \left(-\frac{x^2}{4}\right)^{r-1} \right.$$

$$- \sin^2\frac{\theta}{2} \left\{ \frac{x^2}{8}\,\frac{1}{\Pi(m+r)\,\Pi(r-2)} \left(-\frac{x^2}{4}\right)^{r-2} - \frac{x^4}{2^4.3}\,\frac{1}{\Pi(m+r)\,\Pi(r-3)} \left(-\frac{x^2}{4}\right)^{r-3} \right\}$$

$$- \sin^3\frac{\theta}{2} \left\{ \frac{x}{2}\,\frac{1}{\Pi(m+r)\,\Pi(r-2)} \left(-\frac{x^2}{4}\right)^{r-2} - \tfrac{3}{2}\left(\frac{x}{2}\right)^3 \frac{1}{\Pi(m+r)\,\Pi(r-3)} \left(-\frac{x^2}{4}\right)^{r-3} \right.$$

$$\left. + \tfrac{1}{3}\left(\frac{x}{2}\right)^5 \frac{1}{\Pi(m+r)\,\Pi(r-4)} \left(-\frac{x^2}{4}\right)^{r-4} \right\}$$

$$+ \sin^4\frac{\theta}{2} \left\{ \tfrac{1}{18}\left(\frac{x}{2}\right)^8 \frac{1}{\Pi(m+r)\,\Pi(r-6)} \left(-\frac{x^2}{4}\right)^{r-6} - \tfrac{17}{30}\left(\frac{x}{2}\right)^6 \frac{1}{\Pi(m+r)\,\Pi(r-5)} \left(-\frac{x^2}{4}\right)^{r-5} \right.$$

$$\left. \left. + \tfrac{11}{8}\left(\frac{x}{2}\right)^4 \frac{1}{\Pi(m+r)\,\Pi(r-4)} \left(-\frac{x^2}{4}\right)^{r-4} - \tfrac{2}{3}\left(\frac{x}{2}\right)^2 \frac{1}{\Pi(m+r)\,\Pi(r-3)} \left(-\frac{x^2}{4}\right)^{r-3} \right\} - \&c. \right]$$

whence the result.]

The zeroes of $P_n^{-m}(\cos\theta)$ are then obtained by making the series on the right-hand side vanish. When θ is very small the zeroes are given by the zeroes of $J_m(x)$; if x_0 is one of these zeroes, the corresponding zero of $P_n^{-m}(\cos\theta)$ is given by

$$n = \frac{x_0}{2}\operatorname{cosec}\frac{\theta}{2} \quad \text{or} \quad n = \frac{x_0}{\theta},$$

and for small values of θ this value of n is a first approximation. To obtain a further approximation the values of the Bessel functions

which occur, and of their differential coefficients when $J_m(x) = 0$, are required; these are as follows :—

$$J_m(x) = 0, \quad J'_m(x) = -J_{m+1}(x), \quad J''_m(x) = \frac{1}{x} J_{m+1}(x),$$

$$J'''_m(x) = \left(1 - \frac{m^2+2}{x^2}\right) J_{m+1}(x), \quad J^{\text{iv}}_m(x) = \left\{ \frac{6(m^2+1)}{x^3} - \frac{2}{x} \right\} J_{m+1}(x),$$

$$J'_{m+1}(x) = -\frac{m+1}{x} J_{m+1}(x), \quad J''_{m+1}(x) = \left\{ \frac{(m+1)(m+2)}{x^2} - 1 \right\} J_{m+1}(x),$$

$$J'''_{m+1}(x) = \left\{ \frac{m+2}{x} - \frac{(m+1)(m+2)(m+3)}{x^3} \right\} J_{m+1}(x),$$

$$J_{m+2}(x) = \frac{2(m+1)}{x} J_{m+1}(x), \quad J'_{m+2}(x) = \left\{ 1 - \frac{2(m+1)(m+2)}{x^2} \right\} J_{m+1}(x),$$

$$J''_{m+2}(x) = \left\{ \frac{2(m+1)(m+2)(m+3)}{x^3} - \frac{2m+3}{x} \right\} J_{m+1}(x),$$

$$J_{m+3}(x) = \left\{ \frac{4(m+1)(m+2)}{x^2} - 1 \right\} J_{m+1}(x),$$

$$J'_{m+3}(x) = \left\{ \frac{3m+5}{x} - \frac{4(m+1)(m+2)(m+3)}{x^3} \right\} J_{m+1}(x),$$

$$J''_{m+3}(x) = \left\{ \frac{4(m+1)(m+2)(m+3)(m+4)}{x^4} - \frac{(m+2)(5m+11)}{x^2} + 1 \right\} J_{m+1}(x),$$

$$J_{m+4}(x) = \left\{ \frac{8(m+1)(m+2)(m+3)}{x^3} - \frac{4(m+2)}{x} \right\} J_{m+1}(x),$$

$$J'_{m+4}(x) = \left\{ -1 + \frac{4(m+2)(2m+5)}{x^2} - \frac{8(m+1)(m+2)(m+3)(m+4)}{x^4} \right\} J_{m+1}(x),$$

$$J_{m+5}(x) = \left\{ \frac{16(m+1)(m+2)(m+3)(m+4)}{x^4} - \frac{12(m+2)(m+3)}{x^2} + 1 \right\} J_{m+1}(x),$$

$$J_{m+6}(x) = \left\{ \frac{6(m+3)}{x} - \frac{32(m+2)(m+3)(m+4)}{x^3} + \frac{32(m+1)(m+2)(m+3)(m+4)(m+5)}{x^5} \right\} J_{m+1}(x).$$

[redacted]

To make $J_m(x) - \sin\frac{\theta}{2} J_{m+1}(x) - \ldots$ vanish, assume

$$x = x_0 + a_1 \sin\frac{\theta}{2} + a_2 \sin^2\frac{\theta}{2} + a_3 \sin^3\frac{\theta}{2} + a_4 \sin^4\frac{\theta}{2} + \ldots;$$

then it can be shown, either by successive approximation, or by applying Lagrange's theorem, that

$$a_1 = -1, \quad a_2 = -\frac{x_0}{6}\left(1 + \frac{1-4m^2}{x_0^2}\right), \quad a_3 = 0,$$

$$a_4 = -\frac{17x_0}{360} + \frac{592m^2+40m-13}{180x_0} + \frac{48m^4+6480m^2+28400m+7720}{360x_0^3}, \text{ \&c.}$$

4. *Zeroes of $P_n^{-m}(\mu)$ when $\pi - \theta$ is Small and m is not an Integer.*

To obtain a suitable formula for calculating the zeroes in this case, it is convenient to express $P_n^{-m}(\mu)$ in terms of functions of $-\mu$.

From the relation

$$P_n^m(-\mu \mp o\iota) = e^{\mp n\pi\iota} P_n^m(\mu \pm o\iota) - \frac{2\sin(n+m)\pi}{\pi} e^{-m\pi\iota} Q_n^m(\mu \pm o\iota),*$$

it follows that

$$e^{\frac{1}{2}(m\pi\iota)} P_n^m(-\mu - o\iota) + e^{\frac{1}{2}(3m\pi\iota)} P_n^m(-\mu + o\iota)$$

$$= e^{-n\pi\iota + \frac{1}{2}(m\pi\iota)} P_n^m(\mu + o\iota) + e^{n\pi\iota + \frac{3}{2}(m\pi\iota)} P_n^m(\mu - o\iota)$$

$$- \frac{2\sin(n+m)\pi}{\pi}\left\{e^{-\frac{1}{2}(m\pi\iota)} Q_n^m(\mu + o\iota) + e^{\frac{1}{2}(m\pi\iota)} Q_n^m(\mu - o\iota)\right\},$$

that is, μ being real,

$$e^{m\pi\iota} P_n^m(-\mu) = e^{m\pi\iota}\cos(n+m)\pi . P_n^m(\mu) - \frac{2\sin(n+m)\pi}{\pi} e^{m\pi\iota} Q_n^m(\mu);$$

hence $P_n^m(-\mu) = \cos(n+m)\pi\, P_n^m(\mu) - \frac{2\sin(n+m)\pi}{\pi} Q_n^m(\mu)$.

From the relation

$$P_n^{-m}(\mu) = \frac{\Pi(n-m)}{\Pi(n+m)}\left\{P_n^m(\mu) - \frac{2}{\pi} e^{-m\pi\iota} \sin m\pi Q_n^m(\mu)\right\},$$

---

* Hobson, *Phil. Trans.*, 1896, p. 463.

[redacted]

it follows that when μ is real

$$P_n^{-m}(\mu) = \frac{\Pi(n-m)}{\Pi(n+m)} \left\{ P_n^m(\mu) \cos m\pi - \frac{2}{\pi} \sin m\pi \, Q_n^m(\mu) \right\},$$

that is,

$$Q_n^m(\mu) = \frac{\pi}{2 \sin m\pi} \left\{ P_n^m(\mu) \cos m\pi - \frac{\Pi(n+m)}{\Pi(n-m)} P_n^{-m}(\mu) \right\}.$$

Hence

$$P_n^m(-\mu) = \cos(n+m)\pi P_n^m(\mu)$$
$$- \frac{\sin(n+m)\pi}{\sin m\pi} \left\{ P_n^m(\mu) \cos m\pi - \frac{\Pi(n+m)}{\Pi(n-m)} P_n^{-m}(\mu) \right\}.$$

The zeroes of $P_n^{-m}(\mu)$ when $\pi-\theta$ is small are given by

$$\tan(n-m)\pi = \frac{\sin m\pi . P_n^{-m}(\mu')}{\dfrac{\Pi(n-m)}{\Pi(n+m)} P_n^m(\mu') - \cos m\pi . P_n^{-m}(\mu')},$$

where $$\mu' = \cos(\pi-\theta),$$

that is, by

$$n = m+k+\frac{1}{\pi} \tan^{-1} \left\{ \frac{\sin m\pi . P_n^{-m}(\mu')}{\dfrac{\Pi(n-m)}{\Pi(n+m)} P_n^m(\mu') - \cos m\pi P_n^{-m}(\mu')} \right\},$$

where k has all positive integral values including zero. The above can be expanded by Lagrange's theorem; in particular, when $\pi-\theta = \phi$ is very small,

$$n = m+k+\frac{1}{\pi} \tan^{-1} \left\{ \frac{\Pi(n+m)\,\Pi(-m)}{\Pi(n-m)\,\Pi(m)} \tan^{2m}\frac{\phi}{2} \sin m\pi \right\},$$

that is,

$$n = m+k+\frac{\Pi(2m+k)}{\Pi(m)\,\Pi(m-1)\,\Pi(k)} \tan^{2m}\frac{\phi}{2}.$$

The negative zeroes of $P_n^{-m}(\mu)$ are found, as before, by writing $-n-1$ for n. When m is an integer the above method fails, and to obtain an expression for the zeroes it is necessary to express $Q_n^m(\mu)$ when m is an integer in a suitable form.

5. *Form of $Q_n^m(\mu)$ when m is an Integer.*

From the relation given above

$$Q_n^m(\mu) = \frac{\pi}{2\sin m\pi}\left\{P_n^m(\mu)\cos m\pi - \frac{\Pi(n+m)}{\Pi(n-m)}P_n^{-m}(\mu)\right\},$$

it follows that

$$Q_n^m(\mu) = \frac{\pi}{2\sin m\pi}\left\{\frac{\cos m\pi}{\Pi(-m)}\left(\frac{1+\mu}{1-\mu}\right)^{\frac{1}{2}m} F\left(-n,\, n+1,\, 1-m,\, \frac{1-\mu}{2}\right)\right.$$

$$\left. - \frac{\Pi(n+m)}{\Pi(n-m)}\left(\frac{1-\mu}{1+\mu}\right)^{\frac{1}{2}m} F\left(-n,\, n+1,\, 1+m,\, \frac{1-\mu}{2}\right)\right\}.$$

and the limiting value of the expression on the right-hand side when m is an integer is required. This may be written in the form

$$Q_n^m(\mu)$$

$$= \frac{\pi}{2\sin m\pi}\left\{\cos m\pi\left(\frac{1+\mu}{1-\mu}\right)^{\frac{1}{2}m}\sum_0^{s-1}\frac{\Pi(-n+r-1)\,\Pi(n+r)}{\Pi(n)\,\Pi(-n-1)\,\Pi(r)\,\Pi(r-m)}\left(\frac{1-\mu}{2}\right)^r\right.$$

$$+ \cos m\pi\left(\frac{1+\mu}{1-\mu}\right)^{\frac{1}{2}m}\sum_s^{\infty}\frac{\Pi(-n+r-1)\,\Pi(n+r)}{\Pi(n)\,\Pi(-n-1)\,\Pi(r)\,\Pi(r-m)}\left(\frac{1-\mu}{2}\right)^r$$

$$\left. - \frac{\Pi(n+m)}{\Pi(n-m)}\left(\frac{1-\mu}{1+\mu}\right)^{\frac{1}{2}m}\sum_0^{\infty}\frac{\Pi(-n+r-1)\,\Pi(n+r)}{\Pi(n)\,\Pi(-n-1)\,\Pi(r)\,\Pi(r+m)}\left(\frac{1-\mu}{2}\right)^r\right\},$$

where s is the greatest integer contained in m. Hence

$$Q_n^m(\mu)$$

$$= \tfrac{1}{2}\left(\frac{1+\mu}{1-\mu}\right)^{\frac{1}{2}m}\cos m\pi\sum_0^{s-1}\frac{\Pi(-n+r-1)\,\Pi(n+r)\,\Pi(m-r-1)}{\Pi(n)\,\Pi(-n-1)\,\Pi(r)}\left(\frac{1-\mu}{2}\right)^r\cos r\pi$$

$$+ \frac{\pi}{2\sin m\pi}\left\{\cos m\pi\left(\frac{1+\mu}{1-\mu}\right)^{\frac{1}{2}m}\sum_0^{\infty}\frac{\Pi(-n+r+s-1)\,\Pi(n+r+s)}{\Pi(n)\,\Pi(-n-1)\,\Pi(r+s)\,\Pi(r+s-m)}\left(\frac{1-\mu}{2}\right)^{r+s}\right.$$

$$\left. - \frac{\Pi(n+m)}{\Pi(n-m)}\left(\frac{1-\mu}{1+\mu}\right)^{\frac{1}{2}m}\sum_0^{\infty}\frac{\Pi(-n+r-1)\,\Pi(n+r)}{\Pi(n)\,\Pi(-n-1)\,\Pi(r)\,\Pi(r+m)}\left(\frac{1-\mu}{2}\right)^r\right\}.$$

Proceeding to the limit when $m = s$,

$$Q_n^m(\mu)$$

$$= -\frac{\sin n\pi \cos m\pi}{2\pi}\left(\frac{1+\mu}{1-\mu}\right)^{\frac{1}{2}m}\sum_0^{m-1}\frac{\Pi(-n+r-1)\,\Pi(n+r)\,\Pi(m-r-1)}{\Pi(r)} \times\left(\frac{1-\mu}{2}\right)^r \cos r\pi$$

$$+\tfrac{1}{4}\log\frac{1+\mu}{1-\mu}\left\{\cos m\pi\left(\frac{1+\mu}{1-\mu}\right)^{\frac{1}{2}m}\frac{\Pi(-n+m-1)\,\Pi(n+m)}{\Pi(n)\,\Pi(-n-1)\,\Pi(m)} \times\left(\frac{1-\mu}{2}\right)^m F\left(-n+m,\, n+m+1,\, 1+m,\, \frac{1-\mu}{2}\right)\right.$$

$$\left.+\frac{\Pi(n+m)}{\Pi(n-m)}\left(\frac{1-\mu}{1+\mu}\right)^{\frac{1}{2}m}\frac{1}{\Pi(m)}F\left(-n,\, n+1,\, 1+m,\, \frac{1-\mu}{2}\right)\right\}\sec m\pi$$

$$+\tfrac{1}{2}\left\{\cos m\pi\left(\frac{1+\mu}{1-\mu}\right)^{\frac{1}{2}m}\sum_0^{\infty}\frac{\Pi(-n+r+m-1)\,\Pi(n+r+m)\,\Pi'(r)}{\Pi(n)\,\Pi(-n-1)\,\Pi(r+m)\,\Pi(r)\,\Pi(r)} \times\left(\frac{1-\mu}{2}\right)^{r+m}\right.$$

$$+\frac{\Pi(n+m)}{\Pi(n-m)}\left(\frac{1-\mu}{1+\mu}\right)^{\frac{1}{2}m}\sum_0^{\infty}\frac{\Pi(-n+r+1)\,\Pi(n+r)\,\Pi'(r+m)}{\Pi(n)\,\Pi(-n-1)\,\Pi(r)\,\Pi(r+m)\,\Pi(r+m)} \times\left(\frac{1-\mu}{2}\right)^r$$

$$-\left[\frac{\Pi'(n+m)}{\Pi(n-m)}+\frac{\Pi'(n-m)\,\Pi(n+m)}{\Pi(n-m)\,\Pi(n-m)}\right]\left(\frac{1-\mu}{1+\mu}\right)^{\frac{1}{2}m}$$

$$\left.\times\sum_0^{\infty}\frac{\Pi(-n+r+1)\,\Pi(n+r)}{\Pi(n)\,\Pi(-n-1)\,\Pi(r)\,\Pi(r+m)}\left(\frac{1-\mu}{2}\right)^r\right\}\sec m\pi.$$

Now $\left(\frac{1-\mu}{1+\mu}\right)^m F\left(-n,\, n+1,\, 1+m,\, \frac{1-\mu}{2}\right)$

$$= \left(\frac{1-\mu}{2}\right)^m F\left(-n+m,\, n+m+1,\, 1+m,\, \frac{1-\mu}{2}\right);$$

therefore $Q_n^m(\mu)$

$$= \tfrac{1}{2}P_n^m(\mu)\log\frac{1+\mu}{1-\mu}+\tfrac{1}{2}\left\{2\Pi'(0)-\frac{\Pi'(n+m)}{\Pi(n+m)}-\frac{\Pi'(n-m)}{\Pi(n-m)}\right\}P_n^m(\mu)$$

$$-\frac{\sin n\pi\cos m\pi}{2\pi}\left(\frac{1+\mu}{1-\mu}\right)^{\frac{1}{2}m}\sum_0^{m-1}\frac{\Pi(-n+r-1)\,\Pi(n+r)\,\Pi(m-r-1)}{\Pi(r)} \times\left(\frac{1-\mu}{2}\right)^r\cos r\pi$$

$$-\tfrac{1}{2}\left\{\cos m\pi \left(\frac{1+\mu}{1-\mu}\right)^{\frac{1}{2}m} \sum_{1}^{\infty} \frac{\Pi(-n+r+m-1)\,\Pi(n+r+m)}{\Pi(n)\,\Pi(-n-1)\,\Pi(r+m)\,\Pi(r)} A_r \times \left(\frac{1-\mu}{2}\right)^{r+m}\right.$$

$$\left. + \frac{\Pi(n+m)}{\Pi(n-m)} \left(\frac{1-\mu}{1+\mu}\right)^{\frac{1}{2}m} \sum_{0}^{\infty} \frac{\Pi(-n+r+1)\,\Pi(n+r)}{\Pi(n)\,\Pi(-n-1)\,\Pi(r+m)\,\Pi(r)} B_r \times \left(\frac{1-\mu}{2}\right)^{r}\right\} \sec m\pi,$$

where
$$A_r = \sum_{1}^{r} \frac{1}{k} \quad \text{and} \quad B_r = \sum_{1}^{r+m} \frac{1}{k}.$$

When $m = 0$ this takes the form

$$Q_n(\mu) = \tfrac{1}{2} P_n(\mu) \log \frac{1+\mu}{1-\mu} + \left\{\Pi'(0) - \frac{\Pi'(n)}{\Pi(n)}\right\} P_n(\mu)$$
$$- \sum_{1}^{\infty} \frac{\Pi(-n+r-1)\,\Pi(n+r)}{\Pi(n)\,\Pi(-n-1)\,\Pi(r)\,\Pi(r)} \left(\frac{1-\mu}{2}\right)^{r} A_r.$$

6. *Zeroes of $P_n^{-m}(\mu)$ when $\pi-\theta$ is Small and m is an Integer.*

From the relation

$$P_n^{-m}(-\mu) = \cos(n-m)\pi\, P_n^{-m}(\mu) - \frac{2\sin(n-m)\pi}{\pi} Q_n^m(\mu),$$

the zeroes of $P_n^{-m}(\mu)$ when $\pi-\theta$ is small are given by

$$\frac{2\sin(n-m)\pi}{\pi} \frac{\Pi(n-m)}{\Pi(n+m)} Q_n^m(\mu') - \cos(n-m)\pi P_n^{-m}(\mu') = 0,$$

where
$$\mu' = \cos(\pi-\theta),$$

that is, by $n = m+k+\dfrac{1}{\pi}\tan^{-1}\left\{\dfrac{\Pi(n+m)}{\Pi(n-m)}\,\dfrac{P_n^{-m}(\mu')}{Q_n^m(\mu')}\,\dfrac{\pi}{2}\right\}$,

where k has all positive integral values including zero. As before, a series for n may be obtained by expanding the above by Lagrange's theorem. When $\pi-\theta=\phi$ is very small, and m is different from zero,

$$n = m+k+\frac{\Pi(2m+k)}{\Pi(m)\,\Pi(m-1)\,\Pi(k)} \tan^{2m}\frac{\phi}{2},$$

as before. When m is zero, the zeroes of $P_n(\mu)$ are given by

$$n = k + \frac{1}{\pi}\tan^{-1}\left\{\frac{\pi}{\log\dfrac{1+\mu'}{1-\mu'}}\right\},$$

$\pi - \theta = \phi$ being very small, that is, by

$$n = k + \frac{1}{2 \log \frac{2}{\phi}}.$$

7. *The Zeroes of* $P_n^{-m}(\mu)$ *decrease as* $\cos^{-1}(\mu)$ *increases from* 0 *to* π.

Let ν be a zero of $P_n^{-m}(\cos\theta)$ and $\nu + \delta\nu$ the corresponding zero of $P_n^{-m}\{\cos(\theta + \delta\theta)\}$; then

$$-\sin\theta \frac{\partial P_\nu^{-m}(\mu)}{\partial \mu} \delta\theta + \delta\nu \frac{\partial P_n^{-m}(\mu)}{\partial n} = 0.$$

Now, from § 1,

$$(n'-n)(n+n'+1) \int_\mu^1 P_n^{-m}(\mu) P_{n'}^{-m}(\mu)\, d\mu$$
$$= (1-\mu^2)\left\{ P_n^{-m}(\mu) \frac{dP_{n'}^{-m}(\mu)}{d\mu} - P_{n'}^{-m}(\mu) \frac{dP_n^{-m}(\mu)}{d\mu} \right\};$$

therefore, writing $n = \nu$, $n' = \nu + \delta\nu$,

$$\delta\nu\,(2\nu+1) \int_\mu^1 \{P_\nu^{-m}(\mu)\}^2 d\mu = -(1-\mu^2) \frac{dP_n^{-m}(\mu)}{dn} \frac{dP_n^{-m}(\mu)}{d\mu} \delta\nu,$$

when $n = \nu$, that is,

$$\delta\nu\,(2\nu+1) \int_\mu^1 \{P_\nu^{-m}(m)\}^2 d\mu = -(1-\mu^2)^{\frac{3}{2}} \left(\frac{dP_\nu^{-m}(\mu)}{d\mu} \right)^2 \delta\theta;$$

hence $\frac{d\theta}{d\nu}$ is negative for all values of θ between 0 and π; and therefore, as θ increases from 0 to π, any zero diminishes.

8. *The Zeroes of* $P_n^m(\mu)$.

An infinity of real zeroes can be found, as in § 2, from the expression

$$P_n^m(\mu) = \frac{2}{\sqrt{\pi}} \frac{\Pi(n+m)}{\Pi(n+\frac{1}{2})} \left(\frac{\cos\left\{ (n+\frac{1}{2})\theta - \frac{\pi}{4} + \frac{m\pi}{2} \right\}}{(2\sin\theta)^{\frac{1}{2}}} + \ldots \right),$$

where, in this case, in the series for $n + \frac{1}{2} = x$,

$$x_0 = \frac{\pi}{2\theta}(2k + 2s - m + \tfrac{3}{2}),$$

s being the greatest integer less than m, and k having all positive integral values including zero; the corresponding negative real zeroes are obtained by changing the sign of the right-hand side. A formula for the zeroes when θ is small can be obtained, as in § 3, by writing $-m$ for m in the expression for $P_n^{-m}(\mu)$ in terms of Bessel functions. It appears that when θ is small $P_n^m(\mu)$ has $2s$ complex zeroes, unless m is an integer, when s is the greatest integer less than m; as θ increases from 0 to $\pi/2$, these zeroes become real, two at a time, being all real when $\theta = \pi/2$, and, as θ increases from $\pi/2$ to π, they reappear as complex zeroes, two at a time, being all complex when θ differs but slightly from π. A formula for calculating them can be obtained from the expression for

$$P_n^m(\mu) = \frac{1}{\Pi(-m)}\left(\frac{1-\mu}{1+\mu}\right)^{\frac{1}{2}m} F\left(-n,\, n+1,\, 1-m,\, \frac{1-\mu}{2}\right),$$

which may be written

$$\left(\frac{1-\mu}{1+\mu}\right)^{\frac{1}{2}m} \sum_0^{s-1} \frac{\Pi(n+r)\,\Pi(m-r-1)}{\Pi(n-r)\,\Pi(r)} \sin m\pi \left(\frac{1-\mu}{2}\right)^r$$

$$+\left(\frac{1-\mu}{1+\mu}\right)^{\frac{1}{2}m} \sum_s^{\infty} \frac{\Pi(n+r)\cos r\pi}{\Pi(n-r)\,\Pi(r-m)\,\Pi(r)} \left(\frac{1-\mu}{2}\right)^r;$$

when m is an integer this has real zeroes given by

$$\frac{\Pi(n+s)}{\Pi(n-s)} = 0,$$

and the zeroes continuous with these can be obtained in a similar manner to that given in a previous paper.* The real zeroes when $\pi-\theta$ is a small quantity can be obtained, as above, from the formula

$$\tan(n+m)\,\pi = \frac{\pi}{2}\,\frac{P_n^m(\mu')}{Q_n^m(\mu')}.$$

---

* *Proceedings*, Vol. xxix., p. 583.

The following presents were made to the Library during the Recess:—

"Educational Times," July–October, 1899.

"Indian Engineering," Vol. xxv., Nos. 20–25, May 20–June 24; Vol. xxvi., Nos. 1-11, July 1–Sept. 9, 1899.

Sprague, T. B.—"On the Eight Queen Problems," pamphlet; Edinburgh, 1893–99.

Jamin, J.—"Cours de Physique de l'Ecole Polytechnique—Tables générales," 8vo; Paris, 1891.

Poincaré, H.—"Cinématique et Mécanismes Potentiel et Mécanique des Fluides," roy. 8vo; Paris, 1899.

Lorenz, L.—"Œuvres Scientifiques," revues et annotées par H. Valentiner, Tome II., Fasc. 1, 8vo; Copenhague, 1899.

"Proceedings of the American Philosophical Society," Vol. xxxviii., No. 159; Philadelphia, 1899.

"Mathematical Gazette," No. 17; London, 1899.

"Annales Scientifiques de l'Ecole Normale Supérieure—Table des Matières (1864–1883)"; Paris, 1899.

"Journal de Mathématiques Pures et Appliquées" (Liouville, C. Jordan); Paris, 1897.

"American Journal of Mathematics," Vol. xxi., No. 4; October, 1899.

"Annals of Mathematics," Vol. xii., No. 6, June, 1899; Virginia.

Mittag-Leffler, G. — "Sur la Représentation Analytique d'une Fonction Monogène" ("Acta Mathematica," Tome xxiii.).

Huygens, Ch.—"Œuvres Complètes," Vol. viii., 4to; La Haye, 1899.

"L'Enseignement Mathématique," No. 4 (July, 1899) and No. 5 (September, 1899).

"Reciprocal Polygons," by Jamshedji Edalji; Ahmedabad, 1898 (from the Author).

"Bourne's Reciprocals"; Liverpool, 1899.

The following, bound in half calf, were presented by Mr. Tucker:—

"Philosophie der Arithmetik: psychologische und logische Untersuchungen," von Dr. E. G. Husserl, Band I.; Halle-Saale, 1891.

"Lehrbuch der Algebra," von Heinrich Weber, 2te Auflage, Band II.; Braunschweig, 1899.

"Die mathematischen Elemente der Erkenntnisstheorie: Grundriss einer Philosophie der mathematischen Wissenschaften," von O. Schmitz-Dumont; Berlin, 1878.

The following exchanges were received:—

"Transactions of the Royal Society," Series A, Vol. cxci., 1898; and List of Members, November, 1898.

"Proceedings of the Royal Society," Vol. lxv., Nos. 415–417, 1899.

"Record of the Royal Society," No. 1; 1897.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. **xxiii., St. 5–9**; Leipzig, 1899.

"Rendiconti del Circolo Matematico di Palermo," Tomo **xiii., Fasc. 3, 4; 1899.**

"Bulletin de la Société Mathématique de France," **Tome xxvii., Fasc. 11; Paris,** 1899.

"Bulletin of the American Mathematical Society," **Series 2, Vol. v., Nos. 9, 10**; New York, 1899.

"Bulletin des Sciences Mathématiques," Tome xxiii., Av., **Juin, Juillet, Août.** 1899; "Table des Matières," Tome xxii.; Paris, 1898.

"Rendiconto dell' Accademia delle Scienze Fisiche e **Matematiche," Serie 3,** Vol. iii., Fasc. 7, 1897; Serie 3, Vol. v., Fasc. 5–7, 1899; Napoli.

"Journal für die reine und angewandte Mathematik," **Band cxx., Heft 3, 4**; Berlin, 1899.

"Annali di Matematica," Serie 3, Tomo ii., Fasc. 4; Milano, 1899.

"Archives Néerlandaises," Série 2, Tome iii., Livr. 1; La Haye, 1899.

"Atti della Reale Accademia dei Lincei—Rendiconti," **Sem. 1, Vol. viii.,** Fasc. 10–12; Sem. 2, Vol. viii., Fasc. 1–5; and "**Rendiconto dell' Adunanza** solenne del 4 Giugno 1899, onorata della presenza delle **LL. MM. il Re e la** Regina"; Roma, 1899.

"Berichte über die Verhandlungen der Königl. Sächs. Gesellschaft der **Wissen**schaften zu Leipzig," Bd. iii.–iv.; 1899.

"Nyt Tidsskrift for Matematik," A. Aargang x., Nr. 6, 7; Copenhagen, 1899.

"Revue Semestrielle des Publications Mathématiques," Tome vii., Partie 2 (Oct. 1898–Av. 1899); Amsterdam, 1899.

"Journal of the Institute of Actuaries," Vol. xxxiv., Part 6; July, 1899.

"Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich," **Jahrgang** xliv., Heft 1, 2; 1899.

"Nieuw Archiev voor Wiskunde," Reeks 2, Deel iv., Stuk 2; Amsterdam, 1899.

"Wiskundige Opgaven," Deel vii., Stuk 7; Deel viii., Stuk 1; **Amsterdam,** 1899.

"Proceedings of the Physical Society," Vol. xvi., Pt. 6; London, 1899.

"Annales de la Faculté des Sciences de Marseille," Tome ix., **Fasc.** 1–5; 1899.

"Sitzungsberichte der Königl. Preuss. Akademie der Wissenschaften zu **Berlin,"** Nos. 23–38; 1899.

"Proceedings of the Cambridge Philosophical Society," Vol. x., Pt. 2; 1899.

"Transactions of the Cambridge Philosophical Society," Vol. xvii., Pt. 3; 1899.

"Memoirs and Proceedings of the Manchester Literary and Philosophical **Society,"** Vol. xliii., Pt. 2; 1898–9.

"Nachrichten von der Königl. Gesellschaft der Wissenschaften zu **Göttingen"**: Geschäftliche Mittheilungen, 1898, Pt. 2, 1899, Pt. 1; Math.-Phys. **Klasse,** 1899, Pt. 1; 1899.

"Jahrbuch über die Fortschritte der Mathematik," Bd. **xxviii., Heft 1, 2** (Jahrgang, 1897); Berlin, 1899.

"Acta Mathematica," Vol. xxii., No. 4; Stockholm, 1899.

"Wiadomosci Matematyczne," Tom iii., Zeszyt 3, 4; Warsaw, 1899.

"Proceedings of the Russian Mathematical Society," Vols. xvi., **xix.**; **Odessa,** 1899.

"Proceedings of the Edinburgh Mathematical Society," Vol. xvii., 1898-9.
"Periodico di Matematica," Serie ii., Vol. ii.; Livorno, 1899.
"Supplemento al Periodico di Matematica," Livorno, 1899.

Purchase:

"Official Year Book of the Scientific and Learned Societies of Great Britain and Ireland," 1899 (C. Griffin & Co.).

APPENDIX.

(Session 1898—1899.)

Lord Kelvin's communication (p. 147) is published in the *Philosophical Magazine* for May, August, and October, 1899. The title of it is "On the Application of Force, within a Limited Space, required to produce Spherical Solitary Waves, or Trains of Periodic Waves, of both species, Equivoluminal and Irrotational, in an Elastic Solid."

The following is the text of the address which was presented by Lord Kelvin, on behalf of the Society, to Sir George Gabriel Stokes, on the occasion of the celebration of his Jubilee as Lucasian Professor:—

To Prof. Sir George Gabriel Stokes, Bart., M.A., D.C.L., LL.D., F.R.S.,
Lucasian Professor of Mathematics in the University of Cambridge.

Sir,—The Council of the London Mathematical Society have much pleasure in tendering to you their sincere congratulations on the approaching completion of the fiftieth year of your tenure of the Lucasian Professorship.

They recognize that many of the Memoirs published in the *Proceedings* of the Society bear witness to the great influence which your writings have had in opening new paths of discovery in Mathematics, and in pointing the way to novel applications in the domain of Natural Philosophy, and they desire to express a hope that you may live long to represent in Great Britain the science which you have done so much to promote.

KELVIN, *President.*
R. TUCKER, } *Hon. Secs.*
A. E. H. LOVE, }

[redacted]

The Connecticut Academy of Arts and Sciences, on the occasion of celebrating its centenary, on October 11th, 1899, invited delegates from the Society to attend the "commemorative exercises." The following address was drawn up and signed by the President and Hon. Secretaries. Prof. E. W. Brown undertook to present the address :—

The London Mathematical Society present fraternal greetings to the Connecticut Academy of Sciences on the occasion of the hundredth anniversary of their foundation.

They look back with satisfaction on the exchange of publications which has subsisted between the two bodies ever since their own foundation in the year 1865.

They recognize with much pleasure the importance of the researches in Mathematical and Physical Science given to the world by the Connecticut Academy, in a language which does not convey to them any suggestion of foreign origin. In no country has the value of these researches been earlier or more fully recognized than in Great Britain.

They desire and expect a long career of increasing usefulness and honour for the Connecticut Academy of Sciences, which even now takes rank among the most ancient of the existing learned societies of the world.

KELVIN, *President.*
R. TUCKER, A. E. H. LOVE, } *Hon. Secs.*

Dr. Macaulay has drawn up the accompanying notice of his friend Mr. S. O. Roberts, and placed it at our disposal:—

Samuel Oliver Roberts, son of Samuel Roberts, F.R.S., was born at Boston, in Lincolnshire, on September 19th, 1859, and was educated at the Islington Proprietary School. He went to Cambridge in 1879, having gained a scholarship at St. John's College, and was seventh Wrangler in 1882. In 1884 he became Mathematical Master at the Grammar School of Newcastle-on-Tyne, and in 1888 second Mathematical and Science Master at Merchant Taylors' School. In the present year he was one of the selected candidates for the Head-mastership of the Central Foundation School, but had to resign his candidature owing to the illness from which he was not destined to recover. He became a member of the Mathematical Society in 1885, and occasionally attended the meetings, but his time was too fully occupied to allow of his contributing any papers to the *Proceedings*. He was also an actively interested member of the Physical Society.*

* Mr. Roberts was also an Hon. Secretary of the Mathematical Association, which has taken up and extended the work of the Association for the Improvement of Geometrical Teaching.

His abundant energy was devoted to the interests of his school, and there was no department of school life in which his help was not eagerly sought and readily given. He is described by his colleagues as a born organizer. His pupils gained many remarkable successes at Oxford and Cambridge. He was a good French scholar, and keenly interested in modern history and in all educational questions. Apart from this he devoted much time to the school athletics, had the management of the cricket club, and was a fine chess player.

Towards the close of the year 1888, Mr. Tucker, at the request of Mrs. Spottiswoode and Dr. Hirst, undertook to edit the late Mr. Spottiswoode's Mathematical Papers.* After some little delay the idea was abandoned, the papers remaining in Mr. Tucker's possession. The result of a recent correspondence with Mr. Hugh Spottiswoode was that the Council received a letter from this gentleman in which he expressed a desire to present to the Society the pure mathematical papers—"the only stipulation that I should make would be that I should have free access to the papers at any future time should I desire to reprint them or consult them for any other purpose." The offer was accepted,† and Mr. Tucker was directed to return the thanks of the Council to Mr. Spottiswoode. The physical papers were returned to Mr. Spottiswoode.

The following is a list of the papers retained by the Council:—

From *Crelle's Journal des Mathématiques pures et appliquées*:—

"Mémoire sur quelques formules relatives aux surfaces du second ordre" [Tome XLII.], 1850.

"Mémoire sur les points singuliers d'une courbe à double courbure" [Tome XLII.], 1851.

"Two lettres [*sic*] of the geometrical correspondence between M. Donkin and M. Spottiswoode," March, 1853.

"Sur quelques formules générales dans le calcul des opérations" [Tome LIX.], 1861.

"Note sur la transformation de la cubique ternaire en sa forme canonique" [Tome LXIII.], 1863.

"Elementary Theorems relating to Determinants" [64 pp.], 1851.

"Elementary Theorems relating to Determinants," 2nd edition, rewritten and much enlarged by the Author (this was published in *Crelle's Journal*, Tome LI., 116 pp.), 1855.

"Meditationes Analyticæ" [Pts. 1–5, Pt. 1 is in MS.]‡, Oxford, 1847.

---

* *Cf.* Letter in *Nature*, for December 27th, 1888, p. 197.

† October 12th, 1899. ‡ Made by Mr. Tucker.

From the *Philosophical Transactions* :—

"On an Extended Form of the Index-Symbol in the Calculus of Operations," December. 1859.

"On the Contact of Curves," November, 1861.

"On the Calculus of Symbols " (two memoirs), November, 1861; January, 1862.

"On the Sextactic Points of a Plane Curve," June, 1865.

"On the Contact of Conics with Surfaces," March, 1870.

"On the Contact of Surfaces," February, 1872.

"On Multiple Contact of Surfaces," June, 1875.

"On Hyper-Jacobian Surfaces and Curves," May, 1877.

"On the Forty-eight Coordinates of a Cubic Curve in Space," January, 1881.

From the *Comptes Rendus* :—

"Note sur l'équilibre des forces dans l'espace," January, 1868.

From *Proceedings of the Royal Astronomical Society*, Vol. XXIX. :—

"On a Method for determining Longitude by means of Observations on the Moon's greatest Altitude," December, 1860.

From *Proceedings of Royal Society* :—

"On the Equations of Rotation of a Solid Body about a Fixed Point," [No. 59] April, 1863.

"On the Rings produced by Crystals when submitted to Circularly Polarized Light," [No. 134] 1872.

"On Multiple Contact of Surfaces," [No. 163] 1875.

"An Experiment on Electro-Magnetic Rotation," [No. 168] 1876.

"On the 48 Coordinates of a Cubic Curve in Space," [No. 209] 1880.

"On certain Geometrical Theorems, No. 1," [No. 217] 1881, and "No. 2," [No. 220] 1882, by W. H. L. Russell, with Note on "No. 2," by W. Spottiswoode, [No. 220] 1882.

"Note on Mr. Russell's Paper 'On certain Definite Integrals, No. 10,' " [No. 218] 1882.

"Die Mathematik in ihren beziehungen zu den anderen Wissenschaften," Leipzig, 1879 [translation of his Presidential Address at the Dublin meeting of the British Association, 1878].

From the *Cambridge and Dublin Mathematical Journal* :—

"On certain Geometrical Theorems," May, 1851.

"On the Curvature of Curves in Space," November, 1854.

From the *Quarterly Journal of Pure and Applied Mathematics* :—

"On a Theorem in Statics" (4 pp.).

"Note on Axes of Equilibrium" (4 pp.).

"On Typical Mountain Ranges: an Application of the Calculus of Probabilities to Physical Geography" (read before Royal Geographical Society, April 23rd, 1860).

"On the Súrya Siddhánta, and the Hindu method of Calculating Eclipses" (from *Journal of Royal Asiatic Society*).

"Note on Differential Resolvents," [*Memoirs of the Literary and Philosophical Society of Manchester*] 1864.

From the *Philosophical Magazine* :—

"On the Quaternion Expressions of Coplanarity and Homoconicism" (3 pp.).

"On a Geometrical Theorem" (1 p.), 1850.

"On the Geometrical Interpretation of Quaternions" (10 pp.).

"On a Problem in Combinatorial Analysis," May, 1852.

"Presidential Address to the Mathematical Section, Birmingham Meeting of British Association, 1865."

"On Petzval's Asymptotic Method of Solving Differential Equations," and "On the Reduction of a Decadic Binary Quantic to its Canonical Form" [slip from *British Association Report* for 1861].

From *Proceedings of London Mathematical Society* :—

"Remarks on some recent Generalizations of Algebra" [Nos. 50, 51], Presidential Address.

"On the Contact of Quadrics with other Surfaces" [Vol. v., No. 71].

"On Determinants of Alternate Numbers" [Vol. vii., Nos. 94, 95].

"On Curves having Four-Point Contact with a Triply-Infinite Pencil of Curves" [Vol. viii., Nos. 105, 106].

"On the Twenty-One Coordinates of a Conic in Space" [Vol. x., Nos. 152, 153].

"On the Polar Planes of Four Quadrics" [Vol. xiii., No. 181].

THIRTY-SIXTH SESSION, 1899–1900

(since the Formation of the Society, January 16th, 1865).

November 9th, 1899.

THE SIXTH ANNUAL GENERAL MEETING OF THE LONDON MATHEMATICAL SOCIETY, as incorporated under the Companies Act, 1867, on October 23rd, 1894, held at 22 Albemarle Street, W.

Lord KELVIN, G.C.V.O., President, in the Chair.

Eighteen members present.

The Treasurer gave a short abstract of his Report; its reception was moved by Mr. Kempe, and seconded by Prof. W. Burnside, and carried unanimously.

The President stated that Mr. Gallop would be asked to act as Auditor of the Report, as in the past session.

Mr. Tucker announced that the Council had consented to exchange the *Proceedings* with Dr. Lazzeri for his *Periodico di Matematica per l'insegnamento secondario.* He also mentioned that the Society's losses, by death, during the session had been Prof. Bartholomew Price, Mr. S. O. Roberts, and an honorary member, Prof. Sophus Lie.

Mr. Love said that the number of the members at the beginning of the session was 234, losses by death had been 2, and the new members elected during the session were 17, thus making the present number of members to be 247.

On the motion of Mr. Kempe, a vote of thanks was unanimously carried to Mr. F. W. Russell for his services to the Society as Hon. Librarian. On this gentleman's resignation of the office, Mr. A. E. Western, having expressed his willingness to undertake the duties of the post, was appointed Librarian.

Lord Kelvin briefly stated that the Council, as announced at the June meeting, had awarded the De Morgan Medal to Prof. W. Burnside, and requested Major MacMahon to state the grounds of *the Council's* award. Lord Kelvin then presented the medal, and

Prof. Burnside feelingly thanked the Council for the honour they had conferred upon him.

The ballot was next taken, and the result was stated by the Scrutators, Messrs. A. E. Western and E. W. Barnes, to be that the following gentlemen, nominated by the Council, were elected as the Council for the Session 1899–1900 :—President: The Right Hon. Lord Kelvin; Vice-Presidents: Prof. Elliott, Lt.-Col. Cunningham, and Prof. Lamb; Treasurer: Dr. J. Larmor; Hon. Secs.: Mr. Tucker, and Prof. Love. Other members: Prof. W. Burnside, Dr. Glaisher, Prof. M. J. M. Hill, Dr. Hobson, Mr. Kempe, Dr. Macaulay, Mr. H. M. Macdonald, Major MacMahon, and Mr. Whittaker.

Prof. Burnside communicated a short note by Dr. L. E. Dickson on "The Abstract Group isomorphic with the Symmetric Group on k Letters." Major MacMahon spoke on "The Fundamental Solutions of the Indeterminate Relation $\lambda x \geqslant \mu y$." Mr. Western and Lt.-Col. Cunningham asked a few questions on the subject of the communication.

The following papers were read in abstract:—

Certain Correspondences between Spaces of n Dimensions: Dr. E. O. Lovett.

(i.) On the Form of Lines of Force near a Point of Equilibrium; (ii.) The Reduction of Conics and Quadrics to their Principal Axes by the Weierstrassian Method of Reducing Quadratic Forms; and (iii.) On the Reduction of a Linear Substitution to a Canonical Form, with some Applications to Linear Differential Equations and Quadratic Forms: T. J. I'A. Bromwich.

(i.) On Ampère's Equation $Rr+2Ss+Tt+U(rt-s^2) = V$; and (ii.) The Theory of Automorphic Functions: Prof. A. C. Dixon.

Note on Clebsch's Second Method for the Integration of a Pfaffian Equation: J. Brill.

The following presents were made to the Library:—

"Mathematical Gazette," No. 18; October, 1899.

"Transactions of the Connecticut Academy of Arts and Sciences," Vol. x., Pt. 1; Newhaven, U.S.A., 1899.

"Wiadomosci Matematyczne," Tom III., Zeszyt 5, 6; Warsaw, 1899.

"Annals of Mathematics," Series 2, Vol. I., No. 1; Cambridge, Mass., U.S.A., 1899.

Bubnov, Dr. N.—"Gerberti postea Silvestri II. Papæ Opera Mathematica." 8vo; Berolini, 1899.

Biddle, D.—"Mathematical Questions and Solutions from the 'Educational Times,'" Vol. LXXI., 8vo: London, 1899.

"Educational Times," November, 1899.

"Indian Engineering," Vol. XXVI., Nos. 12–16; Sept. 16–Oct, 14, 1899.

The following exchanges were received :—

"Periodico di Matematica per l'insegnamento secondario," Anno XV., Fasc. 2, Sett.-Ottob.; Livorno, 1899.

"Proceedings of the Royal Society," Vol. LXV., Nos. 418–420; 1899.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XXIII., St. 10; Leipzig, 1899.

"Rendiconti del Circolo Matematico di Palermo," Tomo XIII., Fasc. 5; 1899.

"Bulletin de la Société Mathématique de France," Tome XXVII., Fasc. 3; Paris, 1899.

"Bulletin of the American Mathematical Society," 2nd Series, Vol. VI., No. 1; New York, October, 1899.

"Monatshefte für Mathematik und Physik," Jahrgang X., Pt. 4; Wien, 1899.

"Bulletin des Sciences Mathématiques," Tome XXIII., Sept., 1899; Paris, 1899.

"Journal für die reine und angewandte Mathematik," Bd. CXXI., Heft 1, 2; Berlin, 1899.

"Annali di Matematica," Serie 3, Tomo III., Fasc. 1, 2; Milano, 1899.

"Sitzungsberichte der Physikalisch-medicinischen Societät in Erlangen," Heft 30; 1898.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 2, Vol. VIII., Fasc. 6–8; Roma, 1899.

"Journal of the Institute of Actuaries," Vol. XXXV., Pt. 1; October, 1899.

"Memorie della Regia Accademia in Modena," Serie III., Vol. I., Tavole 16; 1899.

"Nieuw Archief voor Wiskunde," 2 Reeks, Deel IV., Stuk 3; Amsterdam, 1899.

"Transactions of the Canadian Institute," Vol. II., Pt. 2, No. 8; Toronto, September, 1899.

On the Reduction of a Linear Substitution to a Canonical Form.

By T. J. I'A. Bromwich. Received October 27th, 1899. Read November 9th, 1899.

I. *Introductory Remarks.*

It will be seen at once that the main idea of the following note is the same as that of Herr Netto's paper "Zur Theorie der linearen Substitutionen."* That is to say, we pass from the case of a substitution whose characteristic determinant has a root a repeated p times to the case of a substitution with p roots differing but little from a, but all distinct. The change is made by increasing each coefficient of the substitution by a small arbitrary quantity; and so we reduce the substitution to the limiting case of one with all its roots distinct.

The point of divergence between my work and Herr Netto's is that I have shown that it is unnecessary to retain these small changes in the coefficients when we seek to determine the linear functions which reduce the substitution to a canonical form. This simplification will be found of considerable advantage in cases of numerical calculation; from a purely theoretical point of view, it is probably of less importance. To show that the difficulties of calculation are not great, I have worked out at length the particular example given by Prof. Burnside in illustration of his method for reducing linear substitutions.†

II.

Suppose we have the substitution

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n, \\ x_2' &= a_{21}x_1 + \ldots \qquad\quad + a_{2n}x_n, \\ \ldots &\quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ x_n' &= a_{n1}x_1 + \ldots \qquad\quad + a_{nn}x_n. \end{aligned}$$

---

* *Acta Mathematica*, Vol. xvii., p. 265.

† *Proc. Lond. Math. Soc.*, Vol. xxx., p. 180.

Then, forming the quantity

$$l_1 x_1' + l_2 x_2' + \dots + l_n x_n',$$

we shall have that this is θ times

$$l_1 x_1 + l_2 x_2 + \dots + l_n x_n,$$

provided we have

$$\left.\begin{array}{l} (a_{11}-\theta)\, l_1 + a_{21} l_2 + \dots + a_{n1} l_n \qquad = 0 \\ a_{12} l_1 + (a_{22}-\theta)\, l_2 + \dots + a_{n2} l_n \qquad = 0 \\ \dots \qquad \dots \qquad \dots \qquad \dots \qquad \dots \\ a_{1n} l_1 + a_{2n} l_2 + \dots \qquad + (a_{nn}-\theta)\, l_n = 0 \end{array}\right\}. \tag{1}$$

Hence θ is a root of the determinantal equation

$$\Delta = \begin{vmatrix} a_{11}-\theta, & a_{21}, & \dots, & a_{n1} \\ a_{12}, & a_{22}-\theta, & \dots, & a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{1n}, & a_{2n}, & \dots, & a_{nn}-\theta \end{vmatrix} = 0.$$

If the roots of this equation are all distinct, we have n values of θ, and n corresponding determinations of the ratios

$$l_1 : l_2 : \dots : l_n.$$

But this method breaks down if we find that any root of $\Delta = 0$, say $\theta = a$, is repeated. The object of this note is to explain how we can very easily extend our method so as to cover this case. Suppose, then, that $(\theta - a)$ is a p-times repeated factor of Δ and a q-times repeated factor of *every* first minor of Δ; so that $(\theta - a)^{p-q}$ is a first invariant-factor (*Elementartheiler*) of Δ. Solve for the ratios $l_1 : l_2 : \dots : l_n$ from any $(n-1)$ of the equations (1); we shall suppose the last $(n-1)$ to be selected, to avoid verbal confusion, the method of procedure being the same whatever $(n-1)$ equations are chosen. We now have

$$\frac{l_1}{A_{11}} = \frac{l_2}{A_{21}} = \dots = \frac{l_n}{A_{n1}},$$

the capital letters being the first minors of the corresponding small letters in Δ. Every one of the quantities $A_{11}, \dots, A_{n1}$ will contain the factor $(\theta - a)^q$; divide out by this and write $\theta = a + t$. We then have $l_1, l_2, \dots, l_n$ expressed as polynomials in t; one at least of these polynomials must have a term independent of t, which implies that at least one of the minors $A_{11}, \dots, A_{n1}$ is *regular* or is not divisible by a higher power than t^q.*

* If this is not the case, we must use another set of $(n-1)$ equations for the l's.

Now write $\xi = (a_{11}-\theta)\, l_1 + a_{21} l_2 + \ldots + a_{n1} l_n,$

and eliminate $l_1 : l_2 : \ldots : l_n.$

We obtain a determinant equal to zero, which only differs from Δ in having a_{k1} replaced by $a_{k1} - \xi / l_k$, where k is any one of the numbers 1, 2, ..., n which satisfies the condition that A_{k1} is regular.

Thus we have $\Delta - (\xi A_{k1} / l_k) = 0,$

and, replacing θ by $(a+t)$, we see that Δ contains t^p as a factor, and A_{k1}/l_k contains t^q. Hence ξ contains t^{p-q} as a factor. the remaining factor not vanishing with t.

Thus we have the equations

$$a_{11} l_1 + a_{21} l_2 + \ldots + a_{n1} l_n = (a+t)\, l_1$$
$$+ \text{terms of order } t^{p-q} \text{ and higher orders,}$$

$$a_{12} l_1 + a_{22} l_2 + \ldots + a_{n2} l_n = (a+t)\, l_2,$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$a_{1n} l_1 + a_{2n} l_2 + \ldots + a_{nn} l_n = (a+t)\, l_n.$$

Hence $l_1 x_1' + l_2 x_2' + \ldots + l_n x_n' = (a+t)(l_1 x_1 + l_2 x_2 + \ldots + l_n x_n)$

$$+ \text{terms of order } t^{p-q},$$

or, if we expand $l_1 x_1 + \ldots + l_n x_n$ in the form

$$X_1 + X_2 t + X_3 t^2 + \ldots,$$

we shall have

$$X_1' + X_2' t + X_3' t^2 + \ldots = (a+t)(X_1 + X_2 t + X_3 t^2 + \ldots)$$
$$+ \text{terms of order } t^{p-q}.$$

Thus, equating coefficients of corresponding powers of t,* we have

$$X_1' = a X_1,$$
$$X_2' = a X_2 + X_1,$$
$$X_3' = a X_3 + X_2,$$
$$\ldots \quad \ldots \quad \ldots$$
$$X_{p-q}' = a X_{p-q} + X_{p-q-1}.$$

* This step is legitimate, for t is arbitrary, and the series on the two sides will be terminated.

It should be observed that the values of the X's so determined are not unique; for we need not take

$$l_1 = \frac{A_{11}}{t^q}, \text{ \&c.,}$$

and it is only necessary to put

$$l_1 = \frac{A_{11}}{t^q}(\beta_1+\beta_2 t+\beta_3 t^2+\ldots),$$

$$l_2 = \frac{A_{21}}{t^q}(\beta_1+\beta_2 t+\beta_3 t^2+\ldots),$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$l_n = \frac{A_{n1}}{t^q}(\beta_1+\beta_2 t+\beta_3 t^2+\ldots).$$

Then we see that, if

$$(X_1+X_2 t+X_3 t^2+\ldots)(\beta_1+\beta_2 t+\beta_3 t^2+\ldots) = Y_1+Y_2 t+Y_3 t^2+\ldots,$$

so that $$Y_m = \beta_1 X_m+\beta_2 X_{m-1}+\ldots+\beta_m X_1,$$

the series of Y's will satisfy the same equations as the X's; and the Y's contain $(p-q)$ arbitrary constants. There may be a further indeterminacy, according to the particular line in equations (1) whose minors give the l's.

We now proceed to get more linear functions of $x_1, \ldots, x_n$ which possess similar properties. Suppose that $(\theta-a)^r$ is a factor of every second minor of Δ, and, further, that this is the highest power of $(\theta-a)$ that does occur in every second minor.

Solve for the ratios $l_1 : l_2 : \ldots : l_n$ from the last $(n-2)$ equations* of (1); the results will involve one arbitrary quantity (such as the ratio $l_1 : l_2$). We write $\theta = a+t$, and express the ratios in powers of t, dividing by t^r; now put

$$\xi = (a_{11}-\theta)\, l_1+a_{21} l_2+\ldots+a_{n1} l_n,$$

$$\eta = a_{12} l_1+(a_{22}-\theta)\, l_2+\ldots+a_{n2} l_n.$$

---

* We might use equally well any other $(n-2)$ of the equations; the process of selection adopted here is to cancel successively the first, second, third, &c., of equations (1).

Then we see that

$$\begin{vmatrix} a_{12}l_1+a_{k2}l_k-\eta, & a_{22}-\theta, & \ldots, & a_{n2} \\ a_{13}l_1+a_{k3}l_k, & a_{23}, & \ldots, & a_{n3} \\ \ldots & \ldots & \ldots & \ldots \\ a_{1n}l_1+a_{kn}l_k, & a_{2n}, & \ldots, & a_{nn}-\theta \end{vmatrix} = 0,$$

where the two l's in the first column may be any two we please, although here one has been taken to be l_1, simply to save multiplicity of distinguishing suffixes.

Expanding out this determinant, we have

$$\eta \text{ (a second minor of } \Delta) = l_1 \text{ (a first minor)} + l_k \text{ (a first minor)}.$$

Now, at least one of the second minors contains as a factor t raised to no higher power than t^r (or is regular); we here suppose this to be true for the minor obtained by deleting the first and second rows and columns of Δ. All the first minors contain t^q as a factor, while none of the l's contain negative powers of t. Hence the lowest power of t in η is at least t^{q-r}; the same result holds for ξ.

Accordingly we have the equations

$$\begin{aligned} a_{11}l_1+a_{21}l_2+\ldots+a_{n1}l_n &= (a+t)\,l_1 + \text{terms of order } t^{q-r}, \\ a_{12}l_1+a_{22}l_2+\ldots+a_{n2}l_n &= (a+t)\,l_2 + \text{terms of order } t^{q-r}, \\ a_{13}l_1+a_{23}l_2+\ldots+a_{n3}l_n &= (a+t)\,l_3. \\ \ldots \quad \ldots \quad & \ldots \quad \ldots \quad \ldots \\ a_{1n}l_1+a_{2n}l_2+\ldots+a_{nn}l_n &= (a+t)\,l_n. \end{aligned}$$

Thus, if with these values of $l_1, l_2, \ldots, l_n$, we write

$$l_1x_1+\ldots+l_nx_n = X_{p-q+1}+tX_{p-q+2}+t^2X_{p-q+3}+\ldots+t^{q-r-1}X_{p-r}+\ldots,$$

we shall have

$$X'_{p-q+1}+tX'_{p-q+2}+t^2X'_{p-q+3}+\ldots = (a+t)(X_{p-q+1}+tX_{p-q+2}+\ldots)$$
$$+\text{terms of order } t^{q-r}.$$

Hence

$$\begin{aligned} X'_{p-q+1} &= aX_{p-q+1}, \\ X'_{p-q+2} &= aX_{p-q+2}+X_{p-q+1}, \\ \ldots \quad & \ldots \quad \ldots \quad \ldots \\ X'_{p-r} &= aX_{p-r} \quad +X_{p-r-1}. \end{aligned}$$

Just as before, we can construct a set of Y's which satisfy these equa-

tions and contain $(q-r)$ arbitrary constants. But we may add to Y_{p-q+k} any term such as

$$\gamma_1 X_k + \gamma_2 X_{k-1} + \ldots + \gamma_k X,$$

for $X_1, X_2, \ldots, X_k, \ldots$ satisfy the same relations as $X_{p-q+1}, X_{p-q+2}, \ldots, X_{p-q+k} \ldots$, and so we get $(q-r)$ more arbitrary constants. It should be observed that the last $(q-r)$ constants should include the one arbitrary constant that appears in the original solution for $l_1 : l_2 : \ldots : l_n$. We have thus in the general reduction of these $(q-r)$ terms $2(q-r)$ arbitrary constants.

It is now easy to see how we can extend the method proposed so as to deal with terms which arise from minors of higher orders. Suppose h denotes the difference between the index of $(\theta-a)$ in the greatest common measure of all the $(k-1)^{\text{th}}$ minors of Δ, and the corresponding index for the k^{th} minors; so that $(\theta-a)^h$ is the k^{th} invariant-factor (*Elementartheiler*) of Δ. We solve for $l_1 : l_2 : \ldots : l_n$ from the last $(n-k)$ of equations (1)*; then we write $\theta = a + t$, and divide by the extraneous power of t; the values of $l_1 : \ldots : l_n$ so found will satisfy the first k of equations (1) up to terms in t^h. Then we expand $l_1 x_1 + \ldots + l_n x_n$ up to terms in t^{h-1}, and so obtain h linear functions of $x_1, \ldots, x_n$, say $X_{v+1}, X_{v+2}, \ldots, X_{v+h}$. These will satisfy

$$\begin{aligned} X'_{v+1} &= aX_{v+1}, \\ X'_{v+2} &= aX_{v+2} + X_{v+1}, \\ \ldots \quad & \quad \ldots \quad \ldots \\ X'_{v+h} &= aX_{v+h} + X_{v+h-1}. \end{aligned}$$

The most general values which satisfy these equations can be constructed as before explained; and we see that they will contain hk arbitraries, h from each of the k groups of linear functions $(X_1, \ldots, X_{p-q}), (X_{p-q+1}, \ldots, X_{p-r}), \ldots, (X_{v+1}, \ldots, X_{v+h})$.

Our process goes on until we reach a minor of Δ which does not contain $(\theta-a)$ as a factor. We shall then have found

$$(p-q)+(q-r)+(r-s)+\ldots = p$$

linear functions of $x_1, \ldots, x_n$ which reduce to canonical forms those parts of the substitution which correspond to the root $\theta = a$ of $\Delta = 0$.

Proceeding in this way for each root of $\Delta = 0$, we finally obtain n linear functions which will reduce the given substitution to its canonical form.

---

* We assume that one at least of the minors formed from these $(n-k)$ equations is regular.

III.

To give a numerical illustration, and to compare the method with Prof. Burnside's (*Proc. Lond. Math. Soc.*, Vol. xxx., p. 191), let us take his example,

$$\begin{aligned} x_1' &= -2x_1 - x_2 - x_3 + 3x_4 + 2x_5, \\ x_2' &= -4x_1 + x_2 - x_3 + 3x_4 + 2x_5, \\ x_3' &= x_1 + x_2 - 3x_4 - 2x_5, \\ x_4' &= -4x_1 - 2x_2 - x_3 + 5x_4 + x_5, \\ x_5' &= 4x_1 + x_2 + x_3 - 3x_4. \end{aligned}$$

We find the equations for the l's,

$$\begin{aligned} -(2+\theta)\, l_1 - 4l_2 + l_3 - 4l_4 + 4l_5 &= 0, \\ -l_1 + (1-\theta)\, l_2 + l_3 - 2l_4 + l_5 &= 0, \\ -l_1 - l_2 - \theta l_3 - l_4 + l_5 &= 0, \\ 3l_1 + 3l_2 - 3l_3 + (5-\theta)\, l_4 - 3l_5 &= 0, \\ 2l_1 + 2l_2 - 2l_3 + l_4 - \theta l_5 &= 0, \end{aligned}$$

and these give $\qquad \Delta = -(\theta+1)^2(\theta-2)^3.$

The minors of the elements of the first row are found to be, in order,

$$\begin{aligned} &\theta(\theta-2)^3, \\ &-(\theta+1)^2(\theta-2), \\ &-(\theta-2)^3, \\ &3(\theta+1)(\theta-2)^2, \\ &(\theta+1)(\theta-2)(2\theta-7), \end{aligned}$$

so that $(\theta-2)$ is a factor of all these, but $(\theta+1)$ is not. As a matter of fact $(\theta-2)$ is a factor of every first minor; so that for $(\theta-2)$ the second and fifth of the above are regular; for $(\theta+1)$, the first and third. To proceed put $\theta = -1+t$, and, the corresponding difference $(p-q)$ being equal to 2, we have to expand only as far as the terms in t. After dividing by $-3(\theta-2)$, we find, neglecting terms in t^2 and higher powers of t, in agreement with our rule,

$$\begin{aligned} l_1 &= \tfrac{1}{3}(1-t)(3-t)^2 = 3-5t, \\ l_2 &= \tfrac{1}{3}t^3 \qquad\qquad\quad\; = 0, \\ l_3 &= \tfrac{1}{3}(3-t)^2 \qquad\quad\;\; = 3-2t, \\ l_4 &= t(3-t) \qquad\qquad\;\; = 3t, \\ l_5 &= \tfrac{1}{3}t(9-2t) \qquad\quad\, = 3t. \end{aligned}$$

Hence $$X_1 = 3(x_1+x_3).$$
$$X_2 = -5x_1-2x_3+3(x_4+x_5).$$

We shall not have to proceed to the second minors, as $(\theta+1)$ is not a factor of all the first minors.

We next put $\theta = 2+t$, and divide by $(-3t)$: then

$$\begin{aligned} l_1 &= -\tfrac{1}{3}t^2(2+t) &&= 0, \\ l_2 &= \tfrac{1}{3}(3+t)^2 &&= 3+2t. \\ l_3 &= \tfrac{1}{3}t^3 &&= 0, \\ l_4 &= -t(3+t) &&= -3t, \\ l_5 &= \tfrac{1}{3}(3+t)(3-2t) &&= 3-t, \end{aligned}$$

where we expand only to terms in t, because the difference $(p-q)$ is 2.

Thus $$X_3 = 3(x_2+x_3),$$
$$X_4 = 2x_2-3x_4-x_5.$$

In this case we have to go on to solve the last three equations in the l's; but we may simply write $\theta = 2$, and drop the t, for the difference $(q-r)$ is 1; so that the l's need only be calculated to the term independent of t. Hence we take

$$\begin{aligned} -(l_1+l_2)-2l_3-l_4+\ l_5 &= 0, \\ (l_1+l_2)-\ l_3+l_4-\ l_5 &= 0, \\ 2(l_1+l_2)+2l_3+l_4-2l_5 &= 0, \end{aligned}$$

giving $$l_3 = 0,\quad l_4 = 0,\quad l_1+l_2 = l_5.$$

Hence $$X_5 = l_1(x_1+x_5)+l_2(x_2+x_5),$$

and l_1 must not vanish, for, if so,

$$3X_5 = l_2X_3.$$

We now have the given substitution reduced to

$$\begin{aligned} X_1' &= -X_1, \\ X_2' &= -X_2+X_1, \\ X_3' &= 2X_3, \\ X_4' &= 2X_4+X_3, \\ X_5' &= 2X_5, \end{aligned}$$

which is a canonical form of the substitution. We see that the classification of any substitution is given by the indices of the invariant-factors, and hence this substitution is typified by [(2, 1), 2].

We may take the generalized reducing functions

$$\begin{aligned} Y_1 &= \beta_1 X_1, \\ Y_2 &= \beta_1 X_2 + \beta_2 X_1, \\ Y_3 &= \gamma_1 X_3, \\ Y_4 &= \gamma_1 X_4 + \gamma_2 X_3, \\ Y_5 &= \delta_1 X_5, \end{aligned}$$

where β_1, β_2, γ_1, γ_2, δ_1 are arbitrary.

Prof. Burnside's two reductions are given by

$$\text{(i.)}\ \beta_1 = -3, \quad \beta_2 = 0, \quad \gamma_1 = -3, \quad \gamma_2 = 0, \quad l_2 = -l_1.$$

$$\text{(ii.)}\ \beta_1 = \tfrac{1}{3}, \quad \beta_2 = \tfrac{5}{9}, \quad \gamma_1 = \tfrac{1}{3}, \quad \gamma_2 = -\tfrac{2}{9}, \quad l_2 = -l_1.$$

[*February* 15*th*.—Since completing the above I have seen Mr. A. C. Dixon's paper (p. 170 of this volume). His method is quite different from mine.]

Notes on the Theory of Automorphic Functions. *By* A. C. Dixon.

Received November 8th, 1899. Communicated November 9th, 1899. Received, in revised form, January 9th, 1900.

Under the above heading I propose to make some remarks on certain points in the theory of automorphic functions, from the point of view taken by Poincaré in his memoirs (*Acta Mathematica*, Vols. I., III., IV., V.).

In the first place, I show how the theorem given by him that a Fuchsian function exists of the second family and of class 0, and taking assigned values at singular points, may be used to establish the existence theorem on a Riemann surface, so far at least as that theorem relates to uniform functions of position on the surface.

Next I give expressions for Abelian integrals of the first two kinds in terms of series of the type used by Poincaré. Series of the same type are also used to form factorial functions.

It is also shown that a uniform function of the automorphic class exists which will serve as a prime function in the expression of Fuchsian functions as the product of factors. Such have been constructed for automorphic functions existing all over the plane. That which is here given serves for the other class.

The Existence Theorem of Riemann.

1. The existence of a second uniform function of position on a Riemann surface of the ordinary kind, consisting of n spherical sheets connected together, may be proved on the lines of Poincaré's work* as follows.

Let x be the original variable, so that each of the sheets is the inverse of the x-plane, and let $a_1, a_2, a_3, \dots$ be the values of x at the branch-points. Then it is proved by Poincaré (*Acta Mathematica*, Vol. IV., pp. 242–250) that x may be made a Fuchsian function of class (*genre*) 0 of a new variable z, in such a way that the vertices of the fundamental polygon all rest on the circular boundary of the function, the angles at those vertices being accordingly zero, and that only one point in the polygon corresponds to each point on the x-plane or sphere, while the points corresponding to $a_1, a_2, a_3, \dots$ are all vertices of the polygon. The present argument is not affected by the symmetry of the polygon, or the fact that it generally has other vertices as well as those corresponding to the points $a_1, a_2, a_3, \dots$.

This function gives a conformal representation on the Fuchsian polygon of the x-sphere, that is to say, of a *single sheet* of the Riemann surface.

The sides of the polygon, in pairs, represent the parts of a cut in the x-sphere, still a single sheet, the cut passing through, or at least reaching to, the points $a_1, a_2, a_3, \dots$, but not dividing the spherical surface into parts. Let this cut be made right through all the sheets of the Riemann surface. This surface will then be divided into n separate sheets, each consisting of a spherical surface with the one cut, and any one of these sheets is conformally represented on any one of the polygons into which the area enclosed by the circular boundary of the function is divided. Distinguish these polygons as I., II., III., ...

Take the polygon I. to represent the first of these separate sheets, and remove a piece of one of the cuts in such a way as to connect this with another sheet, say the second. As x travels into the second sheet across this cut, z will travel into a new polygon II., adjacent to I. Suppose II. added to I. Now take out another piece of a cut so as to connect the first or second sheet with a third, and suppose III. to be the polygon into which z passes from I. or II. when x crosses this cut into the third sheet. Add III. to I. and II. and

* See especially *Acta Mathematica*, Vol. IV., pp. 301, 302.

carry on this process until there are n polygons joined together, one representing each sheet. Let P_n be the polygon which is made up of these n. Then the whole Riemann surface as bounded by those cuts which are left is conformally represented on P_n by means of the functional relation between x and z. But P_n is a polygon bounded by circular arcs orthogonal to the bounding circle, and all its vertices lie on this bounding circle. Also the correspondence of its sides in pairs is quite definite, being ascertained by noticing the way in which the n sheets must be united again to form the original Riemann surface. Hence P_n is the generating polygon of a Fuchsian group, which is a sub-group of the original one.

Form by Poincaré's method a Fuchsian function of z having this group. Take, for instance, as generating function $\dfrac{1}{z-a}$, where a is a point within the bounding circle. The theta-Fuchsian function Θ thus formed will not vanish identically, nor will it have the pseudo-automorphic property for any greater group than that of P_n. But x is included among the Fuchsian functions belonging to P_n, and therefore Θ divided by the proper power of $\dfrac{dx}{dz}$ will be another of them, necessarily distinct from x because its group is not the same as that of x. Thus we have another Fuchsian function of z, say y, which is a uniform function of position on the Riemann surface, and is therefore connected with x by an algebraic equation to which the given Riemann surface will be appropriate; from the way in which y was formed it follows that every uniform function of position can be expressed rationally in terms of x and y, since y becomes infinite at a point in one sheet and not at the corresponding points in the other sheets.

A simple example of the construction may make the argument clearer. Take a surface of three sheets having one branch line AB (Fig. 1), and no branch-points other than A, B. The closed circuit drawn round B shows the order in which the sheets are connected,

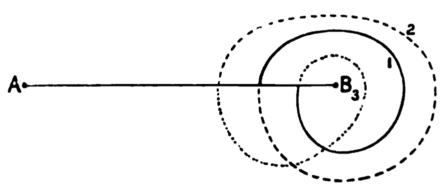


FIG. 1.

the unbroken part being in the first sheet, the broken part in the second, the dotted part in the third.

In Fig. 2, let $acec'bdb'c''a$ be the original Fuchsian polygon, the

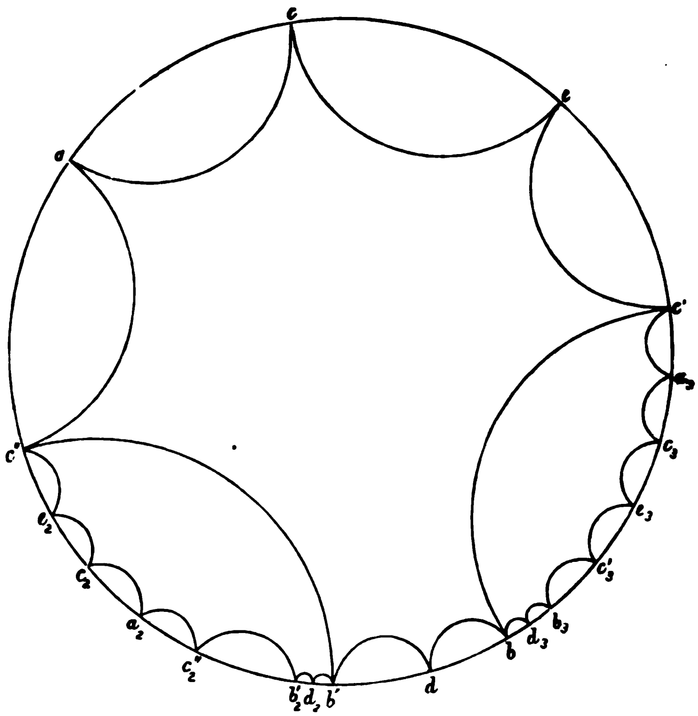


Fig. 2.

pairs of corresponding sides being ac and ac'', ce and $c'e$, $c'b$ and $c''b'$, db and db'; the cycles are then a, bb', $cc'c''$, d, e. Let A, B, C, D, E be the points of the x-plane corresponding to these cycles respectively (Fig. 3); then what is conformally represented on the Fuchsian

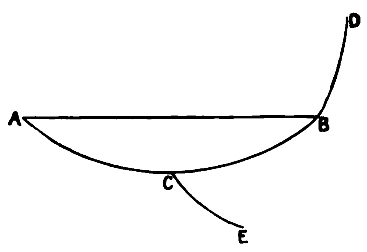


Fig. 3.

polygon is the x-plane, a single sheet only, bounded by a cut consisting of three lines CA, CBD, CE; these are supposed drawn in Fig. 3.

Now distinguish points in the different sheets by suffixes, and suppose the whole surface cut through along the lines CA, CBD, CE; it is thus divided into three sheets whose boundaries are

$$AC_1E_1C_1BD_1BC_2A,$$
$$AC_2E_2C_2BD_2BC_3A,$$
$$AC_3E_3C_3BD_3BC_1A;$$

in fact, the area AC_2B is transferred from the 2nd sheet to the 1st,

,,	AC_3B	,,	,,	3rd	,,	,,	2nd,
,,	AC_1B	,,	,,	1st	,,	,,	3rd.

Take the representation of the first sheet $AC_1E_1C_1BD_1BC_2A$ upon the polygon $acec'bdb'c''a$. Since the passage across BD_1 or C_1E_1 does not lead into another sheet, the polygons adjoining $ace \dots c''a$ along bd, $b'd$, ce, $c'e$ will also represent the first sheet. On the other hand, the passage across AC_2 or C_2B leads into the second sheet, which is therefore represented by either of the polygons adjoining $ace \dots c''a$ along ac'' and $c''b'$, and the passage across AC_1 or C_1B leads into the third sheet, which is therefore represented by either of the polygons adjoining $ace \dots c''a$ along ac and bc'.

Remove the cuts BC_1, BC_2; the three sheets are thus joined together into a simply connected whole whose boundary is

$$AC_1E_1C_1AC_3E_3C_3BD_3BD_1BD_2BC_3AC_2E_2C_2A.$$

Let the polygon adjoining $acec'bdb'c''a$ along $c''b'$ be $a_2c_2e_2c''b'd_2b'_2c''_2a_2$, and that adjoining along $c'b$ be $a_3c_3e_3c'_3b_3d_3bc'a_3$. Then the Riemann surface with the boundary

$$AC_1E_1C_1AC_3E_3C_3BD_3BD_1BD_2BC_3AC_2E_2C_2A$$

is conformally represented on the polygon

$$acec'a_3c_3e_3c'_3b_3d_3bdb'd_2b'_2c''_2a_2c_2e_2c''a.$$

The pairs of corresponding sides are ac and a_3c', ce and $c'e$, a_3c_3 and $a_2c''_2$, c_3e_3 and c'_3e_3, c'_3b_3 and $c''_2b'_2$, b_3d_3 and bd_3, bd and $b'd$, $b'd_2$ and b'_2d_2, a_2c_2 and ac'', c_2e_2 and $c''e_2$; the cycles are aa_3a_2, cc', e, $c_3c''_2c'_3$, e_3, $bb'b'_2b_3$, d_3, d, d_2, c_2c'', e_2.

2. The method here used is the general one (Poincaré, *Acta Mathematica*, Vol. IV., p. 286) for forming sub-groups of a given automorphic group whose generating polygon is, say, R_1. Join together R_1 and any number of the polygons into which it is transformed, say R_2, R_3, ..., R_n, in such a way as to form a new polygon. Let a_1, b_1 be a pair of conjugate sides* of R_1 and a_2, b_2, a_3, b_3, ... the corresponding pairs in R_2, R_3, ..., any that do not form part of the boundary of the whole being left out. In the new polygon make a_1, a_2, a_3, ... conjugate with b_1, b_2, b_3, ... in any order; this will be possible since all have the same non-Euclidian length. Then, subject to the condition that the sum of the angles of any cycle in the new polygon shall be a sub-multiple of 2π, the new polygon will generate a new discontinuous group, included in the original one. If all the vertices of R_1 are on the bounding circle in the Fuchsian case, or if R_1 has no vertices, the condition as to the angles disappears; if, on the other hand, the original group includes elliptic substitutions, or, in fact, if R_1 has finite angles, the condition will generally restrict the order in which b_1, b_2, ... may be taken as conjugate with a_1, a_2, ... For instance, if R_1 has only one cycle, the sum of whose angles is 2π, the new polygon must have at least n cycles.

The consideration of sub-groups of an automorphic group and the associated theory of transformation of automorphic functions is thus closely connected with that of the functions that exist on the surface formed by joining together a number of sheets in Riemann's manner, when each of these sheets is a multiply connected surface, and is, in fact, the original Riemann surface deformed; the new surface is, of course, only equivalent to a spherical Riemann surface of a special class. The order of connexion of the new surface may be the same as that of the old; the theory for such a case will be in close connexion with the transformation theory for Abelian functions, since the Abelian integrals of the first kind on such a multiple surface will be the same as for one of its sheets, but their moduli of periodicity will be multiplied.

As an example, suppose R_1 to have four sides and opposite sides to correspond. Call the vertices a, b, c, d, and let R_2 $(bcef)$ be the region adjoining R_1 along bc, R_3 $(cdgh)$ along cd (Fig. 4). Then R_1, R_2, R_3 may be supposed to form the new polygon; we are to take

* The argument as stated applies to a Fuchsian group; for the modification in the case of a Kleinian group compare Poincaré, *Acta Mathematica*, Vol. III., p. 72, § 5.

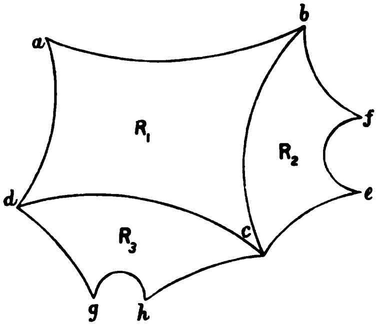


FIG. 4.

as conjugate to *ad*, *dg* either *fe*, *ch* or *ch*, *fe*, and as conjugate to *ab*, *bf* either *gh*, *ce* or *ce*, *gh*. Thus there are four possible arrangements—

I.	*ad*	*dg*	*ab*	*bf*	giving one cycle *afedcbhg*,
	fe	*ch*	*gh*	*ce*	
II.	*ad*	*dg*	*ab*	*bf*	giving one cycle *acbhdfeg*,
	ch	*fe*	*gh*	*ce*	
III.	*ad*	*dg*	*ab*	*bf*	giving one cycle *afhgbedc*,
	fe	*ch*	*ce*	*gh*	
IV.	*ad*	*dg*	*ab*	*bf*	giving three cycles *ac*, *dhf*, *beg*.
	ch	*fe*	*ce*	*gh*	

Now in the original region R_1 there is only one cycle, and the sum of the angles is therefore a sub-multiple of 2π, say $2\pi/q$. Hence, with the arrangements I., II., III., we have the sum of the angles of a cycle equal to $6\pi/q$. These must therefore be rejected unless q is a multiple of 3, but will be admissible if q is a multiple of 3. The arrangement IV. is always admissible, since the sum of the angles in each cycle is $2\pi/q$, as may be readily seen from the figure.

Let S, T be the generating operations of the original group, and suppose, in fact, that R_2 is SR_1, R_3 is TR_1. The relation satisfied by S, T is

$$(STS^{-1}T^{-1})^q = 1.$$

The generating operations for the sub-groups will be*

$$\text{I.}\quad S^3,\ TST^{-1},\ T^3,\ STS^{-1};$$

$$\text{II.}\quad TS,\ S^2T^{-1},\ T^3,\ STS^{-1};$$

$$\text{III.}\quad S^3,\ TST^{-1},\ ST,\ T^2S^{-1}$$

(these three sub-groups will therefore exist if q is divisible by 3);

$$\text{IV.}\quad TS,\ S^2T^{-1},\ ST,\ T^2S^{-1}$$

(this sub-group exists for all values of q).

In this example there were four possible arrangements, and only one of them satisfied the condition as to the angles in general. There is at present nothing to show that in general even one of the possible arrangements will satisfy the condition.

Poincaré's generating polygons do not always absolutely fix the corresponding groups; take, for instance, the case of a Fuchsian group generated by a polygon with no vertices except on the fundamental circle. When this is so, the operations of the new group which is proposed as a sub-group must be chosen from among the operations of the original group; this can always be done.

The Abelian Integrals.

3. I have not seen it observed that some Abelian integrals on the Riemann surface are zeta-Fuchsian or zeta-Kleinian functions of z, the automorphic argument. If u, for instance, is an Abelian integral and is considered as a function of z, say $u(z)$, we have for any generating substitution of the group, say S_1,

$$u(S_1 z) = u(z) + \mu_1,$$

μ_1 being the modulus of periodicity for the cut on the Riemann surface which corresponds to the boundary between R_1, the original region, and $S_1 R_1$. Thus $u(z)$ and 1 are a pair of functions which undergo a homogeneous linear transformation for any substitution of the group, the determinant being, for S_1,

$$\begin{vmatrix} 1 & \mu_1 \\ 0 & 1 \end{vmatrix}.†$$

* Here ST, for instance, means T followed by S.

† Prof. Burnside remarks that we are thus led to a class of self-conjugate sub-groups of the automorphic group. Such a sub-group will, in fact, be made up of all substitutions which leave the function $u(z)$, or a set of such functions, unaltered. Since Abelian integrals of the second kind (not canonical) may be so chosen as to

They have therefore the characteristic property of zeta-Fuchsian functions, as given by Poincaré (*Acta Mathematica*, Vol. v., pp. 227–8). The analytical expression given by him for these functions in the case of groups of the first two and sixth families* (pp. 232–5, 257–264) can be adapted to the present case as follows:—take $p = 2$, $H_1 = 0$. Let the generating substitutions be $S_1, S_2, S_3, \dots$, and suppose any substitution S of the group to be $S_a^{a'} S_b^{b'} S_c^{c'}, \dots$. Take a series of quantities $\mu_1, \mu_2, \mu_3, \dots$, and suppose each to correspond to the substitution with the same suffix; corresponding to S take the quantity

$$\mu = a'\mu_a + b'\mu_b + c'\mu_c + \dots .$$

Suppose also $\mu_1, \mu_2, \dots$ to be such that when S is an identical or elliptic substitution μ is 0; this ensures that, if the same substitution S can be expressed in terms of $S_1, S_2, S_3, \dots$ in two or more ways, there shall be no ambiguity in the value of μ. The group of linear substitutions, such as

$$\begin{matrix} 1 & \mu \\ 0 & 1 \end{matrix}$$

is then isomorphic to the Fuchsian group.†

---

have any set of their moduli of periodicity zero, we are thus led to different self-conjugate sub-groups according to the particular set of moduli that are made to vanish. Any such sub-group will include all the elliptic substitutions; the number of its generating substitutions will usually be infinite.

* It is pointed out below (§ 9) that this restriction is unnecessary.

† The fundamental relations among the substitutions of a Fuchsian group are discussed by Poincaré (*Acta Mathematica*, Vol. i., pp. 45–7). He shows that one such relation arises from each cycle of the first category, and that there are no others. The number of relations that must be satisfied by the moduli $\mu_1, \mu_2, \dots$ to secure the isomorphism is therefore the same as the number of cycles of the first category. With his notation we must, in fact, have

$$\lambda(\mu_{a_1} + \mu_{a_2} + \mu_{a_3} + \dots) = 0,$$

that is to say, the modulus of periodicity for the elliptic substitution $\{z, F(z)\}$ must vanish. There will be one such relation among the moduli $\mu_1, \mu_2, \dots$ for each cycle, but among these relations one will sometimes be a consequence of the rest, as pointed out later in the text.

Or thus: let S, T be any two substitutions of the group, μ, ν the corresponding moduli; then the modulus for the substitution TST^{-1} will be $\nu + \mu - \nu$, or the same as for S. Now, all the elliptic substitutions of the group can be expressed in the form TST^{-1}, where S, T belong to the group, and S is one of the finite number of elliptic substitutions that arise from the respective cycles of vertices of the original polygon. The same holds for parabolic substitutions.

In like manner all the identical relations among the operations of the group are combinations of transformations of a certain finite number.

Form the two series

$$-\Sigma\mu H(Sz\ \left(\frac{dSz}{dz}\right)^m = Z(z),$$

$$\Sigma H(Sz)\left(\frac{dSz}{dz}\right)^m = \Theta(z),$$

the summations being taken over all the substitutions of the group, and H denoting such a rational function that $\Theta(z)$ does not vanish. The absolute convergency of zeta-Fuchsian series for the first two families is established by Poincaré in the memoir referred to, on the supposition that m is great enough (p. 235), and that the zeta-Fuchsian functions whose expression is sought are of the first species (p. 258). Here the last is evidently true, since all the multipliers are unity, whether for critical, or other, substitutions.

Then, taking any substitution S_1, belonging to the group, we have, still following Poincaré (p. 232),

$$Z(S_1z) = -\Sigma\mu H(SS_1z)\left(\frac{dSS_1z}{dS_1z}\right)^m,$$

$$Z(z) = -\Sigma(\mu+\mu_1)H(SS_1z)\left(\frac{dSS_1z}{dz}\right)^m,$$

the last expression containing the same terms as the former series for $Z(z)$, but differently arranged. Hence

$$Z(z)-Z(S_1z)\left(\frac{dS_1z}{dz}\right)^m = -\mu_1\Theta(z),\text{*}$$

$$\frac{Z(z)}{\Theta(z)}-\frac{Z(S_1z)}{\Theta(S_1z)} = -\mu_1.$$

A like result holds for each of the other substitutions of the group. Hence $Z(z)/\Theta(z)$ is a uniform function of z, and its values at corresponding points of different polygons differ by multiples of the moduli $\mu_1, \mu_2, \ldots$, these multiples depending on the particular polygons in question only. Thus $Z(z)/\Theta(z)$ is an Abelian integral belonging to the Riemann surface. This investigation applies to the first, second, and sixth families.

The modulus for a parabolic substitution need not vanish, although

* Thus, if $\Theta(z)$ vanishes identically, $Z(z)$ is a theta-Fuchsian function.

the principal point of such a substitution forms a cycle, or at least may be made to; suppose this to be done. Then the substitution changes one of the sides drawn from this vertex of the polygon into the other. A line drawn between these sides corresponds to a circuit round the corresponding point on the Riemann surface; hence, if the modulus for this substitution does not vanish, the Abelian integral has a logarithmic discontinuity at the corresponding point on the Riemann surface.

The Abelian integrals of the first two kinds will therefore have the moduli corresponding to the parabolic substitutions zero.

If the fundamental polygon is not simply connected, some irreducible circuits on the Riemann surface will correspond to contours round the holes, and an Abelian integral will therefore only be a uniform function of z when its moduli for all such circuits vanish.

4. Take a Fuchsian group of the first, second, or sixth family, giving functions with a natural boundary. The fundamental polygon is here simply connected; the number of moduli is the number of pairs of corresponding sides; the number of relations connecting them is the number of cycles diminished by one, since the aggregate of all the cycles contains each substitution once, and also its inverse once. This is on the supposition that the modulus for each parabolic substitution vanishes. Hence the number of arbitrary moduli is the same as that of irreducible circuits on the Riemann surface,* and the modulus for each irreducible circuit is, in fact, arbitrary.

The difference of two functions whose moduli are the same will clearly be a Fuchsian function, that is, a uniform function of position on the Riemann surface.

Let c be the number of irreducible circuits. Then c linearly independent functions can be formed having certain assigned poles, amongst others, and having different sets of moduli. The poles can be assigned by making them zeroes of $\Theta(z)$ which need not be finite everywhere, and can therefore be made to have any assigned zeroes. Let $p+1$ be the least number of arbitrary poles that can be assigned for a Fuchsian function which is to have no others. Then by subtraction of Fuchsian functions the above set of c functions can be made to have p arbitrary poles and no others. By combining them linearly we can form $c-p$ functions whose residues at these poles

* Compare *Acta Mathematica*, Vol. I., p. 229, lines 1–6.

are zero, and which are therefore everywhere finite. But now consider the integral $\int x\,du$, where x is a Fuchsian function, and u one of these functions that are everywhere finite. The value of this integral round the contour of the polygon is zero, and thus we have a linear relation connecting the residues of x at its different poles. There will be one such relation for each of the $c-p$ functions that may be taken in the place of u, so that, if the residues are not all to vanish, there must be at least $c-p-1$ of them, and we conclude that $p = c-p$ or $c = 2p$. This argument is, of course, loose: but, as the result is so well known, there is no object in labouring the point. The c (or $2p$) independent functions with p arbitrary poles and with moduli as above can thus be combined so as to give p functions without poles and p others having one pole each; these are, of course, the usual Abelian integrals of the first two kinds.

It is clear that in general an Abelian integral of the third kind is not a uniform function of the automorphic argument. If, however, the points of logarithmic discontinuity fall at vertices of the polygon belonging to parabolic cycles, this statement ceases to be true. In fact it was seen above that the function $Z(z)/\Theta(z)$ was an Abelian integral of the third kind if its modulus for any parabolic substitution was other than zero. Now, in the construction of § 1, we may add any points we please to the list of branch-points $a_1, a_2, \ldots$, without affecting the argument, and thus ensure that the logarithmic discontinuities of any finite number of Abelian integrals of the third kind shall fall on the bounding circle: hence we may, if it is desired, suppose the automorphic representation so chosen that any finite number of specified integrals of the third kind are uniform functions of the automorphic argument.

5. Now take an integral of the first kind u, and another of the second kind v_c, having in the polygon a single pole for the value $z = c$, and such that

$$\operatorname*{Lim}_{z=c}\,(z-c)\,v_c\,\frac{du}{dz} = 1.$$

Then $\exp \int_{z_0}^{z} v_c \frac{du}{dz}\,dz$ is a uniform function of z, having simple zeroes at c and the corresponding points in other polygons only, and having no infinity except at the essential singularities. Let us consider the effect on it of any substitution S of the group. Suppose μ_c to be the

corresponding modulus for v_c; then

$$\begin{aligned}\int_{z_0}^{Sz} v_c \frac{du}{dz} dz &= \int_{z_0}^{Sz_0} v_c \frac{du}{dz} dz + \int_{Sz_0}^{Sz} v_c \frac{du}{dz} dz \\ &= \int_{z_0}^{Sz_0} v_c \frac{du}{dz} dz + \int_{z_0}^{z} (v_c+\mu_c) \frac{du}{dz} dz \\ &= \int_{z_0}^{z} v_c \frac{du}{dz} dz + \mu_c (u-u_0) + \int_{z_0}^{Sz_0} v_c \frac{du}{dz} dz.\end{aligned}$$

If, then, we write $\dfrac{\vartheta_c(z)}{\vartheta_c(z_0)}$ for $\exp \displaystyle\int_{z}^{z} v \frac{du}{dz} dz$, we have

$$\vartheta_c(Sz) = \vartheta_c(z) \frac{\vartheta_c(Sz_0)}{\vartheta_c(z_0)} \exp \mu_c (u-u_0).$$

Suppose now that v_c is a multiple of the normal integral of the second kind, and take $2(p+1)$ distinct points $b_1, b_2, \ldots, b_{p+1}$, $c_1, c_2, \ldots, c_{p+1}$. The function

$$\Pi\vartheta_b(z) \div \Pi\vartheta_c(z)$$

is a uniform function of z, and by the change of z into Sz it is multiplied by

$$\exp(\Sigma\mu_b - \Sigma\mu_c)(u-u_0).$$

If, then,
$$\Sigma\mu_b = \Sigma\mu_c$$

for each substitution, that is to say, if p conditions are satisfied,

$$\Pi\vartheta_b(z) \div \Pi\vartheta_c(z)$$

is a uniform function of position on the Riemann surface, having, say, $p+1$ arbitrary poles and one arbitrary zero.

The function ϑ_c is not unique, since u may be any integral of the first kind. The quotient of two such functions, having a common vanishing point, will therefore be a uniform function of z, neither vanishing nor becoming infinite except at an essential singularity; also, if v is an integral of the first kind, $\exp \int v\,du$ will have the same properties.

The function $\vartheta_c(z)$ is clearly very like the prime function of Schottky and Klein. (See, for instance, Prof. Burnside's paper, *Proc. Lond. Math. Soc.*, Vol. XXIII., pp. 289–293.) It differs, however, in not having a pole at infinity, as is to be expected, since infinity is outside the region in which it exists.

[redacted]

Factorial Functions.

6. Pseudo-automorphic functions of the factorial class can also be considered as zeta-Fuchsian or zeta-Kleinian functions. Let the multiplier corresponding to the substitution

$$S \equiv S_a^{a'} S_b^{b'} S_c^{c'} \ldots$$

be
$$M = M_a^{a'} M_b^{b'} M_c^{c'} \ldots .$$

Suppose also that, when S is an elliptic substitution of period k, M is a k^{th} root of unity; that, when S is a parabolic substitution, the modulus $|M|$ is unity (see *Acta Mathematica*, Vol. v., pp. 258, 269); and that, if $S \equiv 1$, then $M = 1$. Form the series

$$\Sigma M^{-1} H(Sz) \left(\frac{dSz}{dz}\right)^m = \phi(z),$$

$$\Sigma H(Sz) \left(\frac{dSz}{dz}\right)^m = \Theta(z),$$

the summations being taken over all the substitutions of the group. The series $\phi(z)$ is a zeta-Fuchsian series. We may therefore use Poincaré's results as to the convergency of such series, already quoted. The subject is further discussed below (§ 9). Then, if S_1 is a substitution belonging to the group, we have

$$\phi(S_1 z) = \Sigma M^{-1} H(SS_1 z) \left(\frac{dSS_1 z}{dS_1 z}\right)^m,$$

$$\phi(z) = \Sigma M_1^{-1} M^{-1} H(SS_1 z) \left(\frac{dSS_1 z}{dz}\right)^m,$$

so that $\phi(S_1 z) \div \phi(z) = M_1 \left(\frac{dS_1 z}{dz}\right)^{-m} = M_1 \Theta(S_1 z) \div \Theta(z).$

Thus $\phi(z) \div \Theta(z)$ is a factorial pseudo-automorphic function. The restrictions on the multipliers are very much like those on the moduli in the former case; if, however, the multipliers for parabolic substitutions are unity, those for irreducible circuits will be arbitrary, and the roots of unity chosen for the elliptic substitutions must satisfy a relation which may compel them all to be unity itself.

If we further choose roots of unity for the arbitrary multipliers, we have a class of factorial functions that will be of great importance in the transformation theory of automorphic functions. (Compare Ritter, *Math. Annalen*, Vol. XLI., pp. 31, 58.)

7. Other expressions for the Abelian integrals have been given, as, for instance, by Prof. Burnside (*Proc. Lond. Math. Soc.*, Vol. XXIII., pp. 66–9), and it is a point worthy of notice that his function

$$\theta(z, a) = \Sigma \frac{1}{Sz-a} \frac{dSz}{dz},$$

if considered as a function of a, z being constant, is an Abelian integral, and, in fact, is practically the normal function of the second kind.* Prof. Burnside's paper treats mainly of functions of the first class; whereas the method used in the present paper is only completely applicable to those of the second class.

8. The following investigations relate to the convergency of some of the series that are used in the paper.

To prove the convergency of the series

$$\Sigma \mu H(Sz) \left(\frac{dSz}{dz}\right)^m,$$

and to show that this convergency is uniform when z varies within proper limits, it is enough to prove the convergency of the series

$$\Sigma \left| \frac{\mu}{\gamma^{2m}} \right|,$$

where
$$Sz = \frac{\alpha z+\beta}{\gamma z+\delta}, \quad \alpha\delta-\beta\gamma = 1.$$

The function H is supposed to have no pole at any essential singularity of the group, and z is supposed not to fall at any point that is transformed by a substitution of the group into ∞ or one of the poles of $H(z)$. Thus a superior limit can be fixed for $|H(Sz)|$ and superior and inferior limits to $\left| z+\frac{\delta}{\gamma} \right|$, if we suppose, as we may, that one of the polygons outside the fundamental circle contains the point ∞, so that γ does not vanish for any of the substitutions except the identical one. Such an exception affecting only an isolated term may be ignored.

Let κ be the greatest among the absolute values of the moduli of periodicity μ for the generating substitutions, and let n be the exponent of the substitution S, that is the least value of $a'+b'+c'+\dots$,

---

* See H. F. Baker, *Abelian Functions*, p. 357, Ex. ii.

when S is expressed in the form $S_a^{a'} S_b^{b'} S_c^{c'} \dots$ in terms of the generating substitutions and their inverses in such a way that $a', b', c' \dots$ are all positive whole numbers. Then

$$|\mu| \not> n\kappa.$$

Thus we have to consider the series $\Sigma n\,|\gamma|^{-2m}$. Now, since $\left|z+\dfrac{\delta}{\gamma}\right|$ has both a superior and an inferior limit, the series

$$\Sigma\,|\gamma|^{-2m}, \quad \Sigma\,|\gamma z+\delta|^{-2m}$$

converge or diverge together. (The identical substitution for which $\gamma = 0$ is, of course, left out here.) All the terms are positive; so, if V_n is the sum of all the terms in $\Sigma\,|\gamma|^{-2m}$ corresponding to substitutions of exponent n, the series

$$V_1 + V_2 + \dots + V_n + \dots$$

is convergent, when $m > 1$. Thus, when $m > 1$,

$$V_n < \frac{1}{n \log n}$$

from and after some term in the series.

Hence $$nV_n^2 < \frac{1}{n\,(\log n)^2}$$

after this term, and therefore

$$V_1^2 + 2V_2^2 + \dots + nV_n^2 + \dots$$

is convergent. Now V_n^2 is greater than the sum of the terms in $\Sigma\,|\gamma|^{-4m}$ which correspond to substitutions of exponent n, and therefore $\Sigma n\,|\gamma|^{-4m}$ is convergent if $m > 1$, or

$$\Sigma \mu H(Sz) \left(\frac{dSz}{dz}\right)^m$$

is absolutely and uniformly convergent if $m > 2$.

9. M. Poincaré's discussions of the convergency of theta-Fuchsian and zeta-Fuchsian series may be adapted to the case of Kleinian groups as follows.

Suppose the plane of the variable z which is unchanged by the operations of the generalized Kleinian group (*Acta Mathematica*,

Vol. III., pp. 52–6) to be turned into a sphere by stereographic projection. Take this to be the sphere

$$x^2+y^2+z^2 = 1.$$

Then any operation of the group turns the point P (x, y, z) into the point P' (x', y', z'), where

$$\frac{x'^2+y'^2+z'^2-1}{x^2+y^2+z^2-1} = \frac{x'}{A} = \frac{y'}{B} = \frac{z'}{C} = \frac{1}{E},$$

A, B, C, E being linear in $x^2+y^2+z^2$, x, y, z, 1. Also, if ds is any infinitesimal arc at the point P, $\dfrac{ds}{x^2+y^2+z^2-1}$ is unchanged by any inversion which leaves the sphere $x^2+y^2+z^2=1$ unchanged, and therefore by every operation of the group. Hence the linear magnification $\dfrac{ds'}{ds}$ is equal to $\dfrac{x'^2+y'^2+z'^2-1}{x^2+y^2+z^2-1}$ and to $\dfrac{1}{E}$.

Now, if P' goes to infinity, P approaches a definite limiting position K, and therefore E can only be a constant multiple of PK^2, which will be positive here, since the inside and outside of the sphere $x^2+y^2+z^2 = 1$ are not interchanged by any of the operations. We may then write

$$E = \eta^2 . PK^2.$$

Take, then, the series $\Sigma \dfrac{1}{E^m}$ and $\Sigma \dfrac{1}{\eta^{2m}}$, the summation being over all the operations of the generalized Kleinian group, and the x, y, z which occur in E being the coordinates of a point P which is inside one of the polyhedra, and does not coincide with the point in that polyhedron which corresponds to ∞. We shall suppose that ∞, and therefore also the origin, is within one of the polyhedra; this can be secured by proper choice of the vertex of the stereographic projection. Thus $E \div \eta^2$ has a superior and an inferior limit, for every position of the point P, including the origin, except those which correspond to ∞ in the different polyhedra. The series $\Sigma \dfrac{1}{E^m}$ and $\Sigma \dfrac{1}{\eta^{2m}}$ will therefore converge or diverge together. Now, if P lies on the sphere $x^2+y^2+z^2 = 1$, and within one of the polyhedra, the series $\Sigma \dfrac{1}{E^2}$ converges, as in M. Poincaré's first proof for the theta-Fuchsian series, since the whole surface of the sphere is finite. Therefore the series $\Sigma \dfrac{1}{\eta^4}$ converges, and so does $\Sigma \dfrac{1}{E^2}$ for all posi-

tions of P, including the origin. Let E_0 be the value of E when P is at the origin; then, since

$$\frac{1}{E} = \frac{x'^2+y'^2+z'^2-1}{x^2+y^2+z^2-1},$$

we have
$$\frac{1}{E_0} = 1-r^2,$$

r being the distance from the origin of the point into which it is transformed by the substitution in question. That is,

$$E_0 = \cosh^2 R,$$

R being the non-Euclidian distance of this point from the origin, so that

$$R = \int_0^r \frac{d\rho}{1-\rho^2} = \tfrac{1}{2}\log\frac{1+r}{1-r}.$$

Now the arguments used by M. Poincaré (*Acta Mathematica*, Vol. v., pp. 233–235, 259–264) apply here, with the substitution of polyhedra and spheres for polygons and circles, and show that, if there is a group of homogeneous linear substitutions isomorphic to the Kleinian group, the coefficients A in any substitution of the new group are all less in absolute value than

$$e^{aR},$$

where R is the quantity just now indicated, and a is a constant. It is, of course, still necessary to suppose that all the multipliers in a substitution of the new group which corresponds to a parabolic substitution in the Kleinian group have modulus unity.

Hence the series $\Sigma \frac{A}{E_0^m}$, and therefore $\Sigma \frac{A}{\eta^{2m}}$, $\Sigma \frac{A}{E^m}$ will be convergent if

$$2m > 4+a,$$

since $\Sigma \frac{1}{E_0^2}$ is known to be convergent.

The ratio of the magnification on the original plane to that on the sphere has finite limits, superior and inferior, if we suppose ∞ on the plane to be contained within one of the polygons. Hence the convergency of the zeta-Kleinian series is assured for values of m exceeding $2+\frac{1}{2}a$.

M. Poincaré's arguments here referred to apply to all families; the absence of closed cycles would, in fact, simplify the discussion. Closed hyperbolic cycles must be removed, as at *Acta Mathematica*, Vol. III., pp. 71, 72.

Note on Clebsch's Second Method for the Integration of a Pfaffian Equation. *By* J. Brill, M.A. Received November 6th, 1899. Communicated November 9th, 1899.

1. In a paper recently published by the Society,* I have shown that there exists a complete set of multilinear differential covariants of a Pfaffian expression, each derivable from the preceding, and not simply a bilinear one, as was formerly supposed. In that paper I gave the form of these covariants in terms of the quantities that occur in the Pfaffian expression and their derivatives. I propose now to discuss the form which they take when expressed in terms of the quantities included in a normal form of the given expression, and to apply my results to complete what is known as "Clebsch's Second Method for the Integration of a Pfaffian Equation."† I shall, of course, assume the results of my former paper, and shall not think it necessary to give any new proofs of known facts in the theory of Pfaffian expressions.

2. In my former paper I indicated a new method for deriving each of the various sets of derived functions, connected with a Pfaffian expression, from the preceding; and stated that the generality of the method could be established by mathematical induction. Since the paper was printed I have hit upon a convenient method for the completion of the induction, and I propose to give a sketch of it here, as the establishment of the identity of the coefficients of the various covariants with the derived functions of the various orders depends upon the generality of the method.

It will only be necessary to deal with the case of the derivation of the derived functions of the odd orders by means of differential operations, as the connexion of those of any even order with those of the preceding odd order is evidently identically similar to the connexion of the allied functions of any order with the preceding set of Pfaffians.

We will thus suppose that the derived functions of order $2r-1$, derived by my method, have been identified with the r^{th} set of

---

* Vol. xxx., pp. 263-271.

† See Forsyth, *Theory of Differential Equations*, Pt. 1, pp. 85, 86, and 209-218.

Pfaffians. From these we form the set of allied functions next in order, which will obviously be identical with my derived functions of order $2r$. We have a set of formulæ of the type

$$[012 \ldots (2r+1)] = X_1 [23 \ldots (2r+1)] - X_2 [134 \ldots (2r+1)] + \ldots$$
$$\ldots + (-1)^{2r} X_{2r+1} [123 \ldots (2r)]. \quad (1)$$

The formulæ for the derivation of our next set of derived functions are of the type

$$\frac{\partial}{\partial x_1}[023 \ldots (2r+2)] - \frac{\partial}{\partial x_2}[0134 \ldots (2r+2)] + \ldots$$
$$\ldots + (-1)^{2r+1} \frac{\partial}{\partial x_{2r+2}} [0123 \ldots (2r+1)] = (r+1) K, \quad (2)$$

in which we have to identify K with

$$[123 \ldots (2r+2)].$$

Now, substituting in equation (2) the values of $[023 \ldots (2r+2)]$, &c., as given by the equations of the type (1), we obtain

$$\begin{aligned}
(r+1)K = \;& [12][345 \ldots (2r+2)] - [13][245 \ldots (2r+2)] + \ldots \\
& \qquad \ldots + (-1)^{2r} [1\,(2r+2)][234 \ldots (2r+1)] \\
& + [23][145 \ldots (2r+2)] - [24][135 \ldots (2r+2)] + \ldots \\
& \qquad \ldots + (-1)^{2r-1} [2\,(2r+2)][134 \ldots (2r+1)] \\
& + [34]\; 125 \ldots (2r+2)] - [35][1246 \ldots (2r+2)] + \ldots \\
& \qquad \ldots + (-1)^{2r-2} [3\,(2r+2)][1245 \ldots (2r+1)] \\
& \qquad + \&c. \qquad (3)
\end{aligned}$$

By means of a well known method of expanding Pfaffians,* we have

$$\begin{aligned}
[123 \ldots (2r+2)] = \;& [12][345 \ldots (2r+2)] - [13][245 \ldots (2r+2)] + \ldots \\
& \qquad \ldots + (-1)^{2r} [1\,(2r+2)][234 \ldots (2r+1)], \\
[123 \ldots (2r+2)] = \;& [12][345 \ldots (2r+2)] + [23][145 \ldots (2r+2)] \\
& - [24][135 \ldots (2r+2)] + \ldots \\
& \qquad \ldots + (-1)^{2r-1} [2\,(2r+2)][134 \ldots (2r+1)], \\
[123 \ldots (2r+2)] = \;& -[13][245 \ldots (2r+2)] + [23][145 \ldots (2r+2)] \\
& + [34][125 \ldots (2r+2)] - [35][1246 \ldots (2r+2)] + \ldots \\
& \qquad \ldots + (-1)^{2r-2} [3\,(2r+2)][1245 \ldots (2r+1)],
\end{aligned}$$

* See Forsyth, p. 95.

and so on. Adding together the $2r+2$ equations so formed, and comparing the right-hand side of the result with that of equation (3), we obtain

$$(2r+2)\,[123\ldots(2r+2)] = 2\,(r+1)\,K;$$

and therefore $$K = [123\ldots(2r+2)].$$

This will provide us with sufficient material for completing the induction.

There are relations among the derived functions of any order. In the case of any set of Pfaffians these relations are of a differential type. In the case of any set of allied functions they are of an algebraic type.

From the expressions for the Pfaffians $[123\ldots(2r)]$, &c., in terms of the preceding set of allied functions, we readily deduce

$$\frac{\partial}{\partial x_1}[234\ldots(2r+1)] - \frac{\partial}{\partial x_2}[134\ldots(2r+1)] + \ldots$$
$$\ldots + (-1)^{2r}\frac{\partial}{\partial x_{2r+1}}[123\ldots(2r)] = 0.$$

Similarly, from the expressions for the functions $[0123\ldots(2r-1)]$, &c., in terms of the preceding set of Pfaffians, we obtain

$$X_1\,[0234\ldots(2r)] - X_2\,[0134\ldots(2r)] + \ldots$$
$$\ldots + (-1)^{2r-1}X_{2r}\,[0123\ldots(2r-1)] = 0.$$

To avoid unnecessary complications, we lay down the rule that our derived functions shall be so constituted that the numbers within the square brackets shall be always in their natural order. Expressions in which this is not the case would be possible, but they would either be the same as the corresponding expressions formed by restoring the natural order of the numbers, or would merely differ from them in sign.

With this restriction, supposing n to be the number of independent variables, the number of functions in the various sets of derived functions will be respectively

$$\frac{n(n-1)}{1.2},\quad \frac{n(n-1)(n-2)}{1.2.3},\quad \ldots,\quad n,\quad 1.$$

The sets are thus finite in number, and always terminate with one containing a single function. If $n = 2m-1$, the final derived function is

$$[0123\ldots(2m-1)],$$

while, if $n = 2m$, we have

$$[123 \ldots (2m)]$$

for the final derived function.

3. We will suppose our Pfaffian expression to be of the form

$$X_1\,dx_1 + X_2\,dx_2 + \ldots + X_n\,dx_n\,;$$

and will assume, in the first instance, that

$$l_1\,da_1 + l_2\,da_2 + \ldots + l_s\,da_s$$

may be taken for a normal form, where s is not greater than $\frac{1}{2}n$ or $\frac{1}{2}(n+1)$, according as n is even or odd.

We have therefore

$$U_1 = l_1 d_1 a_1 + l_2 d_1 a_2 + \ldots + l_s d_1 a_s,$$

and consequently

$$U_{12} = d_{12}(l_1, a_1) + d_{12}(l_2, a_2) + \ldots + d_{12}(l_s, a_s).$$

Now, forming the next covariant by means of the equation

$$U_{123} = U_1 U_{23} - U_2 U_{13} + U_3 U_{12},$$

we readily obtain

$$U_{123} = \Sigma \left\{ -l_p d_{123}(l_q, a_p, a_q) + l_q d_{123}(l_p, a_p, a_q) \right\},$$

where q is greater than p, and where p and q may have all possible values from 1 to s.

For the next covariant we have

$$2U_{1234} = d_1 U_{234} - d_2 U_{134} + d_3 U_{124} - d_4 U_{123}.$$

Thus, from the above expression for U_{123}, it easily follows that

$$2U_{1234} = -2\Sigma d_{1234}(l_p, l_q, a_p, a_q),$$

it being readily verified that all terms containing differentials of the second order disappear. Therefore

$$U_{1234} = -\Sigma d_{1234}(l_p, l_q, a_p, a_q),$$

p and q being supposed in their natural order.

For the next covariant, we have

$$U_{12345} = \Sigma \left\{ -l_p d_{12345}(l_q, l_r, a_p, a_q, a_r) + l_q d_{12345}(l_p, l_r, a_p, a_q, a_r) \right.$$
$$\left. - l_r d_{12345}(l_p, l_q, a_p, a_q, a_r) \right\},$$

p, q, r being supposed to be in their natural order.

In like manner, for the fifth, sixth, and seventh covariants, we have

$$\begin{aligned}
U_{123456} &= -\Sigma d_{123456}\,(l_p,\, l_q,\, l_r,\, a_p,\, a_q,\, a_r),\\
U_{1234567} &= \Sigma \Big\{\ \ l_p d_{1234567}\,(l_q,\, l_r,\, l_t,\, a_p,\, a_q,\, a_r,\, a_t)\\
&\qquad -l_q d_{1234567}\,(l_p,\, l_r,\, l_t,\, a_p,\, a_q,\, a_r,\, a_t)\\
&\qquad +l_r d_{1234567}\,(l_p,\, l_q,\, l_t,\, a_p,\, a_q,\, a_r,\, a_t)\\
&\qquad -l_t d_{1234567}\,(l_p,\, l_q,\, l_r,\, a_p,\, a_q,\, a_r,\, a_t)\Big\};\\
U_{12345678} &= \Sigma d_{12345678}\,(l_p,\, l_q,\, l_r,\, l_t,\, a_p,\, a_q,\, a_r,\, a_t).
\end{aligned}$$

These examples are sufficient to reveal the general types of the successive covariants, and it is easily seen that the generality of those types may be established by induction. In the form in which we have written them, however, there arises a distinct peculiarity with regard to sign. Take first the case of those covariants which contain Pfaffians when expressed by the method of my former paper. The first of these is affected with a positive sign, the next two with negative signs, the next two with positive signs, and so on. Next consider the case of those covariants which, when expressed as in my former paper, contain allied functions. The first two of these will have their first terms negative; the next two will have those terms positive; the next two will have them negative; and so on. The question of the actual sign, however, will not be of much consequence for the purposes for which the results are required.

Comparing the above expressions for the covariants with those given in my former paper, we may readily deduce forms for the expression of the various orders of derived functions in terms of the quantities contained in a normal form of the given Pfaffian expression. As examples, we have

$$\begin{aligned}
[12] &= \Sigma\frac{\partial\,(l_p,\, a_p)}{\partial\,(x_1,\, x_2)},\\
[0123] &= \Sigma\left\{-l_p\frac{\partial\,(l_q,\, a_p,\, a_q)}{\partial\,(x_1,\, x_2,\, x_3)}+l_q\frac{\partial\,(l_p,\, a_p,\, a_q)}{\partial\,(x_1,\, x_2,\, x_3)}\right\},\\
[1234] &= -\Sigma\frac{\partial\,(l_p,\, l_q,\, a_p,\, a_q)}{\partial\,(x_1,\, x_2,\, x_3,\, x_4)},
\end{aligned}$$

&c.

One important point to be noticed, in connexion with the type of normal form assumed above, is that, if s be less than $\frac{1}{2}n$, n being even, or less than $\frac{1}{2}(n+1)$, n being odd, then after a certain point

the covariants will become evanescent. The first covariant that does so will be the $2s^{\text{th}}$. In this case we see that all the derived functions belonging to the $2s^{\text{th}}$ and all subsequent orders vanish. We know that this is the necessary and sufficient condition that the normal form may be of the assumed type.*

4. We have discussed only one of the two possible types of normal form, but the results that arise in the case of the other type are readily deduced from those we have obtained in the preceding article. Thus suppose that we have a normal form of the type

$$d\beta + l_1 da_1 + l_2 da_2 + \ldots + l_s da_s,$$

where s is less than $\frac{1}{2}n$, n being even, or not greater than $\frac{1}{2}(n-1)$, n being odd.

We have

$$\begin{aligned}
U_{12} &= \Sigma d_{12}(l_p, a_p), \\
U_{123} &= \Sigma\{d_{123}(\beta, l_p, a_p)\} \\
&\quad + \Sigma\{-l_p d_{123}(l_q, a_p, a_q) + l_q d_{123}(l_p, a_p, a_q)\}, \\
U_{1234} &= -\Sigma d_{1234}(l_p, l_q, a_p, a_q), \\
U_{12345} &= \Sigma\{-d_{12345}(\beta, l_p, l_q, a_p, a_q)\} \\
&\quad + \Sigma\{-l_p d_{12345}(l_q, l_r, a_p, a_q, a_r) \\
&\qquad + l_q d_{12345}(l_p, l_r, a_p, a_q, a_r) \\
&\qquad - l_r d_{12345}(l_p, l_q, a_p, a_q, a_r)\},
\end{aligned}$$

and so on.

From these we readily deduce the expressions for the derived functions. We obtain

$$[12] = \Sigma \frac{\partial(l_p, a_p)}{\partial(x_1, x_2)},$$

$$[0123] = \Sigma \frac{\partial(\beta, l_p, a_p)}{\partial(x_1, x_2, x_3)} + \Sigma\left\{-l_p \frac{\partial(l_q, a_p, a_q)}{\partial(x_1, x_2, x_3)} + l_q \frac{\partial(l_p, a_p, a_q)}{\partial(x_1, x_2, x_3)}\right\},$$

$$[1234] = -\Sigma \frac{\partial(l_p, l_q, a_p, a_q)}{\partial(x_1, x_2, x_3, x_4)},$$

and so on. It is to be noted that the symbol β occurs in the allied

* See Forsyth, p. 255.

functions only, being entirely absent from the expressions for the Pfaffians.

Here we have further to remark that, if s be less than $\frac{1}{2}n$, n being even, or less than $\frac{1}{2}(n-1)$, n being odd, then after a certain point the covariants become evanescent. The first covariant that does so will be the $(2s+1)^{\text{th}}$, and all the derived functions belonging to the $(2s+1)^{\text{th}}$ and subsequent orders will vanish. As in the former case, we know this to be the condition requisite for the normal form to be of the type assumed.

5. We will consider first the only case to which Clebsch has applied his second method, viz., the unconditioned case in an even number of variables.

If we write $n = 2m$, we have, for this case, $s = m$. Thus the set of derived functions will be complete, ending up with the Pfaffian

$$[123 \dots (2m)].$$

It is, of course, understood that this Pfaffian does not vanish, as otherwise there would be a modification of our normal form.

The normal form will be

$$l_1 da_1 + l_2 da_2 + \dots + l_m da_m.$$

Expressing the previous set of allied functions in terms of the quantities contained in our normal form, we obtain

$$[0234 \dots (2m)] = \pm \Sigma (-1)^{r-1} l_r \frac{\partial (l_1, \dots, l_{r-1}, l_{r+1}, \dots, l_m, a_1, a_2, \dots, a_m)}{\partial (x_2, x_3, x_4, \dots, x_{2m})},$$

$$[0134 \dots (2m)] = \pm \Sigma (-1)^{r-1} l_r \frac{\partial (l_1, \dots, l_{r-1}, l_{r+1}, \dots, l_m, a_1, a_2, \dots, a_m)}{\partial (x_1, x_3, x_4, \dots, x_{2m})},$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$[0123 \dots (2m-1)] \\ = \pm \Sigma (-1)^{r-1} l_r \frac{\partial (l_1, \dots, l_{r-1}, l_{r+1}, \dots, l_m, a_1, a_2, \dots, a_m)}{\partial (x_1, x_2, x_3, \dots, x_{2m-1})}.$$

From these we readily deduce that each of the a's satisfies an equation of the form

$$[0234 \dots (2m)] \frac{\partial a}{\partial x_1} - [0134 \dots (2m)] \frac{\partial a}{\partial x_2} + \dots \\ \dots + (-1)^{2m-1} [0123 \dots (2m-1)] \frac{\partial a}{\partial x_{2m}} = 0. \quad (4)$$

This equation will possess $2m-1$ independent solutions. For our present purpose, we shall require only m solutions, but these cannot be chosen at random. We have $\frac{1}{2}m(m-1)$ other equations connecting the a's in pairs. These are to be deduced from the expressions for the set of Pfaffians immediately preceding the last set of allied functions. For these we have

$$[345 \ldots (2m)] = \pm \Sigma \frac{\partial (l_1, \ldots, l_{r-1}, l_{r+1}, \ldots, l_m, a_1, \ldots, a_{r-1}, a_{r+1}, \ldots, a_m)}{\partial (x_3, x_4, x_5, \ldots, x_{2m})},$$

$$[245 \ldots (2m)] = \pm \Sigma \frac{\partial (l_1, \ldots, l_{r-1}, l_{r+1}, \ldots, l_m, a_1, \ldots, a_{r-1}, a_{r+1}, \ldots, a_m)}{\partial (x_2, x_1, x_5, \ldots, x_{2m})},$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$[145 \ldots (2m)] = \pm \Sigma \frac{\partial (l_1, \ldots, l_{r-1}, l_{r+1}, \ldots, l_m, a_1, \ldots, a_{r-1}, a_{r+1}, \ldots, a_m)}{\partial (x_1, x_4, x_5, \ldots, x_{2m})},$$

and so on. Thus we obtain, as the general type of above-mentioned $\frac{1}{2}m(m-1)$ equations, the equation

$$\Sigma \pm \left[123 \ldots (p-1)(p+1) \ldots (q-1)(q+1) \ldots (2m)\right] \frac{\partial (a, a')}{\partial (x_p, x_q)} = 0; \quad (5)$$

it being understood that any particular term under the Σ is affected with a positive or a negative sign according as the number of displacements required to reduce the series of numbers

$$p, q, 1, 2, 3, \ldots, p-1, p+1, \ldots, q-1, q+1, \ldots, 2m$$

to their proper numerical order is even or odd.

The m equations of the type (4) combined with the $\frac{1}{2}m(m-1)$ equations of the type (5) make up a set of $\frac{1}{2}m(m+1)$ equations. These are the equations obtained by Clebsch.* They flow naturally from our results, which also suggest the natural method of generalization.

To utilize these results for the integration of the equation, we select, in the first place, any solution of (4). We then substitute this for a' in equation (5). We have then to obtain a common integral of (4) and the equation so formed. Having determined this

* See Forsyth, p. 214.

integral, we may also substitute it for a' in equation (5); and we have then to determine a common integral of (4) and two equations of the type (5). Proceeding in this manner, we see that the whole set of $\frac{1}{2}m(m+1)$ equations will have been exhausted by the time we have determined the last integral.

6. Before proceeding to conditioned cases, we will discuss the unconditioned case in an odd number of variables. Thus, let

$$n = 2m+1,$$

in which case we have

$$d\beta + l_1\,da_1 + l_2\,da_2 + \ldots + l_m\,da_m$$

for a normal form.

The final derived function in this case will be

$$[0123\ \ldots\ (2m+1)],$$

which is, of course, supposed finite.

Forming the expressions for the members of the set of Pfaffians immediately preceding, we have

$$[234\ldots(2m+1)] = \pm\frac{\partial(l_1, l_2, \ldots, l_m, a_1, a_2, \ldots, a_m)}{\partial(x_2, x_3, x_4, \ldots, x_{2m+1})},$$

$$[134\ldots(2m+1)] = \pm\frac{\partial(l_1, l_2, \ldots, l_m, a_1, a_2, \ldots, a_m)}{\partial(x_1, x_3, x_4, \ldots, x_{2m+1})},$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$[123\ldots(2m)] = \pm\frac{\partial(l_1, l_2, \ldots, l_m, a_1, a_2, \ldots, a_m}{\partial(x_1, x_2, x_3, \ldots, x_{2m})}.$$

Thus, we see that each of the a's satisfies an equation of the form

$$[234\ldots(2m+1)]\frac{\partial a}{\partial x_1} - [134\ldots(2m+1)]\frac{\partial a}{\partial x_2} + \ldots$$
$$\ldots + (-1)^{2m}[123\ldots(2m)]\frac{\partial a}{\partial x_{2m+1}} = 0. \quad (6)$$

In order to obtain the equations which connect the a's in pairs, we must first write down the expressions for the set of allied func-

tions immediately preceding the last set of Pfaffians. We have

$$[0345 \dots (2m+1)]$$

$$= \pm \Sigma \left\{ (-1)^{r-1} l_r \frac{\partial (l_1, \dots, l_{r-1}, l_{r+1}, \dots, l_m, \alpha_1, \alpha_2, \dots, \alpha_m)}{\partial (x_3, x_4, x_5, \dots, x_{2m+1})} \right.$$

$$\left. + (-1)^{m+r+1} \frac{\partial (\beta, l_1, \dots, l_m, \alpha_1, \dots, \alpha_{r-1}, \alpha_{r+1}, \dots, \alpha_m)}{\partial (x_3, x_4, x_5, \dots, x_{2m+1})} \right\},$$

$$[0245 \dots (2m+1)]$$

$$= \pm \Sigma \left\{ (-1)^{r-1} l_r \frac{\partial (l_1, \dots, l_{r-1}, l_{r+1}, \dots, l_m, \alpha_1, \alpha_2, \dots, \alpha_m)}{\partial (x_2, x_4, x_5, \dots, x_{2m+1})} \right.$$

$$\left. + (-1)^{m+r+1} \frac{\partial (\beta, l_1, \dots, l_m, \alpha_1, \dots, \alpha_{r-1}, \alpha_{r+1}, \dots, \alpha_m)}{\partial (x_2, x_4, x_5, \dots, x_{2m+1})} \right\},$$

... ...

$$[0145 \dots (2m+1)]$$

$$= \pm \Sigma \left\{ (-1)^{r-1} l_r \frac{\partial (l_1, \dots, l_{r-1}, l_{r+1}, \dots, l_m, \alpha_1, \alpha_2, \dots, \alpha_m)}{\partial (x_1, x_4, x_5, \dots, x_{2m+1})} \right.$$

$$\left. + (-1)^{m+r+1} \frac{\partial (\beta, l_1, \dots, l_m, \alpha_1, \dots, \alpha_{r-1}, \alpha_{r+1}, \dots, \alpha_m)}{\partial (x_1, x_4, x_5, \dots, x_{2m+1})} \right\},$$

and so on. Thus we obtain

$$\Sigma \pm \left[0123 \dots (p-1)(p+1) \dots (q-1)(q+1) \dots (2m+1)\right] \frac{\partial (\alpha, \alpha')}{\partial (x_p, x_q)} = 0, \tag{7}$$

where it is understood that the sign of any particular term is determined by the same rule as that laid down in the case of equation (5).

The $\frac{1}{2}m(m+1)$ equations formed by writing down all equations of the types (6) and (7) enable us to determine the α's by a method exactly similar to that adopted in the preceding case. It only remains to determine β.

The m equations connecting β with each of the α's are of the type (7). In fact, we have

$$\Sigma \pm \left[0123 \dots (p-1)(p+1) \dots (q-1)(q+1) \dots (2m+1)\right] \frac{\partial (\beta, \alpha)}{\partial (x_p, x_q)} = 0. \tag{8}$$

The equation, containing β only, differs slightly from the type (6). We have

$$[234\ldots(2m+1)]\frac{\partial\beta}{\partial x_1}-[134\ldots(2m+1)]\frac{\partial\beta}{\partial x_2}+\ldots$$

$$\ldots+(-1)^{2m}[123\ldots(2m)]\frac{\partial\beta}{\partial x_{2m+}}=\pm\frac{\partial(\beta, l_1, \ldots, l_m, a_1, \ldots, a_m)}{\partial(x_1, x_2, x_3, \ldots, x_{2m+1})}$$

$$=[0123\ldots(2m+1)]. \qquad (9)$$

7. We now proceed to discuss the case of a conditioned system in any number of variables. This naturally divides itself into two cases, according to the type of normal form suitable. We shall take first the case in which we have

$$l_1 da_1 + l_2 da_2 + \ldots + l_s da_s$$

for a normal form, s being less than $\frac{1}{2}n$ when n is even, and less than $\frac{1}{2}(n+1)$ when n is odd. Then the derived functions of the $2s^{\text{th}}$ and all following orders vanish. Thus the last set of derived functions that do not vanish are the s^{th} set of Pfaffians. Expressing these in terms of the quantities involved in the normal form, we have

$$[123\ldots(2s)]=\pm\frac{\partial(l_1, l_2, \ldots, l_s, a_1, a_2, \ldots, a_s)}{\partial(x_1, x_2, x_3, \ldots, x_{2s})},$$

$$[234\ldots(2s+1)]=\pm\frac{\partial(l_1, l_2, \ldots, l_s, a_1, a_2, \ldots, a_s)}{\partial(x_2, x_3, x_4, \ldots, x_{2s+1})},$$

&c., &c.,

there being in all $$\frac{n!}{2s!\,(n-2s)!}$$

equations of this type. From these we might deduce a set of equations on the model of (4) and (6). As, however, it is necessary to fall back upon the $(s-1)^{\text{th}}$ set of Pfaffians to obtain the equations connecting the a's in pairs, we will follow strictly the analogy of Article 5, and make use of the $(s-1)^{\text{th}}$ set of allied functions to obtain the required equations. We have

$$[0123\ldots(2s-1)]=\pm\Sigma(-1)^{r-1}l_r\frac{\partial(l_1, \ldots, l_{r-1}, l_{r+1}, \ldots, l_s, a_1, a_2, \ldots, a_s)}{\partial(x_1, x_2, x_3, \ldots, x_{2s-1})},$$

$$[0234\ldots(2s)]=\pm\Sigma(-1)^{r-1}l_r\frac{\partial(l_1, \ldots, l_{r-1}, l_{r+1}, \ldots, l_s, a_1, a_2, \ldots, a_s)}{\partial(x_2, x_3, x_4, \ldots, x_{2s})},$$

&c., &c.,

there being in all
$$\frac{n!}{(2s-1)!\,(n-2s+1)!}$$
equations of this type.

Proceeding to form equations from these on the model of (4) and (6), we obtain a set of the type

$$[0234 \ldots (2s)]\frac{\partial a}{\partial x_1} - [0134 \ldots (2s)]\frac{\partial a}{\partial x_2} + \ldots$$
$$\ldots + (-1)^{2s-1}[0123 \ldots (2s-1)]\frac{\partial a}{\partial x_{2s}} = 0.$$

The total number of equations of this type that can be written down will be the same as the number of sets, each containing $2s$ numbers, that can be selected from the series

$$1,\ 2,\ 3,\ \ldots,\ n,$$

viz.,
$$\frac{n!}{2s!\,(n-2s)!}.$$

These equations will not, however, be all independent. We can express our set of allied functions in the form

$$[0123 \ldots (2s-1)] = \pm\, l_1 \frac{\partial\,(l_2/l_1,\ l_3/l_1,\ \ldots,\ l_s/l_1,\ a_1,\ a_2,\ \ldots,\ a_s)}{\partial\,(x_1,\ x_2,\ x_3,\ \ldots,\ x_{2s-1})},$$

$$[0234 \ldots (2s)] = \pm\, l_1^s \frac{\partial\,(l_2/l_1,\ l_3/l_1,\ \ldots,\ l_s/l_1,\ a_1,\ a_2,\ \ldots,\ a_s)}{\partial\,(x_2,\ x_3,\ x_4,\ \ldots,\ x_{2s})},$$

&c., &c.,

as can be readily proved in the following manner. We have

$$[0123 \ldots (2s-1)]$$
$$= \pm\, l_1 l_2 \ldots l_s \Sigma\, (-1)^{r-1} \frac{\partial\,(\log l_1,\ \ldots,\ \log l_{r-1},\ \log l_{r+1},\ \ldots,\ \log l_s,\ a_1,\ \ldots,\ a_s)}{\partial\,(x_1,\ x_2,\ x_3,\ \ldots,\ x_{2s-1})}.$$

Now

$$\frac{\partial\,(\log l_2,\ \log l_3,\ \ldots,\ \log l_s,\ a_1,\ \ldots,\ a_s)}{\partial\,(x_1,\ x_2,\ x_3,\ \ldots,\ x_{2s-1})} - \frac{\partial\,(\log l_1,\ \log l_3,\ \ldots,\ \log l_s,\ a_1,\ \ldots,\ a_s)}{\partial\,(x_1,\ x_2,\ x_3,\ \ldots,\ x_{2s-1})}$$
$$= \frac{\partial\,(\log l_2/l_1,\ \log l_3,\ \ldots,\ \log l_s,\ a_1,\ \ldots,\ a_s)}{\partial\,(x_1,\ x_2,\ x_3,\ \ldots,\ x_{2s-1})}.$$

Further,

$$\frac{\partial\,(\log l_2/l_1,\ \log l_3,\ \ldots,\ \log l_s,\ a_1,\ \ldots,\ a_s)}{\partial\,(x_1,\ x_2,\ x_3,\ \ldots,\ x_{2s-1})} + \frac{\partial\,(\log l_1,\ \log l_3,\ \log l_4,\ \ldots,\ \log l_s,\ a_1,\ \ldots,\ a_s)}{\partial\,(x_1,\ x_2,\ x_3,\ \ldots,\ x_{2s-1})}$$

$$= \frac{\partial\,(\log l_2/l_1,\ \log l_3,\ \ldots,\ \log l_s,\ a_1,\ \ldots,\ a_s)}{\partial\,(x_1,\ x_2,\ x_3,\ \ldots,\ x_{2s-1})} - \frac{\partial\,(\log l_2/l_1,\ \log l_1,\ \log l_4,\ \ldots,\ \log l_s,\ a_1,\ \ldots,\ a_s)}{\partial\,(x_1,\ x_2,\ x_3,\ \ldots,\ x_{2s-1})}$$

$$= \frac{\partial\,(\log l_2/l_1,\ \log l_3/l_1,\ \log l_4,\ \ldots,\ \log l_s,\ a_1,\ \ldots,\ a_s)}{\partial\,(x_1,\ x_2,\ x_3,\ \ldots,\ x_{2s-1})}.$$

Proceeding in this manner, we eventually obtain

$$[0123 \ldots (2s-1)] = \pm\, l_1 l_2 \ldots l_s \Sigma \frac{\partial\,(\log l_2/l_1,\ \log l_3/l_1,\ \ldots,\ \log l_s/l_1,\ a_1,\ \ldots,\ a_s)}{\partial\,(x_1,\ x_2,\ x_3,\ \ldots,\ x_{2s-1})}$$

$$= \pm\, l_1^s \Sigma \frac{\partial\,(l_2/l_1,\ l_3/l_1,\ \ldots,\ l_s/l_1,\ a_1,\ a_2,\ \ldots,\ a_s)}{\partial\,(x_1,\ x_2,\ x_3,\ \ldots,\ x_{2s-1})}.$$

Considering that our set of allied functions can be expressed in this form, and taking account of known properties of determinants, we see that the said functions are connected by relations of the types

$$[0234 \ldots (2s)][0145 \ldots (2s+1)] - [0134 \ldots (2s)][0245 \ldots (2s+1)] + [01245 \ldots (2s)][0345 \ldots (2s+1)] = 0,$$

$$[0234 \ldots (2s)][0156 \ldots (2s+2)] - [0134 \ldots (2s)][0256 \ldots (2s+2)] + [01245 \ldots (2s)][0356 \ldots (2s+2)] - [0123 = 0,$$

&c., &c.

Thus, the number of independent equations will be $n-2s+1$, and the

[redacted]

whole set will be algebraically deducible from the following:—

$$\left.\begin{array}{l}
[0234\ldots(2s)]\dfrac{\partial a}{\partial x_1}-[0134\ldots(2s)]\dfrac{\partial a}{\partial x_2}+\ldots \\
\qquad\qquad\ldots+(-1)^{2s-1}[0123\ldots(2s-1)]\dfrac{\partial a}{\partial x_{2s}}=0 \\
[0234\ldots(2s-1)(2s+1)]\dfrac{\partial a}{\partial x_1}-[0134\ldots(2s-1)(2s+1)]\dfrac{\partial a}{\partial x_2}+\ldots \\
\qquad\qquad\ldots+(-1)^{2s-2}[0123\ldots(2s-2)(2s+1)]\dfrac{\partial a}{\partial x_{2s-1}} \\
\qquad\qquad\quad+(-1)^{2s-1}[0123\ldots(2s-1)]\dfrac{\partial a}{\partial x_{2s+1}}=0 \\
\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots \\
[0234\ldots(2s-1)\,n]\dfrac{\partial a}{\partial x_1}-[0134\ldots(2s-1)\,n]\dfrac{\partial a}{\partial x_2}+\ldots \\
\qquad\qquad\ldots+(-1)^{2s-2}[0123\ldots(2s-2)\,n]\dfrac{\partial a}{\partial x_{2s-1}} \\
\qquad\qquad\quad+(-1)^{2s-1}[0123\ldots(2s-1)]\dfrac{\partial a}{\partial x_n}=0
\end{array}\right\}. \tag{10}$$

Further, as the ratios of the l's satisfy these equations in addition to the a's, we see that the number of common integrals will be $2s-1$. Thus, the number of equations being $n-2s+1$, we see that they must form a complete system.

We have now to form a set of equations on the model of equations (5) and (7). To do so it will be necessary to write down the expressions for the derived functions of order $2s-3$, *i.e.*, the $(s-1)^{\text{th}}$ set of Pfaffians. We have

$$[123\ldots(2s-2)]=\pm\Sigma\frac{\partial(l_1,\ldots,l_{r-1},l_{r+1},\ldots,l_s,a_1,\ldots,a_{r-1},a_{r+1},\ldots,a_s)}{\partial(x_1,x_2,x_3,\ldots,x_{2s-2})},$$

$$[23\ldots(2s-1)]=\pm\Sigma\frac{\partial(l_1,\ldots,l_{r-1},l_{r+1},\ldots,l_s,a_1,\ldots,a_{r-1},a_{r+1},\ldots,a_s)}{\partial(x_2,x_3,x_4,\ldots,x_{2s-1})}.$$

&c., &c.,

there being in all $\dfrac{n!}{(2s-2)!\,(n-2s+2)!}$

equations of this type.

With the aid of the above expressions we can form a set of equations, similar in form to (5), of which the following may be taken as a type :—

$$\Sigma \pm \left[123 \dots (p-1)(p+1) \dots (q-1)(q+1) \dots (2s)\right] \frac{\partial (a, a')}{\partial (x_p, x_q)} = 0. \quad (11)$$

In this equation the combination (p, q) is to be replaced by every possible selection of two numbers out of the series

$$1, 2, 3, \dots, 2s;$$

and the determination of the sign of any particular term under the Σ is to be made according to the rule laid down in connexion with equation (5). The total number of equations of this type that can be formed will be the same as the number of sets of $2s$ numbers that can be selected from the series

$$1, 2, 3, \dots, n,$$

viz.,
$$\frac{n!}{2s!\,(n-2s)!}.$$

The number of quantities of the form

$$\frac{\partial (a, a')}{\partial (x_p, x_q)}$$

that occur in the whole set of equations is $\frac{1}{2}n\,(n-1)$. Now

$$\frac{n!}{2s!\,(n-2s)!} > \tfrac{1}{2}n\,(n-1),$$

except for the single case $s = 1$. It is in general much greater. Thus it is evident that the members of our set of equations are not all algebraically independent. The number of independent equations cannot be greater than

$$\tfrac{1}{2}n\,(n-1)-1,$$

and is probably less.

This last set of equations will be satisfied by the l's, and consequently by the quantities

$$\frac{l_2}{l_1}, \frac{l_3}{l_1}, \dots, \frac{l_s}{l_1}.$$

Thus our complete set of equations, including equations (10) and

[redacted]

the equations of type (11), will be satisfied by these quantities in addition to the a's. Consequently their general solutions will be of the type

$$\phi\left(\frac{l_2}{l_1},\ \frac{l_3}{l_1},\ \ldots,\ \frac{l_s}{l_1},\ a_1,\ a_2,\ \ldots,\ a_s\right),$$

and therefore the argument used by Forsyth to justify the sufficiency of the equations in the case discussed in Article 5 may also be used here.

8. It now only remains to obtain the necessary equations for the case in which the normal form is

$$d\beta+l_1da_1+l_2da_2+\ldots+l_sda_s,$$

s being less than $\frac{1}{2}n$ if n be even, and less than $\frac{1}{2}(n-1)$ if n be odd. The last set of derived functions that does not become evanescent will be the $2s^{\text{th}}$, *i.e.*, the s^{th} set of allied functions. These may be expressed as follows:—

$$[0123\ldots(2s+1)] = \pm\frac{\partial(\beta,\ l_1,\ \ldots,\ l_s,\ a_1,\ \ldots,\ a_s)}{\partial(x_1,\ x_2,\ x_3,\ \ldots,\ x_{2s+1})},$$

$$[0234\ldots(2s+2)] = \pm\frac{\partial(\beta,\ l_1,\ \ldots,\ l_s,\ a_1,\ \ldots,\ a_s)}{\partial(x_2,\ x_3,\ x_4,\ \ldots,\ x_{2s+2})},$$

&c., &c.,

there being in all $$\frac{n!}{(2s+1)!\,(n-2s-1)!}$$

equations of this form. We could deduce from these a set of equations that are satisfied by the a's, but we shall find it most convenient to follow the analogy of our former work. We must accordingly write down the expressions for the s^{th} set of Pfaffians. We have

$$[123\ldots(2s)] = \pm\frac{\partial(l_1,\ \ldots,\ l_s,\ a_1,\ \ldots,\ a_s)}{\partial(x_1,\ x_2,\ x_3,\ \ldots,\ x_{2s})},$$

$$[234\ldots(2s+1)] = \pm\frac{\partial(l_1,\ \ldots,\ l_s,\ a_1,\ \ldots,\ a_s)}{\partial(x_2,\ x_3,\ x_4,\ \ldots,\ x_{2s+1})},$$

&c., &c.,

there being in all $$\frac{n!}{2s!\,(n-2s)!}$$

equations of this type. Thus we see that the equations satisfied by

the a's will be of the type

$$[234 \dots (2s+1)]\frac{\partial a}{\partial x_1} - [134 \dots (2s+1)]\frac{\partial a}{\partial x_2} + \dots$$

$$\dots + (-1)^{2s}[123 \dots (2s)]\frac{\partial a}{\partial x_{2s+1}} = 0. \quad (12)$$

The number of algebraically independent equations of this type will be $n-2s$, and we may choose a set similar to equations (10) of the preceding article.

The equations to be satisfied by β will be of the type

$$[234 \dots (2s+1)]\frac{\partial \beta}{\partial x_1} - [134 \dots (2s+1)]\frac{\partial \beta}{\partial x_2} + \dots$$

$$\dots + (-1)^{2s}[123 \dots (2s)]\frac{\partial \beta}{\partial x_{2s+1}} = [0123 \dots (2s+1)]. \quad (13)$$

The number of equations of this type that are algebraically independent will be $n-2s$.

In order to form the remaining equations, we have to write down the expressions for the $(s-1)^{\text{th}}$ set of allied functions. We have

$$[0123 \dots (2s-1)]$$

$$= \pm \Sigma \left\{ (-1)^{r-1} l_r \frac{\partial (l_1, \dots, l_{r-1}, l_{r+1}, \dots, l_s, a_1, a_2, \dots, a_s)}{\partial (x_1, x_2, x_3, \dots, x_{2s-1})} \right.$$

$$\left. + (-1)^{s+r+1} \frac{\partial (\beta, l_1, \dots, l_s, a_1, \dots, a_{r-1}, a_{r+1}, \dots, a_s)}{\partial (x_1, x_2, x_3, \dots, x_{2s-1})} \right\},$$

$$[023 \dots (2s)]$$

$$= \pm \Sigma \left\{ (-1)^{r-1} l_r \frac{\partial (l_1, \dots, l_{r-1}, l_{r+1}, \dots, l_s, a_1, a_2, \dots, a_s)}{\partial (x_2, x_3, x_4, \dots, x_{2s})} \right.$$

$$\left. + (-1)^{s+r+1} \frac{\partial (\beta, l_1, \dots, l_s, a_1, \dots, a_{r-1}, a_{r+1}, \dots, a_s)}{\partial (x_2, x_3, x_4, \dots, x_{2s})} \right\},$$

&c., &c.,

there being in all $\dfrac{n!}{(2s-1)!\,(n-2s+1)!}$

equations of this type. From them we readily deduce the remaining

[redacted]

equations. We have a set of the type

$$\Sigma \pm \left[0123 \ldots (p-1)(p+1) \ldots (q-1)(q+1) \ldots (2s+1)\right] \frac{\partial(\alpha, \alpha')}{\partial(x_p, x_q)} = 0, \tag{14}$$

and a set of the type

$$\Sigma \pm \left[0123 \ldots (p-1)(p+1) \ldots (q-1)(q+1) \ldots (2s+1)\right] \frac{\partial(\beta, \alpha)}{\partial(x_p, x_q)} = 0. \tag{15}$$

This accomplishes our present purpose, which is the determination of the forms of certain equations known to exist. The discussion opens up some interesting questions with regard to the general properties of the derived functions of a Pfaffian expression, but these must be left for future investigation.

On Ampère's Equation $Rr+2Ss+Tt+U(rt-s^2) = V$. *By* A. C. Dixon.

Received October 20th, 1899. Communicated November 9th, 1899. Received, in revised form, January 4th, 1900.

1. The methods given hitherto for solving the above equation do not apply in general, but only if certain conditions are satisfied by the coefficients R, S, T, U, V. I propose in this article to investigate a modification of the ordinary Monge and Ampère method which applies to all cases.

It must be remembered that no method can be given which will in all cases give an integral in finite terms. It seems almost certain further that no method will ever be found for reducing the solution of Ampère's equation in general to that of equations of the first order with one dependent variable. Even if such a method were found, it would not enable us to solve the equation in all cases, for there is no general method of solving such equations of the first order, except by continued approximation. The ordinary methods, those of Lagrange, Charpit, and Jacobi, depend on inspection. The

method of continued approximation, as by infinite series, applies to equations of all orders, and not only to those of the first. The statement, often made, that a single function satisfying given partial differential equations of the first order may be considered known is therefore, in the present connexion at least, not well founded. No doubt much more is known of the relations among different solutions in the case of an equation of the first order, but the solutions themselves are equally well known for all orders.

The methods of Monge and Ampère depend for their success on the combination of certain differential expressions so as to form a complete differential. When this can be done by inspection, the methods are of practical use, but hardly otherwise. The method I now propose depends on the combination of the same differential expressions in a particular way, and it will not be of much use in practice unless this can be done by inspection. Its advantage is that it applies universally; that is, that for any equation of Ampère's form any solution, involving three arbitrary constants in such a way that their elimination does not lead to an equation of the first order, satisfies the conditions on which the method depends, and is therefore discoverable by a sufficiently keen inspection.

If the two sets of characteristics coincide, the statement of the method has to be modified, and in this case the practical difficulty is specially marked.

Suppose the equation

$$Rr+2Ss+Tt+U(rt-s^2) = V \tag{1}$$

to have a solution

$$\phi(x, y, z, c_1, c_2, c_3) = 0 \tag{2}$$

involving three arbitrary constants. Then, by taking the first derivatives of this, we form two other equations

$$\frac{\partial\phi}{\partial x}+p\frac{\partial\phi}{\partial z} = 0, \quad \frac{\partial\phi}{\partial y}+q\frac{\partial\phi}{\partial z} = 0, \tag{3}$$

and these three may be supposed solved for c_1, c_2, c_3, so as to give equations

$$u_1 = c_1, \quad u_2 = c_2, \quad u_3 = c_3, \tag{4}$$

unless it is possible to eliminate c_1, c_2, c_3 from (2) and (3), in which case (2) will be a primitive of a differential equation of the first order only, involving no arbitrary constant. This case we shall suppose excluded.

[redacted]

We shall consider the solution rather in the form (4) than (2). If it is a complete primitive depending on five arbitrary constants, then u_1, u_2, u_3 must be supposed to involve two constants as well as x, y, z, p, q.

2. Now, if two of the functions u_1, u_2, u_3, say u_1, u_2, have been found by any means, the third is found without any theoretical difficulty. For the equations

$$u_1 = c_1, \quad u_2 = c_2$$

may be solved for p, q,* and the resulting values substituted in the equation

$$dz = p\,dx + q\,dy.$$

The integration of this must lead to (2), which, when written

$$\phi(x, y, z, u_1, u_2, c_3) = 0,$$

must be equivalent to $\quad u_3 = c_3.$

The problem is therefore reduced to the finding of two functions u_1, u_2 such that the values of p, q given by solving the equations

$$u_1 = c_1, \quad u_2 = c_2$$

may satisfy the conditions

$$Rr + 2Ss + Tt + U(rt - s^2) = V, \tag{1}$$

that is,

$$R\left(\frac{\partial p}{\partial x} + p\frac{\partial p}{\partial z}\right) + 2S\left(\frac{\partial q}{\partial x} + p\frac{\partial q}{\partial z}\right) + T\left(\frac{\partial q}{\partial y} + q\frac{\partial q}{\partial z}\right)$$
$$+ U\left\{\frac{\partial(p, q)}{\partial(x, y)} + p\frac{\partial(p, q)}{\partial(z, y)} + q\frac{\partial(p, q)}{\partial(x, z)}\right\} = V \tag{5}$$

and
$$\frac{\partial q}{\partial x} + p\frac{\partial q}{\partial z} = \frac{\partial p}{\partial y} + q\frac{\partial p}{\partial z}. \tag{6}$$

3. Now (1) may be written (unless $U = 0$)

$$\begin{vmatrix} Ur + T, & Us - S_1 \\ Us - S_2, & Ut + R \end{vmatrix} = 0, \tag{7}$$

---

* Otherwise p, q could be eliminated, and thus there would be a relation among x, y, z, c_1, c_2 not involving c_3. This cannot be since (2) involves c_3.

where S_1, S_2 are such expressions that

$$S_1+S_2=2S, \quad S_1S_2=RT+UV.$$

The equation (7) may be replaced by

$$\left.\begin{aligned} Ur+T+\lambda(Us-S_1)&=0 \\ Us-S_2+\lambda(Ut+R)&=0 \end{aligned}\right\}, \tag{8}$$

or by

$$\left.\begin{aligned} Ur+T+\mu(Us-S_2)&=0 \\ Us-S_1+\mu(Ut+R)&=0 \end{aligned}\right\}, \tag{9}$$

λ, μ being proper multipliers.

Taking first the equations (8), let us suppose quantities A_1, A_2 so chosen that

$$A_1\frac{\partial u_1}{\partial p}+A_2\frac{\partial u_2}{\partial p}=U, \quad A_1\frac{\partial u_1}{\partial q}+A_2\frac{\partial u_2}{\partial q}=U\lambda.$$

Then $\lambda S_1-T=Ur+U\lambda s=A_1\left(r\frac{\partial u_1}{\partial p}+s\frac{\partial u_1}{\partial q}\right)+A_2\left(r\frac{\partial u_2}{\partial p}+s\frac{\partial u_2}{\partial q}\right)$,

so that
$$T-\lambda S_1=A_1\left(\frac{\partial u_1}{\partial x}+p\frac{\partial u_1}{\partial z}\right)+A_2\left(\frac{\partial u_2}{\partial x}+p\frac{\partial u_2}{\partial y}\right),$$

and in like manner, from the second equation (8),

$$\lambda R-S_2=A_1\left(\frac{\partial u_1}{\partial y}+q\frac{\partial u_1}{\partial z}\right)+A_2\left(\frac{\partial u_2}{\partial y}+q\frac{\partial u_2}{\partial z}\right).$$

Thus

$$\begin{aligned} A_1du_1+A_2du_2 &= Udp+U\lambda dq+(T-\lambda S_1)\,dx+(\lambda R-S_2)\,dy \\ &\qquad +\left(A_1\frac{\partial u_1}{\partial z}+A_2\frac{\partial u_2}{\partial z}\right)(dz-p\,dx-q\,dy) \\ &= (U dp+T dx-S_2 dy)+\lambda\,(U dq-S_1 dx+R dy) \\ &\qquad +\left(A_1\frac{\partial u_1}{\partial z}+A_2\frac{\partial u_2}{\partial z}\right)(dz-p\,dx-q\,dy). \end{aligned}$$

Thus a linear combination of du_1, du_2 can be expressed as the sum of multiples of the expressions

$$U dp+T dx-S_2 dy, \quad U dq-S_1 dx+R dy, \quad dz-p\,dx-q\,dy.$$

To this list of three may be added the following, combinations of the first two, for use when U vanishes :—

$$S_1\,dp+T\,dq-V\,dy, \quad R\,dp+S_2\,dq-V\,dx.$$

In like manner it follows from (9) that a linear combination of du_1, du_2 can be expressed as the sum of multiples of

$$U\,dp+T\,dx-S_1\,dy, \quad U\,dq-S_2\,dx+R\,dy, \quad dz-p\,dx-q\,dy,$$

$$S_2\,dp+T\,dq-V\,dy, \quad R\,dp+S_1\,dq-V\,dx.$$

It is easily seen that these combinations are different unless $S_1=S_2$.

4. Conversely, when S_1, S_2 are unequal, if two linear combinations of du_1, du_2 can be expressed as just explained, the equations

$$u_1=c_1, \quad u_2=c_2,$$

if they can be solved for p, q, will furnish a solution. For we must have

$$\begin{aligned} U(A_1\,du_1+A_2\,du_2) = {} & \left(A_1\frac{\partial u_1}{\partial p}+A_2\frac{\partial u_2}{\partial p}\right)(U\,dp+T\,dx-S_2\,dy) \\ & +\left(A_1\frac{\partial u_1}{\partial q}+A_2\frac{\partial u_2}{\partial q}\right)(U\,dq-S_1\,dx+R\,dy) \\ & +U\left(A_1\frac{\partial u_1}{\partial z}+A_2\frac{\partial u_2}{\partial z}\right)(dz-p\,dx-q\,dy) \qquad (10) \end{aligned}$$

and
$$\begin{aligned} U(B_1\,du_1+B_2\,du_2) = {} & \left(B_1\frac{\partial u_1}{\partial p}+B_2\frac{\partial u_2}{\partial p}\right)(U\,dp+T\,dx-S_1\,dy) \\ & +\left(B_1\frac{\partial u_1}{\partial q}+B_2\frac{\partial u_2}{\partial q}\right)(U\,dq-S_2\,dx+R\,dy) \\ & +U\left(B_1\frac{\partial u_1}{\partial z}+B_2\frac{\partial u_2}{\partial z}\right)(dz-p\,dx-q\,dy). \qquad (11) \end{aligned}$$

Now let r, s, σ, t denote

$$\frac{\partial p}{\partial x}+p\frac{\partial p}{\partial z}, \quad \frac{\partial p}{\partial y}+q\frac{\partial p}{\partial z}, \quad \frac{\partial q}{\partial x}+p\frac{\partial q}{\partial z}, \quad \frac{\partial q}{\partial y}+q\frac{\partial q}{\partial z},$$

the derivatives being formed on the supposition that p, q are defined in terms of x, y, z by the equations

$$u_1=c_1, \quad u_2=c_2.$$

[redacted]

Then, the expressions on the left in (10), (11) being put equal to zero, we have, from (10),

$$\left(A_1\frac{\partial u_1}{\partial p}+A_2\frac{\partial u_2}{\partial p}\right)(Ur+T)+\left(A_1\frac{\partial u_1}{\partial q}+A_2\frac{\partial u_2}{\partial q}\right)(U\sigma-S_1)=0, \quad (13)$$

$$\left(A_1\frac{\partial u_1}{\partial p}+A_2\frac{\partial u_2}{\partial p}\right)(Us-S_2)+\left(A_1\frac{\partial u_1}{\partial q}+A_2\frac{\partial u_2}{\partial q}\right)(Ut+R)=0. \quad (14)$$

Now $A_1\frac{\partial u_1}{\partial p}+A_2\frac{\partial u_2}{\partial p}$, $A_1\frac{\partial u_1}{\partial q}+A_2\frac{\partial u_2}{\partial q}$ cannot both vanish, since $\frac{\partial(u_1, u_2)}{\partial(p, q)}$ is supposed not to be zero; we must therefore have

$$\begin{vmatrix} Ur+T, & U\sigma-S_1 \\ Us-S_2, & Ut+R \end{vmatrix}=0$$

or
$$U(rt-s\sigma)+Rr+S_1s+S_2\sigma+Tt=V. \quad (15)$$

In the same way, from (11), we have

$$U(rt-s\sigma)+Rr+S_2s+S_1\sigma+Tt=V. \quad (16)$$

By subtraction, $\qquad (S_1-S_2)(s-\sigma)=0,$

so that, as S_1, S_2 are supposed unequal,

$$s=\sigma.$$

Thus (15), (16) become

$$U(rt-s^2)+Rr+2Ss+Tt=V; \quad (1)$$

that is, the values of p, q given by the equations

$$u_1=c_1, \quad u_2=c_2$$

make the equation $\qquad dz=p\,dx+q\,dy$

integrable, and also cause the condition (1) to be satisfied.

The above proof can easily be modified so as to apply when U vanishes; for the equations (13), (14) may be supplemented by the following two which are consequences of them unless U vanishes:—

$$\left(A_1\frac{\partial u_1}{\partial p}+A_2\frac{\partial u_2}{\partial p}\right)(S_2r+Ts)+\left(A_1\frac{\partial u_1}{\partial q}+A_2\frac{\partial u_2}{\partial q}\right)(S_2\sigma+Tt-V)=0,$$

$$\left(A_1\frac{\partial u_1}{\partial p}+A_2\frac{\partial u_2}{\partial p}\right)(Rr+S_1s-V)+\left(A_1\frac{\partial u_1}{\partial q}+A_2\frac{\partial u_2}{\partial q}\right)(R\sigma+S_1t)=0.$$

Thus, when S_1, S_2 are unequal, we have the following method of solution. Find, by inspection, a combination of the expressions

$$U\,dp+T\,dx-S_2\,dy,\quad U\,dq-S_1\,dx+R\,dy,\quad S_1\,dp+T\,dq-V\,dy,$$

$$R\,dp+S_2\,dq-V\,dx,\quad dz-p\,dx-q\,dy,$$

which can be written in the form

$$A_1\,du_1+A_2\,du_2,$$

and a combination of the same expressions with S_1, S_2 interchanged that can be written in the form $B_1\,du_1+B_2\,du_2$; the differentials du_1, du_2 are to be the same in both cases, but the coefficients A_1, A_2, B_1, B_2 are not restricted, and in particular need not be such that either combination shall be a complete differential. Then, if the equations

$$u_1=c_1,\quad u_2=c_2$$

can be solved for p, q, a solution of the equation (1) will be given by the integral of

$$dz=p\,dx+q\,dy.$$

Although the theory of the method is general, its practical application is, of course, limited to solutions in finite terms, and even when such exist it may not be at all easy to find them by this or any other method. This difficulty is shared, though in a less degree, by the ordinary methods for solving equations of the first order.

5. The equations (8) may be discussed in another way. They may be written

$$U\frac{\partial p}{\partial x}+U\lambda\frac{\partial p}{\partial y}+U(p+\lambda q)\frac{\partial p}{\partial z}=\lambda S_1-T, \tag{17}$$

$$U\frac{\partial q}{\partial x}+U\lambda\frac{\partial q}{\partial y}+U(p+\lambda q)\frac{\partial q}{\partial z}=S_2-\lambda R. \tag{18}$$

Multiply (17) by $\dfrac{\partial u_1}{\partial p}$, (18) by $\dfrac{\partial u_1}{\partial q}$, and add: the result is

$$U\frac{\partial u_1}{\partial x}+U\lambda\frac{\partial u_1}{\partial y}+U(p+\lambda q)\frac{\partial u_1}{\partial z}+(\lambda S_1-T)\frac{\partial u_1}{\partial p}+(S_2-\lambda R)\frac{\partial u_1}{\partial q}=0, \tag{19}$$

whence it follows that du_1 is a linear combination of the expressions

$$\left\| \begin{matrix} dx, & dy, & dz, & dp, & dq \\ U, & U\lambda, & U(p+\lambda q), & \lambda S_1 - T, & S_2 - \lambda R \end{matrix} \right\|, \tag{20}$$

that is, of

$$\begin{gathered} dy - \lambda dx, \\ dz - p\,dx - q\,dy, \\ U\,dp + T\,dx - S_1\,dy, \\ U\,dq - S_2\,dx + R\,dy. \end{gathered}$$

Since u_2 must also satisfy (19), du_2 must also be a combination of these expressions, and thus B_1, B_2 can be so chosen that $B_1du_1+B_2du_2$ is a combination of the last three, a conclusion to which we were before led by considering (9).

6. The matrix (20) may be written in another form by the use of the known value of λ, namely,

$$\left(A_1\frac{\partial u_1}{\partial q}+A_2\frac{\partial u_2}{\partial q}\right) \div \left(A_1\frac{\partial u_1}{\partial p}+A_2\frac{\partial u_2}{\partial p}\right).$$

For

$$U = A_1\frac{\partial u_1}{\partial p}+A_2\frac{\partial u_2}{\partial p},$$

$$T-\lambda S_1 = A_1\frac{\partial u_1}{\partial x}+A_2\frac{\partial u_2}{\partial x}+p\left(A_1\frac{\partial u_1}{\partial z}+A_2\frac{\partial u_2}{\partial z}\right),$$

$$\lambda R - S_2 = A_1\frac{\partial u_1}{\partial y}+A_2\frac{\partial u_2}{\partial y}+q\left(A_1\frac{\partial u_1}{\partial z}+A_2\frac{\partial u_2}{\partial z}\right).$$

Thus, if we use the symbol* $d\Omega$ for $A_1du_1+A_2du_2$, we may write (20) thus

$$\left\| \begin{matrix} dx, & dy, & dz, & dp, & dq \\ \dfrac{\partial\Omega}{\partial p}, & \dfrac{\partial\Omega}{\partial q}, & p\dfrac{\partial\Omega}{\partial p}+q\dfrac{\partial\Omega}{\partial q}, & -\dfrac{\partial\Omega}{\partial x}-p\dfrac{\partial\Omega}{\partial z}, & -\dfrac{\partial\Omega}{\partial y}-q\dfrac{\partial\Omega}{\partial z} \end{matrix} \right\|, \tag{21}$$

in which form it has a striking analogy to the auxiliary matrix used in the theory of partial differential equations of the first order.

* It is not meant that $A_1du_1+A_2du_2$ is a complete differential.

The condition (19) may be written symbolically

$$\frac{\partial(\Omega, u_1)}{\partial(x, p)} + p\frac{\partial(\Omega, u_1)}{\partial(z, p)} + \frac{\partial(\Omega, u_1)}{\partial(y, q)} + q\frac{\partial(\Omega, u_1)}{\partial(z, q)} = 0, \tag{22}$$

and thus reduces to

$$\frac{\partial(u_1, u_2)}{\partial(x, p)} + p\frac{\partial(u_1, u_2)}{\partial(z, p)} + \frac{\partial(u_1, u_2)}{\partial(y, q)} + q\frac{\partial(u_1, u_2)}{\partial(z, q)} = 0, \tag{23}$$

which, as is known, leads directly to $s = \sigma$.

Now, in the argument of § 4, only one of the identities (10), (11) is necessary, if we know by any means that s, σ are equal.

Hence we have modified forms of the original theorem; in fact, $u_1 = c_1$, $u_2 = c_2$ will furnish a solution if multipliers A_1, A_2 can be found such that $A_1 du_1 + A_2 du_2$ is a linear combination of the expressions in the following list:—

$$Udp + Tdx - S_2 dy, \quad Udq - S_1 dx + Rdy,$$

$$S_1 dp + Tdq - Vdy, \quad Rdp + S_2 dq - Vdx, \quad dz - pdx - qdy;$$

and if any one of the following further conditions is satisfied:—

(*a*) du_1 (or du_2) is a linear combination of the expressions got by interchanging S_1, S_2 in the list just given, together with

$$\left(A_1\frac{\partial u_1}{\partial q} + A_2\frac{\partial u_2}{\partial q}\right) dx - \left(A_1\frac{\partial u_1}{\partial p} + A_2\frac{\partial u_2}{\partial p}\right) dy.$$

(*b*) du_1 (or du_2) is a linear combination of the determinants (21).

(*c*) The expressions given for p, q by the equations $u_1 = c_1$, $u_2 = c_2$ are such that $s = \sigma$.

(*d*) The compatibility condition (23).

(*e*) u_1, u_2 are two of the new principal variables in a tangential transformation. (The third principal variable will be u_3.)

In any of these modified forms the theorem applies equally well when $S_1 = S_2$. There is no difficulty in the verification, for we have seen that the above ways of stating the condition are equivalent; the equation (15) is proved as before, and we have now $s = \sigma$.

When the two sets of characteristics coincide, it is then necessary first to find by inspection, or otherwise, two functions u_1, u_2 satisfying the conditions just specified. The solution may then be completed as before, but it is doubtful whether the method in this form is of any practical use.

7. To illustrate the method, let us use it to find the most general solution of the equation

$$rt-s^2=0.$$

Here $A_1du_1+A_2du_2$ must be a combination of dp, dq and $dz-pdx-qdy$, for

$$U=1,\quad R=S_1=S_2=T=V=0.$$

Writing u for $px+qy-z$, we may say that $A_1du_1+A_2du_2$ must be a combination of dp, dq, du, and therefore u_1, u_2, p, q, u must be connected by at least one relation. Since u_1, u_2 are equal to constants, this shows that the equations of the solution yield at least one relation of the form

$$\phi(p,q,u)=0,$$

and we have further $\quad A_1du_1+A_2du_2\equiv d\phi,$

it being understood that u_1, u_2 are substituted for c_1, c_2 in $d\phi$ after formation. Thus du_1, du_2 are linear combinations of the determinants

$$\left\|\begin{matrix} dx, & dy, & dz, & dp, & dq \\ \dfrac{\partial\phi}{\partial p}+x\dfrac{\partial\phi}{\partial u}, & \dfrac{\partial\phi}{\partial q}+y\dfrac{\partial\phi}{\partial u}, & p\dfrac{\partial\phi}{\partial p}+q\dfrac{\partial\phi}{\partial q}+(px+qy)\dfrac{\partial\phi}{\partial u}, & 0, & 0\end{matrix}\right\|.$$

Thus u_1, u_2 must be functions of

$$p,\ q,\ u,\ \text{and}\ \left(\frac{\partial\phi}{\partial p}+x\frac{\partial\phi}{\partial u}\right)\div\left(\frac{\partial\phi}{\partial q}+y\frac{\partial\phi}{\partial u}\right),$$

and the two equations $\quad u_1=c_1,\quad u_2=c_2$

are equivalent to two of the form

$$u=f(p,q),\quad \left(x-\frac{\partial f}{\partial p}\right)\Big/\left(y-\frac{\partial f}{\partial q}\right)=F(p,q), \tag{α}$$

or of the form

$$f(p,q)=0,\quad x\frac{\partial f}{\partial q}-y\frac{\partial f}{\partial p}=F(p,q,u), \tag{β}$$

or to two relations connecting p, q, u. (γ)

The third equation of the solution is to be found by integration from

$$dz=p\,dx+q\,dy,$$

or, what is the same thing,

$$du=x\,dp+y\,dq.$$

Thus γ gives p, q, u all constant.

[redacted]

From (a), $$\left(x-\frac{\partial f}{\partial p}\right)dp+\left(y-\frac{\partial f}{\partial q}\right)dq=0,$$

that is, $$F(p,\,q)\,dp+dq=0.$$

Since the form of F is arbitrary, this gives an arbitrary relation between p, and q; thus, p, q, u may be taken as arbitrary functions of one variable only, and the solution comes out as the result of eliminating v from the equations

$$\begin{cases} z=xF_1(v)+yF_2(v)+F_3(v), \\ 0=xF_1'(v)+yF_2'(v)+F_3'(v). \end{cases}$$

This form includes the solution derived from (γ), and (β) leads to the same. This, then, is the most general solution involving three or more arbitrary constants.

The above example was chosen on account of its familiarity, so as to give an opportunity of comparing the present method with others. Let us now consider the following, which does not, I believe, fall under the rules of Monge and Ampère,

$$2xqr+(yp-2xq)\,s+2pyt+2xy\,(rt-s^2)=-3pq.$$

Here $$S_1+S_2=yp-2xq,$$

$$S_1S_2=4pqxy-6pqxy=-2pqxy.$$

Put then $$S_1=yp,\quad S_2=-2xq.$$

Hence $A_1du_1+A_2du_2$ is to be a combination of

$$2xy\,dp+2py\,dx+2xq\,dy,\quad 2xy\,dq-yp\,dx+2xq\,dy,\quad dz-p\,dx-q\,dy,$$

that is, of

$$dp+p\frac{dx}{x}+\frac{q}{y}\,dy,\quad 2dq-p\frac{dx}{x}+2q\frac{dy}{y},\quad dz-p\,dx-q\,dy.$$

Also $B_1du_1+B_2du_2$ is to be a combination of

$$2xy\,dp+2py\,dx-yp\,dy,\quad 2xy\,dq+2xq\,dx+2xq\,dy,\quad dz-p\,dx-q\,dy,$$

that is, of

$$2x\frac{dp}{p}+2dx-dy,\quad y\frac{dq}{q}+dx+dy,\quad dz-p\,dx-q\,dy.$$

Compare, for instance,

$$\left\{dp+p\frac{dx}{x}+\frac{q}{y}dy\right\}+\lambda\left\{2dq-p\frac{dx}{x}+2q\frac{dy}{y}\right\}$$

with
$$\left\{2x\frac{dp}{p}+2dx-dy\right\}+\mu\left\{y\frac{dq}{q}+dx+dy\right\}.$$

These expressions may be written

$$\left\{\frac{dp}{p}+(1-\lambda)\frac{dx}{x}\right\}p+\left\{2\lambda\frac{dq}{q}+(2\lambda+1)\frac{dy}{y}\right\}q$$

and
$$\left\{2\frac{dp}{p}+(2+\mu)\frac{dx}{x}\right\}x+\left\{\mu\frac{dq}{q}+(\mu-1)\frac{dy}{y}\right\}y.$$

If, then, we put $\qquad \mu=-2\lambda,$

we find as admissible forms these

$$u_1=px^{1-\lambda},$$
$$u_2=q^{2\lambda}y^{2\lambda+1},$$

λ having any constant value.

From the equations $\quad u_1=c_1, \quad u_2=c_2,$

we derive $\qquad p=c_1x^{\lambda-1}, \quad q=c_2^{1/2\lambda}y^{-(1/2\lambda)-1},$

and the solution thus found is

$$z=ax^{\lambda}+by^{-(1/2\lambda)}+c,$$

containing four arbitrary constants, a, b, c, λ.

Of course this is not the most general integral, but it may be the most general one in finite terms.

8. If in any particular case one of the multipliers A_1, A_2, B_1, B_2, say A_2, vanishes, we have substantially the methods of Monge and Ampère. We may take $A_1=1$, so that du_1 is a linear combination of

$$Udp+Tdx-S_2dy, \quad Udq-S_1dx+Rdy, \quad S_1dp+Tdq-Vdy,$$
$$Rdp+S_2dq-Vdx, \quad dz-pdx-qdy,$$

and the process of finding u_2, u_3, as above indicated, is then exactly that of Lagrange and Charpit for the integration of the equation

$$u_1=c_1.$$

The solution derived will involve three arbitrary constants, or one arbitrary constant and an arbitrary function.

[redacted]

9. The following is another method of finding u_1, u_2:—Since p, q are given in terms of x, y, z by

$$u_1 = c_1, \quad u_2 = c_2,$$

we have
$$r = \left\{ \frac{\partial (u_1, u_2)}{\partial (q, x)} + p \frac{\partial (u_1, u_2)}{\partial (q, z)} \right\} \div \frac{\partial (u_1, u_2)}{\partial (p, q)},$$

and so on.

$$rt - s^2 = \left\{ \frac{\partial (u_1, u_2)}{\partial (x, y)} + p \frac{\partial (u_1, u_2)}{\partial (z, y)} + q \frac{\partial (u_1, u_2)}{\partial (x, z)} \right\} \div \frac{\partial (u_1, u_2)}{\partial (p, q)}.$$

Hence the equation (1) gives

$$\begin{aligned} U \left\{ \frac{\partial (u_1, u_2)}{\partial (x, y)} + p \frac{\partial (u_1, u_2)}{\partial (z, y)} + q \frac{\partial (u_1, u_2)}{\partial (x, z)} \right\} & \\ + R \left\{ \frac{\partial (u_1, u_2)}{\partial (q, x)} + p \frac{\partial (u_1, u_2)}{\partial (q, z)} \right\} - 2S \left\{ \frac{\partial (u_1, u_2)}{\partial (p, x)} + p \frac{\partial (u_1, u_2)}{\partial (p, z)} \right\} & \\ + T \left\{ \frac{\partial (u_1, u_2)}{\partial (y, p)} + q \frac{\partial (u_1, u_2)}{\partial (z, p)} \right\} - V \frac{\partial (u_1, u_2)}{\partial (p, q)} = 0. & \qquad (24) \end{aligned}$$

We have also the compatibility condition

$$\frac{\partial (u_1, u_2)}{\partial (p, x)} + p \frac{\partial (u_1, u_2)}{\partial (p, z)} + \frac{\partial (u_1, u_2)}{\partial (q, y)} + q \frac{\partial (u_1, u_2)}{\partial (q, z)} = 0. \qquad (25)$$

Now, if we denote by $d(u_1, u_2)$ the double differential element $du_1 du_2$ as it occurs, for instance, in a double integral, such as $\iint du_1 du_2$, we have

$$\begin{aligned} d(u_1, u_2) = {} & \frac{\partial (u_1, u_2)}{\partial (x, y)} d(x, y) + \frac{\partial (u_1, u_2)}{\partial (y, z)} d(y, z) + \frac{\partial (u_1, u_2)}{\partial (z, x)} d(z, x) \\ & + \frac{\partial (u_1, u_2)}{\partial (x, p)} d(x, p) + \frac{\partial (u_1, u_2)}{\partial (y, p)} d(y, p) + \frac{\partial (u_1, u_2)}{\partial (z, p)} d(z, p) \\ & + \frac{\partial (u_1, u_2)}{\partial (x, q)} d(x, q) + \frac{\partial (u_1, u_2)}{\partial (y, q)} d(y, q) \\ & + \frac{\partial (u_1, u_2)}{\partial (z, q)} d(z, q) + \frac{\partial (u_1, u_2)}{\partial (p, q)} d(p, q). \end{aligned}$$

Thus, by reason of the relations just found among the Jacobians, it follows that $d(u_1, u_2)$ is a linear combination of the determinants

$$\left\| \begin{matrix} d(x,y), & d(y,z), & d(z,x), & d(x,p), & d(y,p), & d(z,p), & d(x,q), & d(y,q), & d(z,q), & d(p,q) \\ U, & -pU, & -qU, & 2S, & T, & 2pS+qT, & -R, & 0, & -pR, & -V \\ 0, & 0, & 0, & -1, & 0, & -p, & 0, & -1, & -q, & 0 \end{matrix} \right\|,$$

that is, of

$$\left\| \begin{matrix} d(x,y), & d(z,y), & d(x,z), & d(x,p)+d(q,y), & d(y,p), & d(z,p)-pd(x,p)-qd(y,p), & d(x,q), & d(z,q)-pd(x,q)-qd(y,q), & d(p,q) \\ U, & pU, & qU, & 2S, & T, & 0, & -R, & 0, & -V \end{matrix} \right\|.$$

Also, if u_1, u_2 are any two functions of x, y, z, p, q such that $d(u_1, u_2)$ is a linear combination of the determinants of this matrix, then

$$u_1 = c_1, \quad u_2 = c_2,$$

if they can be solved for p, q, will furnish a solution.

10. Since $d(u_1, u_3)$ and $d(u_2, u_3)$ must also be linear combinations of the same expressions, it follows that a solution of the form (2) gives three such combinations, independent of each other. A solution given by Ampère's method leads to five independent combinations, for, if $u_1 = c_1$ is the intermediary integral, we have

$$dz - p\,dx - q\,dy \equiv \lambda\, du_1 + \mu\,(du_2 - u_4\, du_3),$$

the other integral equations being, in the case of the general integral,

$$u_2 = f(u_3),$$
$$u_4 = f'(u_3).$$

We have then two complete primitives, among others,

$$u_1 = c_1, \quad u_2 = c_2, \quad u_3 = c_3;$$

$$u_1 = c_1, \quad u_4 = c_2, \quad u_2 - u_4 u_3 = c_3,$$

and the five combinations are

$$d(u_1, u_2),\quad d(u_1, u_3),\quad d(u_1, u_4),\quad d(u_2, u_3),$$
$$d(u_2, u_4) - u_4 d(u_3, u_4).$$

On this subject compare a paper by the writer presented this year to the Royal Society.

10. A point of interest in the theory is that an equation of Ampère's form (1) is changed into another of the same form by any tangential transformation. (Goursat, *Equations aux dérivées partielles du second ordre*, Vol. I., chapter ii.)

Suppose that x, y, z, p, q are expressed in terms of five other variables x', y', z', p', q', in such a way that

$$dz - p\,dx - q\,dy \equiv \rho\,(dz' - p'dx' - q'dy').$$

Then we have r, s, t given by the equations

$$r\frac{dx}{dx'} + s\frac{dy}{dx'} = \frac{dp}{dx'},\quad r\frac{dx}{dy'} + s\frac{dy}{dy'} = \frac{dp}{dy'};$$

$$s\frac{dx}{dx'} + t\frac{dy}{dx'} = \frac{dq}{dx'},\quad s\frac{dx}{dy'} + t\frac{dy}{dy'} = \frac{dq}{dy'};$$

where $\dfrac{d}{dx'}$, $\dfrac{d}{dy'}$ stand for

$$\frac{\partial}{\partial x'} + p'\frac{\partial}{\partial z'} + r'\frac{\partial}{\partial p'} + s'\frac{\partial}{\partial q'}$$

and

$$\frac{\partial}{\partial y'} + q'\frac{\partial}{\partial z'} + s'\frac{\partial}{\partial p'} + t'\frac{\partial}{\partial q'}.$$

Thus r, s, t and $rt - s^2$ are given in terms of x', y', z', p', q', r', s', t' as fractions having a common denominator $\dfrac{d(x, y)}{d(x', y')}$, the numerators and denominator being all linear in r', s', t', $r't' - s'^2$. Thus Ampère's equation (1) is changed into another of the same form.

The new coefficients R', S', T', U', V' are easily found to be as follows:—

$$U' = U\frac{\partial(p, q)}{\partial(p', q')} + R\frac{\partial(p, y)}{\partial(p', q')} + 2S\frac{\partial(x, p)}{\partial(p', q')} + T\frac{\partial(x, q)}{\partial(p', q')} - V\frac{\partial(x, y)}{\partial(p', q')}$$

$$= [p', q'], \text{ say,}$$

and, with a like notation,

$$
\begin{aligned}
R' &= [p', y'] + q'\,[p', z'], \\
2S' &= [q', y'] + q'\,[q', z'] + [x', p'] + p'\,[z', p'], \\
T' &= [x', q'] + p'\,[z', q'], \\
-V' &= [x', y'] + p'\,[z', y'] + q'\,[x', z'].
\end{aligned}
$$

It is natural to expect that $RT+UV-S^2$ will be an invariant for this transformation, and I find, in fact, that

$$R'T'+U'V'-S'^2 = \rho^2\,(RT+UV-S^2).$$

12. We have seen that, if we have a solution

$$u_1 = c_1, \quad u_2 = c_2, \quad u_3 = c_3$$

of Ampère's equation (1), the three expressions u_1, u_2, u_3 in terms of x, y, z, p, q are the principal variables in a tangential transformation. The question then arises: What will be the result of applying this tangential transformation to the equation? Let us then suppose that in the transformation just discussed x', y', z' are such that the equations

$$x' = c_1, \quad y' = c_2, \quad z' = c_3$$

would give a solution. Any two of the three functions x', y', z' must satisfy the equations (24), (25), which are clearly covariant, so that U' is the coefficient of $\dfrac{\partial\,(u_1, u_2)}{\partial\,(x', y')}$ in (24) after transformation, and therefore, on this supposition,

$$U' = 0.$$

We thus have the following theorem:—If a solution of Ampère's equation has been found involving three arbitrary constants, and if other solutions are sought by replacing these constants by parameters which are taken as a new set of principal variables, then the resulting differential equation is of Monge's form

$$Rr+2Ss+Tt = V.$$

This is Imschenetsky's result, given in Forsyth's *Treatise on Differential Equations*, § 271.

13. The two equations

$$
\begin{aligned}
&\alpha_{12}\,(p_1q_2-p_2q_1)+\alpha_{41}\,p_1+\alpha_{13}\,q_1+\alpha_{42}\,p_2+\alpha_{23}\,q_2+\alpha_{34} = 0, \\
&\beta_{12}\,(p_1q_2-p_2q_1)+\beta_{41}\,p_1+\beta_{13}\,q_1+\beta_{42}\,p_2+\beta_{23}\,q_2+\beta_{34} = 0,
\end{aligned}
$$

where the coefficients a, β are functions of x_1, x_2, x_3, x_4, and

$$p_1 = \frac{\partial x_3}{\partial x_1}, \quad q_1 = \frac{\partial x_4}{\partial x_1},$$

$$p_2 = \frac{\partial x_3}{\partial x_2}, \quad q_2 = \frac{\partial x_4}{\partial x_2},$$

may be treated in somewhat the same way as Ampère's equation. We may suppose without loss of generality that*

$$a_{12}a_{34} + a_{31}a_{24} + a_{23}a_{14} = 0,$$

and that either

$$\text{I.} \quad \beta_{12}\beta_{34} + \beta_{31}\beta_{24} + \beta_{23}\beta_{14} = 0$$

or

$$\text{II.} \quad a_{12}\beta_{34} + a_{34}\beta_{12} + a_{31}\beta_{24} + a_{24}\beta_{31} + a_{23}\beta_{14} + a_{14}\beta_{23} = 0,$$

since the given equations may be combined linearly in any way. Then the first of the two equations may be written

$$\begin{vmatrix} a_{12}p_1 - a_{32}, & a_{12}p_2 - a_{13} \\ a_{12}q_1 - a_{42}, & a_{12}q_2 - a_{14} \end{vmatrix} = 0,$$

and replaced by the following pair :—

$$a_{12}(p_1 + \lambda q_1) = a_{32} + \lambda a_{42},$$

$$a_{12}(p_2 + \lambda q_2) = a_{13} + \lambda a_{14}.$$

Hence, if

$$u_1 = c_1, \quad u_2 = c_2$$

constitute a solution, and, if we choose A_1, A_2 so that

$$A_1 \frac{\partial u_1}{\partial x_3} + A_2 \frac{\partial u_2}{\partial x_3} = a_{12},$$

$$A_1 \frac{\partial u_1}{\partial x_4} + A_2 \frac{\partial u_2}{\partial x_4} = \lambda a_{12},$$

we have

$$A_1 \frac{\partial u_1}{\partial x_1} + A_2 \frac{\partial u_2}{\partial x_1} = a_{23} + \lambda a_{24},$$

$$A_1 \frac{\partial u_1}{\partial x_2} + A_2 \frac{\partial u_2}{\partial x_2} = a_{31} + \lambda a_{41},$$

$$A_1 du_1 + A_2 du_2 = (a_{23}dx_1 + a_{31}dx_2 + a_{12}dx_3) + \lambda(a_{24}dx_1 + a_{41}dx_2 + a_{12}dx_4).$$

* It is convenient to assume that two coefficients a (or β) with the same suffixes in different orders are equal with opposite signs. Thus a_{11}, β_{11}, a_{22}, ... are zero.

In case I. the second equation may be treated in the same way as the first, and the converse theorem may be easily proved by reversing the argument as in the case of Ampère's equation (see § 4).

In either case, I. or II., we deduce from the second equation that

$$\sum_{i,j} \beta_{ij} \frac{\partial(u_1, u_2)}{\partial(x_i, x_j)} = 0,$$

which may be written *symbolically*

$$\sum_{i,} \beta_{ij} \frac{\partial(\Omega, u)}{\partial(x_i, x_j)} = 0,$$

where u is u_1 or u_2, and $d\Omega$ denotes $A_1 du_1 + A_2 du_2$.

Thus du is a linear combination of the determinants of the matrix of four columns

$$\left\| \begin{matrix} \ldots, & dx_j, & \ldots \\ \ldots, & \sum_{i} \beta_{ij} \dfrac{\partial\Omega}{\partial x_i}, & \ldots \end{matrix} \right\|.$$

In this matrix write the columns as rows, substituting the known expressions for $\dfrac{\partial\Omega}{\partial x_1}$, ..., in terms of λ. It thus becomes

$$\left\| \begin{matrix} dx_1, & \beta_{21}(a_{31}+\lambda a_{41}) + \beta_{31} a_{12} + \beta_{41} \lambda a_{12} \\ dx_2, & \beta_{12}(a_{23}+\lambda a_{24}) + \beta_{32} a_{12} + \beta_{42} \lambda a_{12} \\ dx_3, & \beta_{13}(a_{23}+\lambda a_{24}) + \beta_{23}(a_{31}+\lambda a_{41}) + \beta_{43} \lambda a_{12} \\ dx_4, & \beta_{14}(a_{23}+\lambda a_{24}) + \beta_{24}(a_{31}+\lambda a_{41}) + \beta_{34} a_{12} \end{matrix} \right\|,$$

and the following rows may be added, being merely combinations of those just written :—

$$\beta_{23} dx_1 + \beta_{31} dx_2 + \beta_{12} dx_3, \quad \lambda a_{12}(\beta_{23}\beta_{41} + \beta_{31}\beta_{42} + \beta_{12}\beta_{43}),$$

$$\beta_{24} dx_1 + \beta_{41} dx_2 + \beta_{12} dx_4, \quad a_{12}(\beta_{23}\beta_{14} + \beta_{31}\beta_{24} + \beta_{12}\beta_{34}),$$

$$a_{23} dx_1 + a_{31} dx_2 + a_{12} dx_3, \quad \lambda a_{12}(a_{23}\beta_{41} + a_{41}\beta_{23} + a_{31}\beta_{42} + a_{42}\beta_{31} + a_{12}\beta_{43} + a_{43}\beta_{12}),$$

$$a_{24} dx_1 + a_{41} dx_2 + a_{12} dx_4, \quad a_{12}(a_{23}\beta_{14} + a_{14}\beta_{23} + a_{31}\beta_{24} + a_{24}\beta_{31} + a_{12}\beta_{34} + a_{34}\beta_{12}).$$

Thus in case I. we are led to the same result as before; in case II. we find that du_1 and du_2 are linear combinations of the expressions

$$a_{23} dx_1 + a_{31} dx_2 + a_{12} dx_3,$$

$$a_{24} dx_1 + a_{41} dx_2 + a_{12} dx_4,$$

$$(\beta_{23} dx_1 + \beta_{31} dx_2 + \beta_{12} dx_3) + \lambda(\beta_{24} dx_1 + \beta_{41} dx_2 + \beta_{12} dx_4),$$

λ being the ratio of the coefficients of the first two expressions in that combination of them which is equal to a linear combination of du_1, du_2.

The resulting methods are then as follows :—

CASE I.—Find by inspection two functions u_1, u_2 such that one linear combination of their differentials is equal to a linear combination of

$$a_{23}\,dx_1+a_{31}\,dx_2+a_{12}\,dx_3, \quad a_{24}\,dx_1+a_{41}\,dx_2+a_{12}\,dx_4,$$

and another linear combination of their differentials to a linear combination of

$$\beta_{23}\,dx_1+\beta_{31}\,dx_2+\beta_{12}\,dx_3, \quad \beta_{24}\,dx_1+\beta_{41}\,dx_2+\beta_{12}\,dx_4\,;$$

then
$$u_1=c_1, \quad u_2=c_2$$
will constitute a solution.

CASE II.—Find by inspection two functions u_1, u_2, such that one linear combination of their differentials is equal to a linear combination of

$$a_{23}\,dx_1+a_{31}\,dx_2+a_{12}\,dx_3, \quad a_{24}\,dx_1+a_{41}\,dx_2+a_{12}\,dx_4,$$

say to
$$a_{23}\,dx_1+a_{31}\,dx_2+a_{12}\,dx_3+\lambda\,(a_{24}\,dx_1+a_{41}\,dx_2+a_{12}\,dx_4),$$

and that another linear combination of their differentials is equal to a linear combination of these same two expressions together with

$$\beta_{23}\,dx_1+\beta_{31}\,dx_2+\beta_{12}\,dx_3+\lambda\,(\beta_{24}\,dx_1+\beta_{41}\,dx_2+\beta_{12}\,dx_4)\,;$$

then
$$u_1=c_1, \quad u_2=c_2$$
will constitute a solution.

In either case all solutions of the form

$$u_1=c_1, \quad u_2=c_2$$

must satisfy the condition here specified.

The Abstract Group isomorphic with the Symmetric Group on k Letters. By Prof. L. E. Dickson, Ph.D. Received October 31st, 1899. Communicated November 9th, 1899.

At the annual meeting of the American Mathematical Society, August 25th and 26th, I presented a very elementary proof of the following theorem due to Prof. Moore (*Proc. Lond. Math. Soc.*, Vol. XXVIII., pp. 357–366):—

The abstract group $G(k)$ generated by the operators $B_1, B_2, \ldots, B_{k-1}$ with the generational relations

(1) $B_1^2 = B_2^2 = \ldots = B_{k-1}^2 = 1$,

(2) $B_i B_j = B_j B_i \quad (i = 1, 2, \ldots, k-3;\ j = i+2, i+3, \ldots, k-1)$,

(3) $B_j B_{j+1} B_j = B_{j+1} B_j B_{j+1} \quad (j = 1, 2, \ldots, k-2)$,

is simply isomorphic with the symmetric substitution-group on k letters.

The symmetric group $G_{k!}^{(k)}$ on the letters $l_1, l_2, \ldots, l_k$ may be generated by the transpositional substitutions

$$S_d \equiv (l_d\, l_{d+1}) \quad (d = 1, 2, k-1),$$

which satisfy the relations (1), (2), (3) prescribed for the generators B_d of the abstract group $G(k)$, and conceivably also other relations not derivable therefrom. The order $O(k)$ of $G(k)$ is therefore $\overline{\gtrless}\ k!$.

Denoting by G the sub-group $G(k-1)$ generated by $B_1, B_2, \ldots, B_{k-2}$, we consider the following sets of operators belonging to the group $G(k)$:

$$\begin{aligned} O_k &\equiv G, \\ O_{k-1} &\equiv GB_{k-1}, \\ O_{k-2} &\equiv GB_{k-1}B_{k-2}, \\ &\ldots \quad \ldots \quad \ldots \\ O_1 &\equiv GB_{k-1}B_{k-2}\ldots B_1. \end{aligned}$$

We proceed to show that these sets of operators are merely permuted amongst themselves by applying as right-hand multipliers the generators of $G(k)$. For example, B_1 as a right-hand multiplier

leaves unaltered the rows $O_k, O_{k-1}, \ldots, O_3$, in virtue of relation (2), but permutes O_2 with O_1, since $B_1^2 = 1$.

In general, B_r leaves fixed every row except O_{r+1} and O_r, which are permuted; viz.,

$$O_{r+1} B_r \equiv G B_{k-1} \ldots B_{r+1} B_r \equiv O_r,$$

$$O_r B_r \equiv G B_{k-1} \ldots B_r B_r = G B_{k-1} \ldots B_{r+1} \equiv O_{r+1}.$$

Further, if $i > r+1$, we find, on applying (2) to move B_r to the left of $B_i, \ldots, B_{k-1}$,

$$O_i B_r \equiv G B_{k-1} \ldots B_i B_r = G B_r B_{k-1} \ldots B_i \equiv O_i.$$

Lastly, if $i < r$, on moving B_r to the left of $B_i, B_{i+1}, \ldots, B_{r-2}$, by (2), we have

$$O_i B_r \equiv G B_{k-1} \ldots B_i B_r = G B_{k-1} \ldots B_{r+1} B_r B_{r-1} B_r B_{r-2} \ldots B_{i+1} B_i.$$

By (3), we may replace $B_r B_{r-1} B_r$ by $B_{r-1} B_r B_{r-1}$. We then move the first B_{r-1} to the left of $B_{r+1}, \ldots, B_{k-1}$ and merge it into G, giving

$$O_i B_r = G B_{k-1} \ldots B_{r+1} B_r B_{r-1} B_{r-2} \ldots B_i \equiv O_i.$$

Hence the right-hand multiplier B_r gives rise to the transposition $(O_r O_{r+1})$ on the k sets $O_1, \ldots, O_k$. It follows that the product of any operator of these k sets by an arbitrary operator of $G(k)$ is again an operator of these sets. Taking for the former operator the identity, we see that these sets include all the operators of the group $G(k)$. The number of operators in $G(k)$ is therefore at most k times the number in $G(k-1)$. Hence

$$O(k) \leqq k \cdot O(k-1) \leqq \ldots \leqq k!$$

Combining this result with the earlier one, we have $O(k) = k!$. The proof of the holohedric isomorphism of $G(k)$ and $G_{k!}^{(k)}$ is therefore complete.

As shown by the order of the group, the above k sets must be wholly distinct, so that we have in the table $O_1, \ldots, O_k$ a rectangular array for the abstract group $G(k)$.

From the standpoint of purely abstract group theory, it is interesting to verify from the relations (1), (2), (3) that the sets $O_1, \ldots, O_k$ are wholly distinct, and hence that the order $O(k) = k!$. Suppose, indeed, that an operator $G_1 B_{k-1} B_{k-2} \ldots B_r$ of the set O_r were identical with an operator $G_2 B_{k-1} B_{k-2} \ldots B_{r-s}$ of the set O_{r-s} $(s \geqq 1)$. It would

follow from (1) and (2) that

$$G_2^{-1}G_1 = B_{k-1} \dots B_{r-s} B_r \dots B_{k-1} = B_{k-1} \dots B_r B_{r-1} B_r \dots B_{k-1} B_{r-2} \dots B_{r-s},$$

on moving to the right $B_{r-2}, \dots, B_{r-s}$ over $B_r, \dots, B_{k-1}$. But we can verify by induction, using (3), that

$$B_{i+j} \dots B_{i+1} B_i B_{i+1} \dots B_{i+j} = B_i B_{i+1} \dots B_{i+j-1} B_{i+j} B_{i+j-1} \dots B_{i+1} B_i.$$

Indeed, supposing the formula true for a certain value of j, we have, by (2),

$$B_{i+j+1} \dots B_{i+1} B_i B_{i+1} \dots B_{i+j+1} = B_i \dots B_{i+j-1} (B_{i+j+1} B_{i+j} B_{i+j+1}) B_{i+j-1} \dots B_i.$$

Applying (3), this becomes

$$B_i \dots B_{i+j-1} B_{i+j} B_{i+j+1} B_{i+j} B_{i+j-1} \dots B_i,$$

so that the formula would be true for $j+1$. Applying this formula to our earlier result, we get

$$G_2^{-1}G_1 = B_{r-1} B_r \dots B_{k-2} B_{k-1} B_{k-2} \dots B_r B_{r-1} B_{r-2} \dots B_{r-s}.$$

Hence would B_{k-1} be expressed in terms of the operators $B_1, \dots, B_{k-2}$, contrary to its assumed independence of them.

Thursday, December 14th, 1899.

Prof. ELLIOTT, Vice-President, in the Chair, and subsequently Dr. MACAULAY and Dr. J. LARMOR.

Eight members present.

Prof. Jamshedji Edalji, Gujarat College, Ahmedabad; Prof. Wendell M. Strong, Yale College, Newhaven, U.S.A.; and Mr. Ronald W. H. T. Hudson, B.A. St. John's College, Cambridge, were elected members.

Mr. Tucker announced the recent decease of Major-General F. Close, R.A., who was elected a member April 13th, 1871.

The Auditor's report having been read, the adoption of the Treasurer's report, coupled with votes of thanks to the Treasurer and Auditor, was moved by Dr. Macaulay, seconded by Mr. W. F. Sheppard, and carried unanimously.

Mr. Sheppard communicated a portion of his papers entitled (1) "A Method for Extending the Accuracy of certain Mathematical Tables," (2) "Central Difference Formulæ."

Mr. Basset spoke on the subject of Circular Cubics, and Dr. Macaulay asked a question bearing upon his paper, entitled "The Theorem of Residuation, being a general treatment of the Intersections of Plane Curves at Multiple Points."

The following papers were communicated by reading their titles, viz.:—

The Genesis of the Double Gamma Functions, by Mr. E. W. Barnes.

On the Expression of Spherical Harmonics as Fractional Differential Coefficients, by Mr. J. Rose Innes.

Sums of Greatest Integers, by Mr. G. B. Mathews.

The following presents were made to the Library:—

"Educational Times," December, 1899.

"Indian Engineering," Vol. xxvi., Nos. 17-21, Oct. 21-Nov. 18, 1899.

"Bulletins de l'Académie Royale de Belgique," Année 67, Série 3, Tome xxxiv., 1897; Année 68, Série 3, Tome xxxv., 1898; Année 69, Série 3, Tome xxxvi., 1898; "Tables Générales du Recueil des Bulletins," Serie 3, Tomes i.-xxx., 1881-95; Bruxelles, 1898.

"Annuaire de l'Académie Royale de Belgique," 1898 and 1899; Bruxelles.

"Math.-naturwissenschaftliche Mitteilungen," 2te Serie, Band i., Heft 3; Notelets, viz., "Ueber das Theorem von Fagnano und die Chasles'chen Kreise," by Dr. O. Böklen; "Ergänzungen zu dem von E. Czuber gegebenen Litteraturverzeichnis ueber Wahrscheinlichkeitsrechnung," by Dr. E. Wölffing, also Answers, &c.; Stuttgart, October, 1899.

Offprints from the "Acta Societatis Scientiarum Fennicæ":—

Mellin, H. J.—"Ueber hypergeometrische Reihen höherer Ordnungen," Tom xxiii., No. 7.

Lindelöf, E.—"Sur la Forme des Intégrales des Equations différentielles," Tom xxii., No. 7.

Mellin, H. J.—"Zur Theorie zweier allgemeinen Klassen bestimmter Integrale," Tom xxii., No. 2.

The following exchanges were received:—

"Proceedings of the Royal Society," Vol. lxv., No. 421, 1899.

"Rendiconti del Circolo Matematico di Palermo," Tomo xiii., Fasc. 6; Nov., Dic., 1899.

"Bulletin of the American Mathematical Society," Series 2, Vol. vi., No. 2, November; New York, 1899.

"Bulletin des Sciences Mathématiques," Tome xxiii., October; Paris, 1899.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 2, Vol. viii., Fasc. 9, 10; Roma, 1899.

[redacted]

"Nyt Tidsskrift for Matematik," A. Aargang x., Nr. 8, B. Aargang x., Nr. 3; Copenhagen, 1899.

"Prace Matematyczno-Fizyczne," Tom x.; Warsaw, 1899–1900.

"Proceedings of the Cambridge Philosophical Society," Vol. x., Pt. 3; 1899.

"Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen," Heft 2; 1899.

"Proceedings of the Royal Irish Academy," Vol. v., No. 3; Dublin, 1899.

Sums of Greatest Integers. *By* G. B. MATHEWS. Received November 22nd, 1899. Communicated December 14th, 1899.

1. This note is submitted not so much for the sake of the results themselves—though they are curious enough—as on account of their connexion with a well known *crux* in the theory of numbers.

Let p be any odd prime, and let $r_1, r_2, \ldots, r_{p-1}$ be the least positive residues (mod p) of $1^2, 2^2, 3^2, \ldots, (p-1)^2$. Each distinct residue occurs twice, because $s^2 \equiv (p-s)^2$, so that $r_s = r_{p-s}$. The residues may be divided into two classes (α) and (β), according as they are less or greater than $\frac{1}{2}p$. For instance, if $p = 7$, the classes are

$$(\alpha)\quad 1,\ 2,$$

$$(\beta)\quad 4.$$

I shall write $n(\alpha)$, $n(\beta)$ for the number of residues in the classes (α), (β) respectively. In every case $n(\alpha)+n(\beta) = \frac{1}{2}(p-1)$; if $p \equiv 1 \pmod 4$, $n(\alpha) = n(\beta)$; but, if $p \equiv 3 \pmod 4$, $n(\alpha) > n(\beta)$, and the difference $n(\alpha)-n(\beta)$ is precisely the number of properly primitive forms of determinant $-p$. The difficulty is to prove this second result without the use of transcendental analysis; in what follows the difficulty is not removed, but an expression for $n(\alpha)-n(\beta)$ is given in a purely arithmetical shape which is quite

independent of the theory of forms, and may possibly suggest a new proof that $n(\alpha) > n(\beta)$, when $p \equiv 3 \pmod 4$.

2. The residue of x^2 will belong to the class (α) so long as

$$0 < x^2 < \tfrac{1}{2}p,$$

that is, so long as $$0 < x < \sqrt{\tfrac{1}{2}p}.$$

If we write $E\left(\frac{p}{2}\right)^{\frac{1}{2}}$ for the greatest integer in $\sqrt{\frac{1}{2}p}$, the series of residues begins with $E\left(\frac{p}{2}\right)^{\frac{1}{2}}$ numbers belonging to the class (α). These are followed by $E(p)^{\frac{1}{2}} - E\left(\frac{p}{2}\right)^{\frac{1}{2}}$ residues of class (β); these again by $E\left(\frac{3p}{2}\right)^{\frac{1}{2}} - E(p)^{\frac{1}{2}}$ residue of class (α), and so on alternately. If we observe that the last square taken into account is

$$(p-1)^2 = (p-2)\,p+1,$$

and that its residue belongs to the class (α), it will follow that we may write

$$2n(\alpha) = E\left(\frac{p}{2}\right)^{\frac{1}{2}} + E\left(\frac{3p}{2}\right)^{\frac{1}{2}} - E(p)^{\frac{1}{2}} + E\left(\frac{5p}{2}\right)^{\frac{1}{2}} - E(2p)^{\frac{1}{2}} + \dots$$

$$\dots + E\left(\frac{2p-5}{2}\,p\right)^{\frac{1}{2}} - E(\overline{p-3}\,.\,p)^{\frac{1}{2}}$$

$$+ (p-1) - E(\overline{p-2}\,.\,p)^{\frac{1}{2}},$$

$$2n(\beta) = E(p)^{\frac{1}{2}} - E\left(\frac{p}{2}\right)^{\frac{1}{2}} + E(2p)^{\frac{1}{2}} - E\left(\frac{3p}{2}\right)^{\frac{1}{2}} + \dots$$

$$\dots + E(\overline{p-2}\,.\,p)^{\frac{1}{2}} - E\left(\frac{2p-5}{2}\,p\right)^{\frac{1}{2}}.$$

Hence $$n(\alpha) - n(\beta) = X - Y,$$

where $$X = E\left(\frac{p}{2}\right)^{\frac{1}{2}} + E\left(\frac{3p}{2}\right)^{\frac{1}{2}} + \dots + E\left(\frac{2p-5}{2}\,p\right)^{\frac{1}{2}} + \frac{p-1}{2},$$

$$Y = E(p)^{\frac{1}{2}} + E(2p)^{\frac{1}{2}} + E(3p)^{\frac{1}{2}} + \dots + E(\overline{p-2}\,.\,p)^{\frac{1}{2}}.$$

[redacted]

3. As a numerical example, when $p = 11$ (omitting the sign E for brevity),

$$X = \left(\frac{11}{2}\right)^{\frac{1}{2}} + \left(\frac{33}{2}\right)^{\frac{1}{2}} + \left(\frac{55}{2}\right)^{\frac{1}{2}} + \ldots + \left(\frac{187}{2}\right)^{\frac{1}{2}} + 5$$

$$= 2+4+5+6+7+7+8+9+9+5 = 62,$$

$$Y = (11)^{\frac{1}{2}} + (22)^{\frac{1}{2}} + (33)^{\frac{1}{2}} + \ldots + (99)^{\frac{1}{2}}$$

$$= 3+4+5+6+7+8+8+9+9 = 59,$$

whence
$$X - Y = 3,$$

which is right, the distinct residues being 1, 3, 4, 5, 9.

The values of X, Y for the first few values of p are given by the table:—

$p =$	3	5	7	11	13	17	19,
$X =$	2	8	20	62	88	160	206,
$Y =$	1	8	19	59	88	160	203.

4. Expressions for Y may be deduced from formulæ given by J. Hacks (*Acta Math.*, Vol. x., p. 39); thus, when

$$p \equiv 1 \pmod 4,$$

$$Y = \tfrac{2}{3}(p-1)(p-2),$$

and X has then the same value. If

$$p \equiv 3 \pmod 4,$$

$$Y = \frac{(p-1)(4p-11)}{6} + \frac{2R}{p},$$

where
$$2R = r_1 + r_2 + \ldots + r_{p-1};$$

and hence, if $h(p)$ is the number of properly primitive classes of determinant $-p$,

$$X = \frac{(p-1)(4p-11)}{6} + \frac{2R}{p} + h(p).$$

As a numerical example, when $p = 19$, the residues are 1, 4, 5, 6, 7, 9, 11, 16, 17; whence

$$h(19) = 3, \quad 2R/19 = 8;$$

and therefore
$$X = \frac{18.65}{6} + 8 + 3 = 206,$$
$$Y = 203,$$
agreeing with the values calculated independently.

The Genesis of the Double Gamma Functions. *By* E. W. BARNES, B.A., Fellow of Trinity College, Cambridge. Received December 5th, 1899. Communicated December 14th, 1899.

1. The following paper is the natural sequence of results obtained in two previous papers.

The "Theory of the Gamma Function"* contained a discussion of the function defined by the formula

$$\frac{1}{e^{\gamma z}\,\Gamma(z+1)} = \prod_{m=1}^{\infty} \left\{ \left(1+\frac{z}{m}\right) e^{-\frac{z}{m}} \right\},$$

and it is evident that the expression on the right-hand side of this equality may be regarded as the positive half of the product expression for $\sin \pi z$; we may, in fact, term it the "halb-sinus" with Betti.†

Again, in the "Theory of the G Function,"‡ it was shown that

$$G(z) = e^{r(z)}\, z \prod_{m}{}_{0}^{\infty} \prod_{n}{}' \left\{ \left(1+\frac{z}{m+n}\right) e^{-\frac{z}{m+n}+\frac{z^2}{2(m+n)^2}} \right\},$$

where $r(z)$ is a quadratic function of z.

If now we can associate with the letter m, each time that it occurs in this product, a complex constant τ which is not real and negative, we shall obtain a product which may be regarded as the positive quarter of the product expression for Weierstrass's function $\sigma(z)$, and which will be therefore a natural extension of the Γ function.

---

* *Messenger of Mathematics*, Vol. XXIX., pp. 64 *et seq.*

† Klein (quoting Betti), *Ueber die hypergeometrische Function* (1894), p. 126.

‡ *Quarterly Journal of Mathematics*, Vol. XXXI., pp. 264 *et seq.*

Such a product we call a double gamma function $G(z \mid \tau)$. It is such that by suitable choice of the associated exponential factor, it satisfies the difference equation

$$f(z+1) = \Gamma\left(\frac{z}{\tau}\right) f(z).$$

It is evident that, when $\tau = 1$, the function reduces to the G function $G(z)$.

The existence of such functions has been surmised by Méray,* and indicated by Pincherle,† while Alexeiewsky‡ appears to have investigated some of their properties. The first two have not considered in detail any of the properties of these functions; and the last, so far as his results are accessible to me, does not appear to have entered into the essentials of the theory. He makes, for example, no mention of the gamma modular constants $C(\tau)$ and $D(\tau)$. The present notation was adopted before I had seen Alexeiewsky's paper, and his function $H(x, a)$ would be written $G(z \mid \tau)$ in the notation of this paper.

As indicated in the title, I only consider in the present paper the genesis of the double gamma functions. Several different product expressions are given for $G(z \mid \tau)$; the gamma modular constants are shown to be transcendental functions of τ; it is shown that $G(z \mid \tau)$ satisfies the two difference equations

$$f(z+1) = \Gamma\left(\frac{z}{\tau}\right) f(z),$$

$$f(z+\tau) = (2\pi)^{\frac{\tau-1}{2}} \tau^{-z+\frac{1}{2}} \Gamma(z) f(\tau);$$

and, finally, the connexion is indicated between these functions, Appell's generalization of the Eulerian functions, and the theta functions.

In a subsequent paper I propose to give in complete detail a symmetric theory of double gamma functions, in which τ is replaced by parameters ω_1 and ω_2, as in the theory of elliptic functions.

---

* Méray, *L'Analyse Infinitesimale*, Deuxième Partie, concluding pages.

† Pincherle, *Comptes Rendus*, Tome CVI., p. 266.

‡ Alexeiewsky, *Ann. de l'Imp. Univ. de Charkow*, 1889, as quoted in the *Jahrbuch über die Fortschritte der Mathematik*, Vol. XXII., p. 439. A synopsis of this paper appears in the *Leipzig Berichte*, 1894, Vol. XLVI., pp. 268–295.

2. We will first take the product

$$G(z \mid \tau) = Ae^{a\frac{z}{\tau}+b\frac{z^2}{2\tau^2}} \frac{z}{\tau} \overset{\infty}{\underset{m}{\Pi_0}} \underset{n}{\Pi'} \left\{ \left(1+\frac{z}{m\tau+n}\right) e^{-\frac{z}{m\tau+n}+\frac{z^2}{2(m\tau+n)^2}} \right\},$$

in which τ is any constant, real or complex, which is not real and negative, and in which a and b are functions of τ only.

We notice that each term of the product is of Weierstrass's form, and that, by Eisenstein's theorem, the product is absolutely convergent.

We proceed to transform this product into one of different form. Since we may group the terms of the product as we please, we have

$$G(z \mid \tau) = Ae^{a\frac{z}{\tau}+b\frac{z^2}{2\tau^2}} \frac{z}{\tau} \prod_{m=1}^{\infty} \left\{ \left(1+\frac{z}{m\tau}\right) e^{-\frac{z}{m\tau}+\frac{z^2}{2m^2\tau^2}} \right\}$$

$$\times \prod_{m=0}^{\infty} \prod_{n=1}^{\infty} \left\{ \frac{\left(1+\frac{z+m\tau}{n}\right)}{\left(1+\frac{m\tau}{n}\right)} \frac{e^{-\frac{z+m\tau}{n}}}{e^{-\frac{m\tau}{n}}} e^{-\frac{z}{m\tau+n}+\frac{z}{n}+\frac{1}{2}\frac{z^2}{(m\tau+n)^2}} \right\},$$

and hence, remembering that

$$\frac{1}{\Gamma(z)} = e^{\gamma z} z \prod_{m=1}^{\infty} \left\{ \left(1+\frac{z}{m}\right) e^{-\frac{z}{m}} \right\},$$

we see that

$$G(z \mid \tau) = Ae^{a\frac{z}{\tau}+b\frac{z^2}{2\tau^2}} \frac{z}{\tau} \frac{e^{-\frac{\gamma z}{\tau}}}{\Gamma\left(\frac{z}{\tau}+1\right)} e^{\frac{z^2}{2\tau^2}\sum_{m=1}^{\infty}\frac{1}{m^2}}$$

$$\times \prod_{m=0}^{\infty} \left\{ \frac{\Gamma(1+m\tau)}{\Gamma(1+z+m\tau)} e^{-\gamma z+z\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+m\tau}\right)+\frac{z^2}{2}\sum_{n=0}^{\infty}\frac{1}{(m\tau+n)^2}} \right\}$$

$$= Ae^{(a-\gamma)\frac{z}{\tau}+\frac{z^2}{2\tau^2}\left(b+\frac{\pi^2}{6}\right)} \Gamma^{-1}\left(\frac{z}{\tau}\right)$$

$$\times \prod_{m=0}^{\infty} \left\{ \frac{\Gamma(1+m\tau)}{\Gamma(1+z+m\tau)} e^{-\gamma z+z\sum_{n=1}^{\infty}\frac{m\tau}{n(n+m\tau)}+\frac{z^2}{2}\sum_{n=0}^{\infty}\frac{1}{(m\tau+n)^2}} \right\}.$$

But, from the product expression

$$\Gamma^{-1}(1+m\tau) = e^{\gamma m\tau} \prod_{n=1}^{\infty} \left\{ \left(1+\frac{m\tau}{n}\right) e^{-\frac{m\tau}{n}} \right\},$$

we obtain, if we put

$$\psi(x) = \frac{d}{dx}\log\Gamma(x)$$

in conformity with Gauss's notation,

$$-\psi(1+m\tau) = \gamma + \sum_{n=1}^{\infty} \left\{ \frac{1}{m\tau+n} - \frac{1}{n} \right\},$$

and, if

$$\psi'(x) = \frac{d}{dx}\psi(x),$$

$$\psi'(1+m\tau) = \sum_{n=1}^{\infty} \frac{1}{(m\tau+n)^2}.$$

Thus we may write

$$\Gamma\left(\frac{z}{\tau}\right) G(z \mid \tau) = Ae^{(a-\gamma)\frac{z}{\tau}+\frac{z^2}{2\tau^2}\left(b+\frac{\pi^2}{6}\right)}$$

$$\times \prod_{m=0}^{\infty} \left\{ \frac{\Gamma(1+m\tau)}{\Gamma(z+1+m\tau)} e^{z\psi(1+m\tau)+\frac{z^2}{2}\psi'(1+m\tau)} \right\}.$$

A form equivalent to this is incorrectly given by Alexeiewsky.

We may conveniently modify this expression slightly.

Since

$$\Gamma(z+1) = z\Gamma(z),$$

$$\psi(1) = -\gamma \quad \text{and} \quad \psi'(1) = \frac{\pi^2}{6},$$

we shall have

$$\Gamma\left(\frac{z}{\tau}\right) G(z \mid \tau) = Ae^{(a-\gamma)\frac{z}{\tau}+\frac{z^2}{2\tau^2}\left(b+\frac{\pi^2}{6}\right)} \frac{1}{\Gamma(z+1)} e^{-\gamma z+\frac{z^2}{2}\frac{\pi^2}{6}}$$

$$\times \prod_{m=1}^{\infty} \left\{ \frac{\Gamma(m\tau)}{\Gamma(z+m\tau)} \frac{m\tau}{z+m\tau} e^{\frac{z}{\tau m}+z\psi(\tau m)-\frac{z^2}{2\tau^2 m^2}+\frac{z}{2}\psi'(\tau m)} \right\},$$

and thus

$$\Gamma\left(\frac{z}{\tau}\right) G(z \mid \tau) = Ae^{(a-\gamma)\frac{z}{\tau}+\frac{z^2}{2\tau^2}\left(b+\frac{\pi^2}{6}\right)} e^{-\gamma z+\frac{z^2}{2}\frac{\pi^2}{6}-\frac{z^2}{2\tau^2}\frac{\pi^2}{6}}$$

$$\times \Gamma\left(\frac{z}{\tau}+1\right) e^{\gamma\frac{z}{\tau}} \prod_{m=1}^{\infty} \left\{ \frac{\Gamma(m\tau)}{\Gamma(z+m\tau)} e^{z\psi(m\tau)+\frac{z^2}{2}\psi'(m\tau)} \right\};$$

or, finally,

$$G(z \mid \tau) = \frac{A}{\Gamma(z+1)} e^{(a-\gamma)z+\frac{z^2}{2}\left(\frac{b}{\tau^2}+\frac{\pi^2}{6}\right)} \frac{z}{\tau} \prod_{m=1}^{\infty} \left\{ \frac{\Gamma(m\tau)}{\Gamma(z+m\tau)} e^{z\psi(m\tau)+\frac{z^2}{2}\psi'(m\tau)} \right\}.$$

It will be noted that this last product, as all employed in the transformation, is absolutely convergent.

Before we proceed to show that, for suitable values of a and b, the function $G(z \mid \tau)$ satisfies the difference equation

$$f(z+1) = \Gamma\left(\frac{z}{\tau}\right) f(z),$$

it is neccssary to interpolate two algebraical limit theorems analogous to those considered in the "Theory of the G Function," §§ 6 and 7.

3. We will first show that, when τ is not real and negative,

$$\underset{m=\infty}{\mathrm{Lt}} \left\{\psi(\tau) + \psi(2\tau) + \dots + \psi(m\tau)\right\}$$
$$= C(\tau) + (m + \tfrac{1}{2} - \tfrac{1}{2}\tau)\log m - m + \frac{1-2\tau}{12\tau^2 m} + \dots,$$

where $C(\tau)$ is a definite finite function of τ, independent of m. Unless the contrary is explicitly stated, the logarithms have their principal values, in which the imaginary part lies between $\pm\pi i$.

As there is no formula to express $\psi(\overline{m+1}\,\tau)$ in terms of $\psi(m\tau)$ for general values of τ, we cannot extend the method formerly employed. We therefore use the Maclaurin sum-formula,*

$$\Sigma u_x = C + \int u_x\,dx + \tfrac{1}{2}u_x + \tfrac{1}{12}\frac{du_x}{dx} - \tfrac{1}{720}\frac{d^3u_x}{dx^3} + \dots.$$

Put $u_x = \psi(\tau x)$, and we obtain

$$\sum_{x=1}^{m} \psi(\tau x) = C' + \frac{1}{\tau}\log\Gamma(\tau m) + \tfrac{1}{2}\psi(\tau m) + \frac{\tau}{12}\psi'(\tau m) + \dots.$$

Now, provided τ be not real and negative, we have, by Stieltjes' theorem,† when m is very large, the asymptotic equality

$$\log\Gamma(\tau m) = \tfrac{1}{2}\log 2\pi + (\tau m - \tfrac{1}{2})\log\tau m - \tau m + \frac{1}{12\tau m} - \dots;$$

and therefore the derived asymptotic equalities

$$\psi(\tau m) = \log\tau m - \frac{1}{2\tau m} - \frac{1}{12\tau^2 m^2}\dots,$$

$$\psi'(\tau m) = \frac{1}{\tau m} + \frac{1}{2\tau^2 m^2} + \dots.$$

---

* Boole, *Finite Differences*, § 2, p. 90.

† "Theory of the Gamma Function," Part IV.

Hence we have, when m is very large,

$$\psi(\tau)+\psi(2\tau)+\ldots+\psi(m\tau)$$

$$= C'+\log\frac{\sqrt{2\pi}}{\tau}+\frac{\tau m-\frac{1}{2}}{\tau}\log\tau m+\tfrac{1}{2}\log\tau m+\frac{1}{12\tau^2 m}-\frac{1}{4\tau m}+\frac{1}{12\tau m}+\ldots$$

$$= C(\tau)+\left(m+\tfrac{1}{2}-\frac{1}{2\tau}\right)\log\tau m-m+\frac{1-2\tau}{12\tau^2 m}+\ldots,$$

where $C(\tau)$ is a definite function of τ independent of m.

From the "Theory of the G Function," § 6, we see that

$$C(1)=\tfrac{1}{2}.$$

4. We will next show that, when τ is not real and negative, and m is a large positive integer,

$$\psi'(\tau)+\psi'(2\tau)+\ldots\ \psi'(m\tau)=D(\tau)+\frac{1}{\tau}\log\tau m+\frac{\tau-1}{2\tau^2 m}+\ldots,$$

where $D(\tau)$ is a definite function of τ independent of m.

On putting $u_x=\psi'(\tau x)$ in the Maclaurin sum formula, we have at once

$$\psi'(\tau)+\psi'(2\tau)+\ldots+\psi'(m\tau)$$

$$= D(\tau)+\frac{1}{\tau}\,\psi(m\tau)+\tfrac{1}{2}\psi'(m\tau)+\frac{\tau}{12}\psi''(m\tau)+\ldots.$$

Hence, using the asymptotic equalities,

$$\psi(\tau m)=\log\tau m-\frac{1}{2\tau m}-\frac{1}{12\tau^2 m^2}\ldots,$$

$$\psi'(\tau m)=\frac{1}{\tau m}+\frac{1}{2\tau^2 m^2}+\ldots,$$

we find

$$\psi'(\tau)+\psi'(2\tau)+\ldots+\psi'(m\tau)$$

$$= D(\tau)+\frac{1}{\tau}\log\tau m-\frac{1}{2\tau^2 m}+\ldots+\frac{1}{2\tau m}+\frac{1}{4\tau^2 m^2}+\ldots$$

$$= D(\tau)+\frac{1}{\tau}\log\tau m+\frac{\tau-1}{2\tau^2 m}+\ldots.$$

On making $\tau=1$, we see that

$$D(1)=1+\gamma.$$

5. The forms $C(\tau)$ and $D(\tau)$ will enter into the theory of double gamma functions from whatever side we may approach it. From the value which $D(\tau)$ assumes when $\tau = 1$, it might be anticipated that this function cannot be expressed in finite form or in terms of elementary transcendents. We proceed to show that this is actually the case; by an analogous process the same theorem might be proved to hold with regard to $C(\tau)$.

Suppose that m and n are large positive integers, and that $\frac{m}{n}$ is very small, and consider the function

$$\sum_{m_1=0}^{m} \sum_{n_1=0}^{n}{}' \frac{1}{(m_1\tau+n_1)^2},$$

the accent denoting that the term in the summation for which $\left.\begin{matrix} m_1 = 0 \\ n_1 = 0 \end{matrix}\right\}$ is to be omitted.

We have seen that

$$\psi'(1+m_1\tau) = \sum_{n_1=1}^{\infty} \frac{1}{(m_1\tau+n_1)^2} = \sum_{n_1=1}^{n} \frac{1}{(m_1\tau+n_1)^2} + \sum_{n_1=1}^{\infty} \frac{1}{(m_1\tau+n+n_1)^2},$$

and hence

$$\sum_{n_1=1}^{n} \frac{1}{(m_1\tau+n_1)^2} = \psi'(1+m_1\tau) - \psi'(1+m_1\tau+n),$$

so that, by the asymptotic equality used in § 3,

$$\begin{aligned} \sum_{n_1=1}^{n} \frac{1}{(m_1\tau+n_1)^2} &= \psi'(1+m_1\tau) - \frac{1}{1+m_1\tau+n} + \dots \\ &= \psi'(1+m_1\tau) - \frac{1}{n} + \frac{(\dots)}{n^2} + \dots, \end{aligned}$$

since m_1 is small compared with m. Thus

$$\begin{aligned} &\sum_{m_1=0}^{m} \sum_{n_1=0}^{n}{}' \frac{1}{(m_1\tau+n_1)^2} \\ &= \sum_{m_1=1}^{m} \frac{1}{(m_1\tau)^2} + \sum_{n_1=1}^{n} \frac{1}{n_1^2} + \sum_{m_1=1}^{m} \left\{ \psi'(1+m_1\tau) - \frac{1}{n} + \frac{(\dots)}{n^2} + \dots \right\} \\ &= \frac{\pi^2}{6} - \frac{1}{n} + \dots + \sum_{m_1=1}^{m} \left\{ \psi'(m_1\tau) - \frac{1}{n} + \frac{(\dots)}{n^2} + \dots \right\} \\ &= \frac{\pi^2}{6} + D(\tau) + \frac{1}{\tau}\log(\tau m) - \frac{m+1}{n} + \frac{\tau-1}{2\tau^2 m} + \dots . \end{aligned}$$

Now

$$\sum_{m_1=-m}^{m} \sum_{n_1=-n}^{n}{}' \frac{1}{(m_1\tau+n_1)^2}$$

$$= 2 \sum_{m_1=0}^{m} \sum_{n_1=0}^{n}{}' \frac{1}{(m_1\tau+n_1)^2} + 2 \sum_{m_1=0}^{m} \sum_{n_1=0}^{n}{}' \frac{1}{(-m_1\tau+n_1)^2}$$

$$- \sum_{m_1=1}^{m} \frac{1}{(m_1\tau)^2} - \sum_{n_1=1}^{n} \frac{1}{n^2},$$

as a graphical representation of the terms of the series will readily show.

Hence, when m and n are large positive integers, and τ is any complex quantity,

$$\sum_{m_1=-m}^{m} \sum_{n_1=-n}^{n}{}' \frac{1}{(m_1\tau+n_1)^2}$$

$$= \quad 2\left\{D(\tau)+\frac{\pi^2}{6}+\frac{1}{\tau}\log \tau m-\frac{\tau-1}{2\tau^2 m}-\frac{m+1}{n}\dots\right\}$$

$$+2\left\{D(-\tau)+\frac{\pi^2}{6}-\frac{1}{\tau}\log(\tau m)\mp\frac{\pi\iota}{\tau}+\frac{\tau+1}{2\tau^2 m}-\frac{m+1}{n}+\dots\right\}$$

$$-\frac{\pi^2}{6}\left(1+\frac{1}{\tau^2}\right)-\frac{1}{m\tau^2}-\frac{1}{n}-\dots$$

$$= \quad 2\left\{D(\tau)+D(-\tau)\right\}+\frac{\pi^2}{6}\left(1-\frac{1}{\tau^2}\right)\mp\frac{2\pi\iota}{\tau}$$

+ terms which vanish when m and n become infinite.

The upper or lower sign must be taken as $R(\iota\tau)$ is positive or negative. Now, by a theorem* due to Forsyth, when m and n become infinite, $\frac{m}{n}$ being small,

$$\mathrm{Lt} \sum_{m_1=-m}^{m} \sum_{n_1=-n}^{n}{}' \frac{1}{\left(m_1\frac{\iota K'}{K}+n_1\right)^2} = \left\{\frac{E}{K}-\tfrac{1}{3}(1+k^2)\right\} 4K^2,$$

---

* Forsyth, "Some Doubly Infinite Converging Series," *Quarterly Journal of Mathematics*, Vol. XXI., p. 263.

in the usual notation of elliptic functions. Hence, if

$$\tau = \frac{\iota K'}{K},$$

so that $R(\iota\tau)$ is negative,

$$D(\tau)+D(-\tau) = -\frac{\pi^2}{12} - \frac{\pi i}{\tau} + \frac{\pi^2}{12\tau^2} + 2EK - \tfrac{2}{3}K^2(1+k'^2),$$

an expression which is usually called a modular function of τ, and which does not in general admit of representation in finite form by elementary transcendents.

We propose then to call $C(\tau)$ and $D(\tau)$ double gamma modular functions of τ.

We shall subsequently express equivalent symmetrical functions as definite integrals.

6. We are now in a position to prove that, for suitable values of the τ-functions a and b,

$$G(z+1 \mid \tau) = \Gamma\left(\frac{z}{\tau}\right) G(z \mid \tau).$$

We have established in § 2 that, if

$$a' = a - \gamma\tau,$$

$$b' = b + \frac{\pi^2\tau^2}{6},$$

$$G(z \mid \tau) = \frac{A}{\tau\Gamma(z)} e^{a'\frac{z}{\tau} + b'\frac{z^2}{2\tau^2}} \prod_{m=1}^{\infty} \left\{ \frac{\Gamma(m\tau)}{\Gamma(z+m\tau)} e^{z\psi(m\tau) + \frac{z^2}{2}\psi'(m\tau)} \right\}.$$

Hence

$$\frac{G(z+1 \mid \tau)}{G(z \mid \tau)} = \frac{1}{z} e^{\frac{a'}{\tau} + b'\frac{2\tau+1}{2\tau^2}} \prod_{m=1}^{\infty} \left\{ \frac{1}{z+m\tau} e^{\psi(m\tau) + \frac{2z+1}{2}\psi'(m\tau)} \right\}$$

$$= \frac{1}{z} e^{\frac{a'}{\tau} + b'\frac{2z+1}{2\tau^2}} \prod_{m=1}^{\infty} \left\{ \frac{1}{\left(1+\frac{z}{m\tau}\right) e^{-\frac{z}{m\tau}}} \frac{e^{\frac{z}{m\tau} + \psi(m\tau) + \frac{2z+1}{2}\psi'(m\tau)}}{m\tau} \right\}$$

$$= \frac{1}{z} e^{\frac{a'}{\tau} + \frac{b'(2z+1)}{2\tau^2}} \Gamma\left(\frac{z}{\tau}+1\right) e^{\gamma\frac{z}{\tau}}$$

$$\times \operatorname*{Lt}_{m=\infty} \left\{ \frac{m^{-\frac{z}{\tau}}}{m!\,\tau^m} e^{-\frac{\gamma z}{\tau} + \sum_{r=1}^{m} \left[\psi(r\tau) + \frac{2z+1}{2}\psi'(r\tau)\right]} \right\},$$

and thus

$$\frac{G(z+1 \mid \tau)}{\Gamma\left(\frac{z}{\tau}\right) G(z \mid \tau)} = \frac{1}{\tau} e^{\frac{a'}{\tau}+\frac{\gamma z}{\tau}+\frac{b'}{2\tau^2}(2z+1)}$$

$$\times \underset{m=\infty}{\mathrm{Lt}} \left\{ \frac{1}{(2\pi)^{\frac{1}{2}}} m^{-m-\frac{1}{2}} \tau^{-m} e^{m-\frac{\gamma z}{\tau}} m^{-\frac{z}{\tau}} \right\}$$

$$\times \underset{m=\infty}{\mathrm{Lt}} \left\{ e^{C(\tau)+\left(m+\frac{1}{2}-\frac{1}{2\tau}\right) \log m\tau - m + (z+\frac{1}{2})[D(\tau)+\frac{1}{\tau}\log \tau m]} \right\},$$

on using the limits which have been investigated in §§ 3 and 4. Hence

$$\frac{G(z+1 \mid \tau)}{\Gamma\left(\frac{z}{\tau}\right) G(z \mid \tau)} = \frac{\tau^{\frac{z}{\tau}-\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}} e^{C(\tau)+\frac{1}{2}D(\tau)+\frac{a'}{\tau}+\frac{b'}{2\tau^2}+z\left[D(\tau)+\frac{b'}{\tau^2}\right]},$$

and thus we shall have for $G(z \mid \tau)$ the difference equation

$$G(z+1 \mid \tau) = \Gamma\left(\frac{z}{\tau}\right) G(z \mid \tau)$$

provided we choose a' and b' so that

$$D(\tau) + \frac{b'}{\tau^2} + \frac{1}{\tau} \log \tau = 0$$

and $$C(\tau) + \tfrac{1}{2} D(\tau) + \frac{a'}{\tau} + \frac{b'}{2\tau^2} = \tfrac{1}{2} \log (2\pi\tau)$$

and thus we must take

$$a' = \frac{\tau}{2} \log (2\pi\tau) + \tfrac{1}{2} \log \tau - \tau C(\tau),$$

$$b' = -\tau \log \tau - \tau^2 D(\tau);$$

or, finally, $$a = \frac{\tau}{2} \log (2\pi\tau) + \tfrac{1}{2} \log \tau - \tau C(\tau) + \gamma\tau,$$

$$b = -\tau \log \tau - \tau^2 D(\tau) - \frac{\pi^2\tau^2}{6}.$$

We have now, by § 2,

$$G(z \mid \tau) = \frac{Az}{\tau} e^{-zC(\tau)-\frac{z^2}{2}D(\tau)+\gamma z-\frac{\pi^2 z^2}{12}}$$

$$\times (2\pi\tau)^{\frac{z}{2}} \tau^{\frac{z-z^2}{2\tau}} \prod_{m=0}^{\infty} \prod_{n=0}^{\infty}{}' \left\{ \left(1+\frac{z}{\Omega}\right) e^{-\frac{z}{\Omega}+\frac{z^2}{2\Omega^2}} \right\},$$

where $$\Omega = m\tau + n.$$

When $\tau = 1$, we have, using the values of $C(1)$ and $D(1)$ given in §§ 3 and 4, respectively,

$$a = \gamma - \tfrac{1}{2} + \tfrac{1}{2}\log 2\pi,$$

$$b = -\left(\frac{\pi^2}{6} + 1 + \gamma\right),$$

and hence, when $\tau = 1$, we have

$$G(z \mid 1) = Az\,(2\pi)^{\frac{z}{2}}\, e^{z(\gamma - \frac{1}{2}) - \frac{z^2}{2}\left(\frac{\pi^2}{6} + 1 + \gamma\right)}$$

$$\times \prod_{m=0}^{\infty} \prod_{n=0}^{\infty}{}' \left\{ \left(1 + \frac{z}{m+n}\right) e^{-\frac{z}{m+n} + \frac{z^2}{2(m+n)^2}} \right\},$$

an expression which agrees with that previously found for $G(z)$.

As a *corollary*, note that we have incidentally proved that

$$\prod_{m=1}^{\infty} \left\{ \frac{1}{z + m\tau}\, e^{\psi(m\tau) + \frac{2z+1}{2}\psi'(m\tau)} \right\} = \Gamma\left(\frac{z}{\tau} + 1\right) \frac{\tau^{\frac{z}{\tau} + \frac{1}{2}}}{(2\pi)^{\frac{1}{2}}}\, e^{C(\tau) + (z + \frac{1}{2}) D(\tau)}.$$

7. We now proceed to determine the constant A by assigning the condition that

$$G(1 \mid \tau) = 1.$$

We have, from § 2,

$$G(z \mid \tau) = \frac{A}{\tau \Gamma(z)}\, e^{a' \frac{z}{\tau} + b' \frac{z^2}{2\tau^2}} \prod_{m=1}^{\infty} \left\{ \frac{\Gamma(m\tau)}{\Gamma(z + m\tau)}\, e^{z\psi(m\tau) + \frac{z^2}{2}\psi'(m\tau)} \right\},$$

and it has just been seen in § 6 that

$$a' = \frac{\tau}{2}\log 2\pi\tau + \tfrac{1}{2}\log\tau - \tau\, C(\tau),$$

$$b' = -\tau\log\tau - \tau^2 D(\tau).$$

Hence, if we make $z = 1$, and assign the condition $G(1 \mid \tau) = 1$, we find

$$1 = \frac{A}{\tau}\, e^{\frac{a'}{\tau} + \frac{b'}{2\tau^2}} \prod_{m=1}^{\infty} \left\{ \frac{1}{m\tau}\, e^{\psi(m\tau) + \frac{1}{2}\psi'(m\tau)} \right\}.$$

But, if we put $z = 0$ in the corollary to § 6, we have

$$\prod_{m=1}^{\infty} \left\{ \frac{1}{m\tau}\, e^{\psi(m\tau) + \frac{1}{2}\psi'(m\tau)} \right\} = \frac{1}{(2\pi)^{\frac{1}{2}}}\, \tau^{\frac{1}{2}}\, e^{C(\tau) + \frac{1}{2}D(\tau)}.$$

We thus find $A = 1$.

[redacted]

8. It is possible to give a third product expression for $G(z \mid \tau)$. To obtain this expression we take the formula of § 2,

$$G(z \mid \tau) = e^{a\frac{z}{\tau}+b\frac{z^2}{2\tau^2}}\frac{z}{\tau}\prod_{m=0}^{\infty}\prod_{n=0}^{\infty}{}' \left\{\left(1+\frac{z}{m\tau+n}\right)e^{-\frac{z}{m\tau+n}+\frac{z^2}{2(m\tau+n)^2}}\right\},$$

where (§ 6) $\qquad a = \frac{\tau}{2}\log 2\pi\tau + \frac{1}{2}\log\tau + \gamma\tau - C(\tau),$

$$b = -\tau\log\tau - \tau^2 D(\tau) - \frac{\pi^2\tau^2}{6},$$

and we write it in the form

$$G(z \mid \tau) = e^{a\frac{z}{\tau}+b\frac{z^2}{2\tau^2}}\frac{z}{\tau}\prod_{n=1}^{\infty}\left\{\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}+\frac{z^2}{2n^2}}\right\}$$

$$\times\prod_{n=0}^{\infty}\prod_{m=1}^{\infty}\left\{\frac{1+\dfrac{z+n}{m\tau}}{1+\dfrac{n}{m\tau}}\,\frac{e^{-\frac{z+n}{m\tau}}}{e^{-\frac{n}{m\tau}}}\,e^{\frac{z}{m\tau}-\frac{z}{m\tau+n}+\frac{z^2}{2(m+\tau n)^2}}\right\},$$

as we may obviously do, since each term is of Weierstrass's form; and now we have

$$G(z \mid \tau) = e^{a\frac{z}{\tau}+b\frac{z^2}{2\tau^2}}\frac{z}{\tau}e^{-\gamma z}\frac{1}{z\Gamma(z)}e^{\frac{\pi z^2}{12}}$$

$$\times\prod_{n=0}^{\infty}\left\{\frac{\Gamma\left(1+\dfrac{n}{\tau}\right)}{\Gamma\left(1+\dfrac{z+n}{\tau}\right)}e^{-\gamma\frac{z}{\tau}+z\sum\limits_{m=1}^{\infty}\left(\frac{1}{m\tau}-\frac{1}{m\tau+n}\right)+\frac{z^2}{2}\sum\limits_{m=1}^{\infty}\frac{1}{(m\tau+n)^2}}\right\}.$$

But, if, as usual, $\qquad \psi(z) = \frac{d}{dz}\log\Gamma(z),$

$$\psi'(z) = \frac{d^2}{dz^2}\log\Gamma(z),$$

we have $\qquad -\psi\left(1+\frac{n}{\tau}\right) = \gamma + \sum_{m=1}^{\infty}\left(\frac{\tau}{n+m\tau}-\frac{1}{m}\right),$

$$\psi'\left(1+\frac{n}{\tau}\right) = \sum_{m=1}^{\infty}\frac{\tau^2}{(n+m\tau)^2},$$

and hence

$$G(z \mid \tau) = e^{a\frac{z}{\tau}+b\frac{z^2}{2\tau^2}} \frac{1}{\tau\Gamma(z)} e^{-\gamma z+\frac{\pi^2 z^2}{12}}$$

$$\times \prod_{n=0}^{\infty} \left\{ \frac{\Gamma\left(1+\frac{n}{\tau}\right)}{\Gamma\left(1+\frac{z+n}{\tau}\right)} e^{\frac{z}{\tau}\psi\left(1+\frac{n}{\tau}\right)+\frac{z^2}{2\tau^2}\psi'\left(1+\frac{n}{\tau}\right)} \right\}$$

We may slightly modify this expression by writing

$$G(z \mid \tau) = e^{z\left(\frac{a}{\tau}-\gamma\right)+\frac{z^2}{2}\left(\frac{b}{\tau^2}+\frac{\pi^2}{6}\right)} \frac{1}{\tau\Gamma(z)\,\Gamma\left(1+\frac{z}{\tau}\right)} e^{\frac{z}{\tau}\psi(1)+\frac{z^2}{2\tau^2}\psi'(1)}$$

$$\times \prod_{n=1}^{\infty} \left\{ \frac{\Gamma\left(\frac{n}{\tau}\right)}{\Gamma\left(\frac{z+n}{\tau}\right)} \frac{n}{z+n} e^{\frac{z}{\tau}\psi\left(\frac{n}{\tau}\right)+\frac{z^2}{2\tau^2}\psi'\left(\frac{n}{\tau}\right)+\frac{z}{n}-\frac{z^2}{2n^2}} \right\},$$

and now, since

$$\psi(1) = -\gamma,$$

$$\psi'(1) = \frac{\pi^2}{6},$$

and

$$G(z+1 \mid \tau) = \Gamma\left(\frac{z}{\tau}\right) G(z \mid \tau),$$

we obtain, finally,

$$G(z+1 \mid \tau) = e^{z\left(\frac{a-\gamma}{\tau}\right)+\frac{z^2}{2\tau^2}\left(b+\frac{\pi^2}{6}\right)} \prod_{n=1}^{\infty} \left\{ \frac{\Gamma\left(\frac{n}{\tau}\right)}{\Gamma\left(\frac{z+n}{\tau}\right)} e^{\frac{z}{\tau}\psi\left(\frac{n}{\tau}\right)+\frac{z^2}{2\tau^2}\psi'(n)} \right\},$$

which yields, on substituting the values a and b,

$$G(z+1 \mid \tau) = (2\pi\tau)^{\frac{z}{2}} \tau^{\frac{z-z^2}{2\tau}} e^{z\left\{\gamma-\frac{\gamma}{\tau}-C(\tau)\right\}+\frac{z^2}{2}\left\{\frac{\pi^2}{6}(1-\tau^2)-D(\tau)\right\}}$$

$$\times \prod_{n=}^{\infty} \left\{ \frac{\Gamma\left(\frac{n}{\tau}\right)}{\Gamma\left(\frac{z+n}{\tau}\right)} e^{\frac{z}{\tau}\psi\left(\frac{n}{\tau}\right)+\frac{z^2}{2\tau^2}\psi'\left(\frac{n}{\tau}\right)} \right\}.$$

9. Recapitulating the results which have now been obtained, we see that a solution of

$$f(z+1) = \Gamma\left(\frac{z}{\tau}\right) f(z), \tag{1}$$

with the condition $$f(1) = 1,$$

is given by

$$G(z \mid \tau) = (2\pi\tau)^{\frac{z}{2}} \tau^{\frac{z-z^2}{2\tau}} e^{z\{\gamma\ C(\tau)\} - \frac{z^2}{2}\{\frac{\pi^2}{6}+D(\tau)\}} \frac{z}{\tau}$$

$$\times \prod_m^{\infty}{}_0 \prod_{n=0}^{\infty}{}' \left\{ \left(1 + \frac{z}{m\tau+n}\right) e^{-\frac{z}{m\tau+n}+\frac{z^2}{2(m\tau+n)^2}} \right\},$$

where $C(\tau)$ and $D(\tau)$ are certain double gamma modular constants.

The general solution of the difference equation (1) is

$$G(z \mid \tau) \times F(e^{2\pi i z}),$$

where $F(e^{2\pi i z})$ is any function of z simply periodic of period unity.

The function $G(z \mid \tau)$ may also be expressed as an infinite product of gamma functions of arguments differing by multiples of τ in the form

$$G(z \mid \tau) = (2\pi\tau)^{\frac{z}{2}} \tau^{\frac{z-z^2}{2\tau}} e^{-zC(\tau)-\frac{z^2}{2}D(\tau)} \frac{1}{\tau\Gamma(z)}$$

$$\times \prod_{m=1}^{\infty} \left\{ \frac{\Gamma(m\tau)}{\Gamma(z+m\tau)} e^{z\psi(m\tau)+\frac{z^2}{2}\psi'(m\tau)} \right\},$$

and again as an infinite product of gamma functions of arguments differing by multiples of $\frac{1}{\tau}$ in the form

$$G(z+1 \mid \tau) = (2\pi\tau)^{\frac{z}{2}} \tau^{\frac{z-z^2}{2\tau}} e^{z\{\gamma - \frac{\gamma}{\tau} - C(\tau)\} + \frac{z^2}{2}\{\frac{\pi^2}{6}(1-\tau^2) - D(\tau)\}}$$

$$\times \prod_{n=1}^{\infty} \left\{ \frac{\Gamma\left(\frac{n}{\tau}\right)}{\Gamma\left(\frac{z+n}{\tau}\right)} e^{\frac{z}{\tau}\psi(n\tau)+\frac{z^2}{2\tau^2}\psi'\left(\frac{n}{\tau}\right)} \right\}.$$

We might at this stage obtain the first terms of the value to which $G(z \mid \tau)$ tends, as z tends to real positive infinity, employing a method similar to that used in the theory of the G function, §§ 3 and 4. The results of such an investigation would, however, be incomplete, and it is therefore more convenient to employ the more powerful methods which will subsequently be adopted.

10. It is now possible for us to prove the fundamentally important theorem

$$G(z+\tau \mid \tau) = (2\pi)^{\frac{\tau-1}{2}} \tau^{-z+\frac{1}{2}} \Gamma(z)\, G(z \mid \tau),$$

where
$$\tau^{-z+\frac{1}{2}} = e^{(-z+\frac{1}{2}) \log \tau},$$

the principal value of the logarithm being taken. This theorem might be expected *a priori*; for we have seen that $G(z \mid \tau)$ satisfies the difference equation

$$G(z+1 \mid \tau) = \Gamma\left(\frac{z}{\tau}\right) G(z \mid \tau),$$

and we have also seen that $G(z \mid \tau)$ can be expressed as products of factors essentially characterized by $\Gamma\left(\frac{z+n}{\tau}\right)$ and $\Gamma(z+m\tau)$ respectively. And the former type bears the same relation to the second difference equation as does the latter type to the difference equation which we proceed to investigate.

Take the formula

$$G(z+\tau \mid \tau) = \frac{1}{\tau\Gamma(z+\tau)} e^{a'\frac{z+\tau}{\tau} + b'\frac{(z+\tau)^2}{2\tau^2}} \times \prod_{m=1}^{\infty} \left\{ \frac{\Gamma(m\tau)}{\Gamma(z+\tau+m\tau)} e^{(z+\tau)\psi(m\tau) + \frac{(z+\tau)^2}{2}\psi'(m\tau)} \right\},$$

and write it in the form

$$G(z+\tau \mid \tau) = \frac{1}{\tau} e^{a'\frac{z}{\tau} + b'\frac{z^2}{2\tau^2}} e^{a' + b'\frac{2\tau+\tau}{2\tau}} \times \underset{p=\infty}{\mathrm{Lt}} \prod_{m=1}^{p} \left\{ \frac{\Gamma(m\tau)}{\Gamma(z+m\tau)} e^{z\psi(m\tau) + \frac{z^2}{2}\psi'(m\tau)} \right\}$$
$$\times \underset{p=\infty}{\mathrm{Lt}} \left\{ \frac{e^{\tau \sum_{m=1}^{p} \psi(m\tau) + \tau\frac{2z+\tau}{2} \sum_{m=1}^{p} \psi'(m\tau)}}{\Gamma(z+\overline{p+1}\,\tau)} \right\}.$$

Then we shall have

$$\frac{G(z+\tau \mid \tau)}{\Gamma(z)\, G(z \mid \tau)} = e^{a' + b'\frac{2z+\tau}{2\tau}} \underset{p=\infty}{\mathrm{Lt}} \left\{ \frac{e^{\tau \sum_{m=1}^{p} \psi(m\tau) + \tau\frac{2z+\tau}{2} \sum_{m=1}^{p} \psi'(m\tau)}}{\Gamma(z+\overline{p+1}\,\tau)} \right\},$$

and, with the proviso that τ be not real and negative, which holds throughout the present investigation, the last written limit may be put in the form

$$\underset{p=\infty}{\mathrm{Lt}} \left\{ \frac{e^{\tau C(\tau) - p\tau + \tau\frac{2\tau+\tau}{2} D(\tau)} (\tau p)^{\tau p + \frac{\tau-1}{2} + \frac{2z+\tau}{2}}}{(2\pi)^{\frac{1}{2}} (z+\tau p+\tau)^{z+\tau p+\tau-\frac{1}{2}} e^{-z-\tau p-\tau}} \right\} = \frac{1}{(2\pi)^{\frac{1}{2}}} e^{\tau C(\tau) + \tau\frac{2z+\tau}{2} D(\tau)}.$$

[redacted]

We have therefore

$$\frac{G(z+\tau \mid \tau)}{\Gamma(z)\, G(z \mid \tau)} = \frac{1}{(2\pi)^{\frac{1}{2}}}\, e^{b'\frac{z}{\tau}+a'+\frac{b'}{2}+\tau C(\tau)+\tau\frac{2z+\tau}{2} D(\tau)},$$

and, on utilizing the values of a' and b' given in § 6, we find, finally,

$$\frac{G(z+\tau \mid \tau)}{\Gamma(z)\, G(z \mid \tau)} = \tau^{-z+\frac{1}{2}}\,(2\pi)^{\frac{\tau-1}{2}},$$

the result stated.

We note that the transcendental double gamma modular constants have disappeared from the final equation.

11. We proceed now to find the value of $G(\tau, \tau)$, and obtain Alexeiewsky's form of the second difference equation for $G(z \mid \tau)$.

Make $z = 0$ in the expression for $G(z \mid \tau)$ as a double product, and we have

$$\operatorname*{Lt}_{z=0}\left\{\frac{G(z \mid \tau)}{z}\right\} = \frac{1}{\tau};$$

and therefore $$\operatorname*{Lt}_{z=0}\left\{G(z \mid \tau)\,\Gamma(z)\right\} = \frac{1}{\tau}.$$

Make now $z = 0$ in the identity

$$\frac{G(z+\tau \mid \tau)}{\Gamma(z)\, G(z \mid \tau)} = \tau^{-z+\frac{1}{2}}\,(2\pi)^{\frac{\tau-1}{2}},$$

and we have $$\tau\, G(\tau \mid \tau) = (2\pi)^{\frac{\tau-1}{2}}\,\tau^{\frac{1}{2}},$$

so that $$G(\tau \mid \tau) = (2\pi)^{\frac{\tau-1}{2}}\,\tau^{-\frac{1}{2}}.$$

Substitute this value, and we have Alexeiewsky's formula

$$G(z+\tau \mid \tau) = \Gamma(z)\, G(z \mid \tau)\,\frac{G(\tau \mid \tau)}{\tau^{z-1}}.$$

We note that, when $$\tau = 1,$$

we have $$G(\tau, \tau) = 1,$$

and the equation just written becomes

$$G(z+1) = \Gamma(z)\, G(z).$$

12. We now see that $G(z \mid \tau)$ satisfies *two* difference equations

$$f(z+1) = \Gamma\left(\frac{z}{\tau}\right) f(z)$$

and $$f(z+\tau) = (2\pi)^{\frac{\tau-1}{2}} \tau^{-z+\frac{1}{2}} \Gamma(z) f(z \mid \tau).$$

It is this fact which leads to an entirely new conception of double gamma functions.

For, if we write $$\Psi(z \mid \tau) = \frac{d}{dz} \log G(z \mid \tau),$$

$$\Psi'(z \mid \tau) = \frac{d}{dz} \Psi(z \mid \tau),$$

we shall have for $\Psi(z \mid \tau)$ the two difference equations

$$f(z+1) = \frac{1}{\tau} \psi\left(\frac{z}{\tau}\right) + f(z),$$

$$f(z+\tau) = \psi(z) + f(z) - \log \tau,$$

where, as usual, $$\psi(z) = \frac{d}{dz} \log \Gamma(z),$$

and for $\Psi'(z \mid \tau)$ we have the difference equations

$$f(z+1) = f(z) + \frac{d^2}{dz^2} \log \Gamma\left(\frac{z}{\tau}\right),$$

$$f(z+\tau) = f(z) + \frac{d^2}{dz^2} \log \Gamma(z).$$

The symmetry of these equations suggests that we write

$$\tau = \frac{\omega_2}{\omega_1},$$

and take absolutely symmetrical difference equations

$$f(z+\omega_1) = f(z) - \psi_1^{(1)}(z \mid \omega_1),$$

$$f(z+\omega_2) = f(z) - \psi_1^{(1)}(z \mid \omega_2),$$

[where $$\psi_1^{(1)}(z \mid \omega_1) = \frac{d^2}{dz^2} \log \Gamma_1(z \mid \omega_1),$$

in the notation of the "Theory of the Gamma Function,"] from which to build up a symmetrical double gamma function. It is on such lines that I propose to develop the theory of the function in a subsequent paper.

13. It is advisable, however, while still retaining the present notation to connect the double gamma function with certain functions already introduced into analysis. With this object in

view we will consider the function

$$T(z \mid \tau) = G(z+1 \mid \tau)\, G(-z \mid -\tau).$$

We have the difference equation

$$G(z+1 \mid \tau) = \Gamma\left(\frac{z}{\tau}\right) G(z \mid \tau),$$

and hence we derive

$$G(-z+1 \mid -\tau) = \Gamma\left(\frac{z}{\tau}\right) G(-z \mid -\tau).$$

Thus $$T(z+1 \mid \tau) = T(z \mid \tau),$$

so that $T(z \mid \tau)$ is a function of z simply periodic of period unity.

Take next the second difference equation

$$G(z+\tau \mid \tau) = (2\pi)^{\frac{\tau-1}{2}}\, \tau^{-z+\frac{1}{2}}\, \Gamma(z)\, G(z \mid \tau).$$

We obtain at once

$$G(-z-\tau \mid -\tau) = (2\pi)^{\frac{-\tau-1}{2}}\, (-\tau)^{-z+\frac{1}{2}}\, \Gamma(-z)\, G(-z \mid -\tau).$$

Remembering that their principal values are always to be assigned to the many valued functions involved, we see that

$$\frac{T(z+\tau \mid \tau)}{T(z \mid \tau)} = \frac{1}{2\pi}\, \frac{\pi}{\sin \pi(z+1)}\, e^{\pm \pi\iota(z+\frac{1}{2})},$$

the upper or lower sign being taken as $R(\iota\tau)$ is positive or negative. Therefore

$$\frac{T(z+\tau \mid \tau)}{T(z \mid \tau)} = \frac{1}{1-e^{\mp 2\pi\iota z}},$$

with the same determination of the signs.

A simply periodic solution (of period unity) of this equation is

$$\prod_{m=0}^{\infty} \left\{1-e^{\mp 2\pi\iota(z+m\tau)}\right\},$$

and therefore $T(z \mid \tau)$ is included among the functions

$$P(z) \prod_{m=0}^{\infty} \left\{1-e^{\mp 2\pi\iota(z+m\tau)}\right\},$$

where $P(z)$ is an arbitrary doubly periodic function of z of periods 1 and τ.

Now $G(z \mid \tau)$ is an integral transcendental function of z with zeroes given by

$$z = -(m\tau+n), \begin{cases} m = 0, 1, \ldots, \infty, \\ n = 0, 1, \ldots, \infty; \end{cases}$$

and therefore $T(z \mid \tau) = G(z+1 \mid \tau)\, G(-z \mid -\tau)$

is a transcendental integral function of z with zeroes given by

$$z = -(m\tau+n), \quad \begin{cases} m = 0, 1, \ldots, \infty, \\ n = -\infty, \ldots, -1, 0, 1, \ldots, \infty, \end{cases}$$

as may be at once seen from a graphical representation of the zeroes of its factors.

But $\prod\limits_{m=0}^{\infty} \left\{1-e^{\mp 2\pi\iota(z+m\tau)}\right\}$ is a transcendental integral function with exactly these zeroes. And hence $P(z)$ is a doubly periodic function with no zeroes, and is therefore a constant.

Hence we may write

$$T(z \mid \tau) = K \frac{\prod\limits_{m=0}^{\infty} \left\{1-e^{\mp 2\pi\iota(z+m\tau)}\right\}}{\prod\limits_{m=1}^{\infty} \left\{1-e^{\mp 2\pi\iota m\tau}\right\}},$$

where K is independent of z.

Now
$$\operatorname*{Lt}_{z=0} \left\{ \frac{G(z \mid \tau)}{z} \right\} = \frac{1}{\tau}.$$

Hence
$$\operatorname*{Lt}_{z=0} \left\{T(z \mid \tau)\right\} = \operatorname*{Lt}_{z=0} \left\{ \frac{\tau}{\tau} \cdot \frac{z}{\tau} \right\} = \operatorname*{Lt}_{z=0} \left\{ \frac{z}{\tau} \right\},$$

and
$$\operatorname*{Lt}_{z=0} K \frac{\prod\limits_{m=0}^{\infty} \left\{1-e^{\mp 2\pi\iota(z+m\tau)}\right\}}{\prod\limits_{m=1}^{\infty} \left\{1-e^{\mp 2\pi\iota m\tau}\right\}} = \operatorname*{Lt}_{z=0} K \left\{1-e^{\mp 2\pi\iota z}\right\}$$
$$= \operatorname{Lt} \left\{ \pm 2\pi\iota z K \right\}.$$

Thus
$$K = \pm \frac{1}{2\pi\iota\tau},$$

and hence we obtain, finally,

$$T(z \mid \tau) = \pm \frac{1}{2\pi\iota\tau} \frac{\prod\limits_{m=0}^{\infty} \left\{1-e^{\mp 2\pi\iota(z+m\tau)}\right\}}{\prod\limits_{m=1}^{\infty} \left\{1-e^{\mp 2\pi\iota m\tau}\right\}}.$$

In the notation of Appell's generalized Eulerian functions,* we may

* Appell, "Generalisation des fonctions Eulériennes," *Math. Ann.*, Bd. XIX., pp. 84–102.

write this in the form

$$T(z \mid \tau) = \frac{1}{\tau} \frac{O(\mp z \mid 1, \mp\tau)}{\left\{\frac{d}{dz} O(\mp z \mid 1, \mp\tau)\right\}_{z=0}},$$

either the upper or the lower signs being taken throughout as $R(\iota\tau)$ is positive or negative, where

$$O(z \mid 1, \tau) = \prod_{m=0}^{\infty} \left\{1 - e^{2\pi i (z + m\tau)}\right\}.$$

The theorem just proved is the natural extension to double gamma functions of the relation

$$\Gamma(z)\, \Gamma(1-z) = \frac{\pi}{\sin z\pi}.$$

Note that the product

$$O(z \mid 1, \tau) = \prod_{m=0}^{\infty} \left\{1 - e^{2\pi\iota(z+m\tau)}\right\}$$

is only convergent, provided $R(\iota\tau)$ is positive and $|\tau| < 1$, or provided $R(\iota\tau)$ is negative and $|\tau| > 1$; while $T(z \mid \tau)$ expressed as a product of two double gamma functions is always convergent provided τ be complex.

14. We may, however, give a single infinite product for $T(z \mid \tau)$ which shall be valid for all complex values of τ.

For this purpose we take the expression

$$G(z \mid \tau) = \frac{1}{\tau\Gamma(z)} e^{a' \frac{z}{\tau} + b' \frac{z^2}{2\tau^2}} \prod_{m=1}^{\infty} \left\{ \frac{\Gamma(m\tau)}{\Gamma(z+m\tau)} e^{z\psi(m\tau) + \frac{z^2}{\tau}\psi'(m\tau)} \right\},$$

where (§ 6) $$a' = \frac{\tau}{2} \log 2\pi\tau + \tfrac{1}{2} \log \tau - \tau C(\tau),$$

$$b' = -\tau \log \tau - \tau^2 D(\tau),$$

and now, if we write

$$T(z \mid \tau) = \Gamma\left(\frac{z}{\tau}\right) G(z \mid \tau)\, G(-z \mid -\tau),$$

we obtain

$$T(z \mid \tau) = \frac{-\Gamma\left(\frac{z}{\tau}\right)}{\Gamma(z)\,\Gamma(-z)} e^{\frac{z}{\tau}[a'(\tau) + a'(-\tau)] + \frac{z^2}{2\tau^2}[b'(\tau) + b'(-\tau)]}$$

$$\times \prod_{m=1}^{\infty} \left\{ \frac{\Gamma(m\tau)\,\Gamma(-m\tau)}{\Gamma(z+m\tau)\,\Gamma(-z-m\tau)} e^{z[\psi(m\tau) - \psi(-m\tau)] + \frac{z^2}{2}[\psi'(m\tau) - \psi'(-m\tau)]} \right\}.$$

Now
$$\Gamma(z)\,\Gamma(-z) = \frac{-\pi}{z \sin z\pi},$$

and hence
$$\psi(z) - \psi(-z) = -\frac{1}{z} - \pi \cot z\pi,$$

$$\psi'(z) + \psi'(-z) = \frac{1}{z^2} + \frac{\pi^2}{\sin^2 \pi z}.$$

Thus we obtain

$$T(z \mid \tau) = \frac{\pi}{\tau z^2 \sin z\pi}\, e^{\frac{z}{\tau}[a'(\tau)+a'(-\tau)] + \frac{z^2}{2\tau^2}[b'(\tau)+b'(-\tau)] - \gamma\frac{z}{\tau} + \frac{\pi^2 z^2}{12\tau^2}}$$
$$\times \prod_{m=1}^{\infty} \left\{ \frac{\sin \pi (z+m\tau)}{\sin \pi m\tau}\, e^{-\pi z \cot m\pi\tau + \frac{\pi^2 z^2}{2 \sin^2 \pi m\tau}} \right\},$$

and, on substituting the values of a' and b', we have. finally,

$$T(z \mid \tau) = \frac{\pi}{z^2 \sin z\pi}\, \tau^{\frac{z}{\tau}-1}\, e^{\pm\frac{\pi\iota z}{2}\left(\frac{1}{\tau}-1\right) - \frac{\gamma z}{\tau} \pm \frac{\pi\iota z^2}{2\tau} + \frac{z^2\pi^2}{12\tau^2}}$$
$$\times e^{-z[C(\tau)-C(-\tau)] - \frac{z^2}{2}[D(\tau)+D(-\tau)]} \prod_{m=1}^{\infty} \left\{ \frac{\sin\pi(z+m\tau)}{\sin\pi m\tau}\, e^{-\pi z \cot m\pi\tau + \frac{\pi^2 z^2}{2\sin^2 \pi m\tau}} \right\},$$

an expression for $T(z \mid \tau)$ valid for all values of τ, except those which are entirely real. The upper or lower sign is to be taken as $R(\iota\tau)$ is positive or negative.

15. Let us finally consider the relation of the double gamma functions to the theta functions.

For this purpose we take

$$\Sigma(z \mid \tau) = T(z \mid \tau)\, T(z-\tau \mid -\tau)$$
$$= G(1+z \mid \tau)\, G(-z \mid -\tau)\, G(z-\tau+1 \mid -\tau)\, G(-z+\tau \mid \tau),$$

a product of four double gamma functions.

We have seen that $T(z \mid \tau)$ is a function of z simply periodic of period unity, and hence the same is true of $\Sigma(z \mid \tau)$. Thus

$$\Sigma(z \mid \tau) = \Sigma(z+1 \mid \tau).$$

Again, we have $\Sigma(z+\tau \mid \tau)$

$$= G(1+z+\tau \mid \tau)\, G(-z-\tau \mid -\tau)\, G(1+z \mid -\tau)\, G(-z \mid \tau),$$

and hence
$$\frac{\Sigma(z+\tau \mid \tau)}{\Sigma(z \mid \tau)}$$
$$=\frac{G(1+z+\tau \mid \tau)}{G(1+z \mid \tau)} \frac{G(-z \mid \tau)}{G(\tau-z \mid \tau)} \frac{G(-z-\tau \mid -\tau)}{G(-z \mid -\tau)} \frac{G(1+z \mid -\tau)}{G(1+z-\tau \mid -\tau)}.$$

But we have seen (§ 10) that
$$\frac{G(z+\tau \mid \tau)}{G(z \mid \tau)}=\Gamma(z)\, \tau^{-z+\frac{1}{2}}(2\pi)^{\frac{\tau-1}{2}},$$

and, since their principal values are always assigned to the many valued functions involved,
$$\frac{G(z-\tau \mid -\tau)}{G(z \mid -\tau)}=\Gamma(z)\, e^{\pm(-z+\frac{1}{2})\pi\iota}\, \tau^{-z+\frac{1}{2}}(2\pi)^{\frac{\tau-1}{2}},$$

the upper or lower sign being taken as $R(\iota\tau)$ is positive or negative. Hence
$$\frac{\Sigma(z+\tau \mid \tau)}{\Sigma(z \mid \tau)}=\frac{\Gamma(1+z)\, \tau^{-z-\frac{1}{2}}(2\pi)^{\frac{\tau-1}{2}}}{\Gamma(-z)\, \tau^{z+\frac{1}{2}}(2\pi)^{\frac{\tau-1}{2}}} \frac{\Gamma(-z)(e^{\pm\pi i}\tau)^{z+\frac{1}{2}}(2\pi)^{\frac{-\tau-1}{2}}}{\Gamma(1+z)(e^{\pm\pi i}\tau)^{-z-\frac{1}{2}}(2\pi)^{\frac{-\tau-1}{2}}}$$
$$= e^{\pm 2\pi\iota(z+\frac{1}{2})}$$
$$=-e^{\pm 2\pi i z}.$$

Thus we see that $\Sigma(z \mid \tau)$ satisfies the two difference equations characteristic of the theta functions
$$f(z+1)=f(z),$$
$$f(z+\tau)=-e^{\pm 2\pi\iota z} f(z).$$

Now it has been shown (§ 13) that
$$T(z \mid \tau)=\pm \frac{1}{2\pi\iota\tau} \frac{\prod_{m=0}^{\infty}\left\{1-e^{\mp 2\pi\iota(z+m\tau)}\right\}}{\prod_{m=1}^{\infty}\left\{1-e^{\mp 2\pi\iota m\tau}\right\}}.$$

From the reduction just obtained for $\dfrac{\Sigma(z+\tau \mid \tau)}{\Sigma(z \mid \tau)}$ we see that the function
$$T(+z-\tau \mid -\tau)=G(z-\tau+1 \mid -\tau)\, G(-z+\tau \mid \tau)$$

is such that
$$\frac{T(z \mid -\tau)}{T(z-\tau \mid -\tau)}=1-e^{\pm 2\pi i z},$$

a difference relation which can at once be obtained from the relation
$$\frac{T(z \mid \tau)}{T(z+\tau \mid \tau)}=1-e^{\mp 2\pi i z},$$

by merely changing τ into $-\tau$. For it is evident that such a change involves the opposite prescription for $R(\iota\tau)$.

We may obtain the same result and at the same time a useful verification of our formulæ if we take the difference equations

$$\frac{T(z \mid -\tau)}{T(z-\tau \mid -\tau)} = 1-e^{\pm 2\pi\iota z},$$

$$T(z-\tau+1 \mid -\tau) = T(z-\tau \mid -\tau),$$

and proceed as in § 13.

We readily find that

$$T(z-\tau \mid -\tau) = \pm\frac{1}{2\pi\iota\tau}\,\frac{\prod\limits_{m=1}^{\infty}\left\{1-e^{\pm 2\pi\iota(z-m\tau)}\right\}}{\prod\limits_{m=1}^{\infty}\left\{1-e^{\mp 2\pi\iota m\tau}\right\}},$$

for this expression satisfies the requisite functional relations; its zeroes are given by

$$z = m\tau+n, \quad \begin{cases} m = 1, 2, \ldots, \infty, \\ n = -\infty, \ldots, -1, 0, 1, \ldots, \infty, \end{cases}$$

just as are those of $G(-z+\tau \mid \tau)\, G(1+z-\tau \mid -\tau)$; and each side reduces to $\pm\dfrac{1}{2\pi\iota\tau}$ when $z=0$.

We now have

$$\Sigma(z \mid \tau) = \frac{e^{\mp 2\pi\iota z}-1}{(2\pi\tau)^2}\,\frac{\prod\limits_{m=1}^{\infty}\left\{\left(1-e^{\mp 2\pi\iota(z+m\tau)}\right)\left(1-e^{\pm 2\pi\iota(z-m\tau)}\right)\right\}}{\prod\limits_{m=1}^{\infty}\left\{\left(1-e^{\mp 2\pi\iota m\tau}\right)^2\right\}}.$$

Thus, if we put $q = e^{\mp\pi\iota\tau}$, the upper or lower sign being taken as $R(\iota\tau)$ is positive or negative, we find

$$\Sigma(z \mid \tau) = \frac{e^{\mp 2\pi\iota z}-1}{(2\pi\tau)^2}\prod_{m=1}^{\infty}\left\{\frac{1-2q^{2m}\cos 2\pi z+q^{4m}}{(1-q^{2m})^2}\right\}.$$

Assume now that $R(\iota\tau)$ is negative; then, with the notation of the theta functions adopted by Tannery and Molk, we have*

$$\vartheta_1(z) = 2q_0 q^{\frac{1}{4}}\sin z\pi\prod_{m=1}^{\infty}\left\{1-2q^{2m}\cos 2z\pi+q^{4m}\right\},$$

where $$q_0 = \prod_{m=1}^{\infty}\left\{1-q^{2m}\right\}.$$

---

* *Fonctions Elliptiques*, Tome II., p. 252.

Hence we see that

$$\Sigma(z \mid \tau) = \frac{e^{2\pi iz}-1}{(2\pi\tau)^2}\,\frac{\vartheta_1(z)}{2q_0^3 q^{\frac{1}{4}} \sin \pi z}$$

$$= \frac{ie^{\pi iz}}{(2\pi\tau)^2}\,\frac{\vartheta_1(z)}{q_0^3 q^{\frac{1}{4}}}$$

$$= \frac{ie^{\pi iz}}{2\pi\tau^2}\,\frac{\vartheta_1(z)}{\vartheta_1'(0)},$$

since*

$$\vartheta_1'(0) = 2\pi q_0^3 q^{\frac{1}{4}}.$$

Finally, then, when $R(\iota\tau)$ is negative,

$$\Sigma(z \mid \tau) = -\frac{\pi i e^{\pi iz}}{2(\log q)^2}\,\frac{\vartheta_1(z)}{\vartheta_1'(0)}.$$

In this manner we have expressed $\vartheta_1(z)$ as a product of four double gamma functions. And it is now evident that we may build up all four theta functions by means of the functions $G(z \mid \tau)$. And from quotients of such products of double gamma functions we may form the Jacobian elliptic functions sn z, cn z, and dn z.

At this stage we are naturally conducted to the consideration of the formation of Weierstrass's σ function, which is in essence a theta function symmetrical in ω_1 and ω_2—the two parameters whose quotient is τ. And such considerations lead to the formation of the analogous symmetrical double gamma function which will be discussed in a following paper.

The Theorem of Residuation, being a general treatment of the Intersections of Plane Curves at Multiple Points. *By* F. S. MACAULAY. Received and read December 14th, 1899.

I.

1. The following paper contains some developments of a theory which appears to be capable of considerable extension, and which is founded essentially, both as regards methods and applications, on

---

* *Fonctions Elliptiques*, Tome II., p. 257.

Noether's celebrated theorem.* Its aim is to deal in a perfectly general manner with the intersections of plane algebraic curves at common multiple points, no matter what may be the nature of the singularities of the curves and their contacts with one another at each common multiple point. For an introduction I may refer to Sections I. and II. of a former paper on the "Theorem of Residuation, &c." (*Proc. Lond. Math. Soc.*, Vol. XXXI., pp. 15–26). In the present paper I have supplied the proofs (§ 32) that were wanting in the earlier one, and have given in § 40 a geometrical solution of the problem of finding the general linear systems S and K (defined §§ 31, 32) when the cutting system C (including C') is any given pencil of curves. The most general problem in connexion with residuation on a base-curve is that of finding the same systems S and K when the cutting system C is any given linear system of curves. This problem is not solved in the paper; but I think I have succeeded in indicating most of the lines along which the solution of the general problem may be looked for.

2. We denote the general polynomial of the n^{th} order in two variables x, y by $\Sigma^n z_q^p x^p \eta^q$, where η stands for y/x, and p, q have all positive integral values, such that $n \geqslant p \geqslant q$. We call p the *index*, and q the *suffix*, of the coefficient z_q^p. The index p in z_q^p is not, of course, the index of a power; and no confusion need arise in this respect, since we have only to deal with *linear* relations among the coefficients. The notation z_q^p for the coefficients is convenient because it gives prominence to the index p, which is the order of the term of which z_q^p is the coefficient.

---

* "Ueber einen Satz aus der Theorie der algebraischen Functionen" (*Mathematische Annalen*, Vol. VI., 1873, pp. 351–359). Part of this paper (pp. 352, 353) gives an incomplete proof for the "simple" case of the theorem stated in § 12 below, depending on a method of merely counting the number of unknowns in a system of linear equations. Noether shows no reason why the method is applicable to the "simple" case, and to the "simple" case alone, being content with the assertion that it fails for all other cases. It is possible that, intending this part of the paper to be chiefly illustrative, he purposely omitted to complete the proof. The omission has, however, proved to be an unfortunate one, for the proof has been reproduced elsewhere as complete and sufficient, and important applications have been made of the results stated to follow from the reasoning, *e.g.*, in the proof of the Brill-Noether theorem of residuation (*Math. Ann.*, Vol. VII., p. 273), and in Clebsch-Lindemann (*Leçons sur la Géométrie*, Benoist's translation, Vol. II., 1880, p. 49, ll. 8–11). Picard and Simart (*Théorie des Fonctions Algébriques de deux variables indépendantes*, Vol. II., 1900, pp. 5–7), by deducing the particular from the general case, are careful to avoid the errors we have referred to.

3. If the curve $\Sigma^n z_q^p x^p \eta^q = 0$ passes through a given point x_1, y_1, there must be one equation among the coefficients, viz. $\Sigma^n z_q^p x_1^p \eta_1^q = 0$. We call this the *equation to the point* x_1, y_1, it being understood that n has as high a finite value as we please. We may then ask the question:—What are the conditions that a given equation satisfied by the coefficients $\Sigma \lambda_q^p z_q^p = 0$, may be the equation to a fixed point? An obvious answer is that there must be two fixed quantities x_1, η_1 such that λ_q^p bears a constant ratio to $x_1^p \eta_1^q$ for all values of p, q. But there is another answer, which we take as the *definition of the equation to a point*, viz., that the equation itself $\Sigma^n \lambda_q^p z_q^p = 0$ shall hold for the coefficients of $P\Sigma^n z_q^p x^p \eta^q$, P being any arbitrary polynomial in x, y. Suppose this condition to hold; and take $P = x$. Then $P\Sigma z_q^p x^p \eta^q = \Sigma z_q^p x^{p+1} \eta^q$. Hence we have $\Sigma \lambda_q^{p+1} z_q^p = 0$; and this must be the same as $\Sigma \lambda_q^p z_q^p = 0$. The ratio $\lambda_q^{p+1} : \lambda_q^p$ is therefore constant for all values of p, q. Similarly, taking $P = y = x\eta$, we find that the ratio $\lambda_{q+1}^{p+1} : \lambda_q^p$ must be constant for all values of p, q. Putting the former ratio equal to x_1, and the latter to $x_1\eta_1$, we have

$$\lambda_q^p = x_1^{p-q} \lambda_q^q = x_1^{p-q} (x_1\eta_1)^q \lambda_0^0 = x_1^p \eta_1^q \lambda_0^0.$$

This proves that the definition gives *sufficient* conditions; and it is obvious that the definition gives no other than *necessary* conditions.

4. We extend the definition of the equation to a fixed point in one respect, by saying that there may be any finite number of equations to the point, instead of one only; and we restrict the definition in another respect, by saying that the indices of the coefficients z_q^p entering the equations shall not exceed a certain definite number, independent of the order, n, of the polynomial, and consequently less than n if n is sufficiently great. In all other respects the definition is the same as in § 3, viz., that the equations to a point must be such that, if satisfied by the coefficients of $\Sigma z_q^p x^p \eta^q$, they are also satisfied by the coefficients of $P\Sigma z_q^p x^p \eta^q$. The effect of restricting the definition as above is that the point is brought to the origin, or rather that, for the purpose of dealing with the equations to the point, the origin is to be transferred to the point. But another important effect is that it evidently removes the necessity of our confining ourselves to polynomials, or finite power series. We shall, in fact, in much of the paper, deal generally with ordinary power series, which we shall understand as including, but not limited to, polynomials.

5. In investigating the equations to a point we take the origin at the point; but, regarding the equations to the point as supplying conditions for a curve, we may evidently again move the origin to any fixed position we please; and, if the equations to the point are known, we shall have known equations for the coefficients of the curve when referred to the fixed origin. We can thus, so to speak, collect the conditions supplied by different complex points for a curve, by moving the origin to the points in succession, and bringing it back to a fixed position. We call a point, regarded as supplying conditions for a curve, a *base-point;* and we retain the term as a convenient one, even when we are not directly dealing with its effect for a curve. The number of the independent equations to the point is called the *degree* of the base-point. Base-points of degree 1 are simple, or ordinary, points; and the degree of a base-point is, in any case, the number of ordinary points to which it is equivalent. A curve whose coefficients satisfy the equations to a base-point is said to *pass through* the base-point; and the order of the multiple point which any such curve must have at the point is called the *order* of the base-point. Any group of base-points (including ordinary points) is called a *point-base*, its *degree* being the sum of the degrees of its base-points, and its *order* being the order of the lowest curve which *passes through* all its base-points. We name the number of the ordinary points of intersection of two given curves which are absorbed at a point the *degree of their intersection** at the point. Thus the degree of the intersection of two curves having contact of order m at a point, but not having multiple points thereat, is $m+1$; and the degree of the intersection of two curves at a point, where they have respectively an i-fold and a j-fold point, is ij in the "simple" case.

6. The general character of the equations to a base-point situated at the origin is easily found from the definition (§ 4). Any such equation $\Sigma\lambda_q^p z_q^p = 0$ must not only hold for the general curve $\Sigma z_q^p x^p \eta^q = 0$ which passes through the base-point, but also for the curve $P\Sigma z_q^p x^p \eta^q = 0$, where P is an arbitrary polynomial. The equation must hold then for the curve

$$x^l \eta^m \Sigma z_q^p x^p \eta^q \equiv \Sigma z_{q-m}^{p-l} x^p \eta^q = 0,$$

* We do not use the term *multiplicity of the intersection* because we take *multiplicity* as meaning degree of variation, or order of infinity, or dimension.

l and m being any two fixed positive integers such that $l \geqslant m$. The equation $\Sigma\lambda_q^p z_{q-m}^{p-l} = 0$ must therefore be included in, or deducible from, the set of equations to the base-point. To repeat: in the equation $\Sigma\lambda_q^p z_{q-m}^{p-l} = 0$, l and m are two fixed positive integers such that $l \geqslant m$, and p, q have all positive integral values such that $n \geqslant p \geqslant q$. It is to be observed also that z_{q-m}^{p-l} is zero unless $p \geqslant l$, $q \geqslant m$, and $p-l \geqslant q-m$, *i.e.*, unless $p-l \geqslant q-m \geqslant 0$.

7. A set of equations which are all derivable from one equation $\Sigma\lambda_q^p z_q^p = 0$, viz., the whole set of equations $\Sigma\lambda_q^p z_{q-m}^{p-l} = 0$, where $l \geqslant m \geqslant 0$ and $n \geqslant p \geqslant q \geqslant 0$, we call a *one-set* system of equations, and the corresponding base-point a *one-set point;* while the equation $\Sigma\lambda_q^p z_q^p = 0$ is called the *prime equation* of the one-set. The equations to any base-point at the origin are then expressible by means of a comparatively small number, t, of independent one-set systems, viz., those whose prime equations are $\Sigma\lambda_q^p z_q^p = 0$, $\Sigma\mu_q^p z_q^p = 0$, &c. The aggregate of any t one-sets, which cannot be reduced to any less number of one-sets, we call a *t-set system* of equations, or a *t-set point*, as the case may be.

8. To find the equations to the whole base-point common to any number of given curves

$$C_1 \equiv \Sigma a_q^p x^p \eta^q = 0, \quad C_2 \equiv \Sigma b_q^p x^p \eta^q = 0, \quad \dots$$

at the origin it would suffice to find the general solution of the following equations, the λ's being the unknowns,

$$\Sigma\lambda_q^p a_q^p = 0, \quad \Sigma\lambda_q^p b_q^p = 0, \quad \dots,$$

and their derivatives

$$\Sigma\lambda_q^p a_{q-m}^{p-l} = 0, \quad \Sigma\lambda_q^p b_{q-m}^{p-l} = 0, \quad \dots .$$

All that can be asserted as regards the λ's is that, provided $C_1, C_2, \dots$ have no common factor which vanishes at the origin, λ_q^p is necessarily zero when p exceeds a certain unknown finite limit. The general values of the λ's which do not vanish may involve a large number of independent linear arbitrary parameters. If these general values be substituted in the equation $\Sigma\lambda_q^p z_q^p = 0$, the coefficients of the independent parameters of the λ's must all be zero; and these coefficients, equated to zero, give the required independent equations to the whole base-point common to $C_1, C_2, \dots$ at the origin. It seems extremely

difficult, by a direct investigation of such a system of equations, to prove any properties such as those mentioned in § 9; but it is useful to keep the method in mind. It is evident that the required equations to the base-point are the equations which are identically satisfied by the coefficients of $C_1P_1+C_2P_2+\ldots$, where $P_1, P_2, \ldots$ are arbitrary power series; for the equations hold separately for $C_1P_1, C_2P_2, \ldots$, and therefore for $C_1P_1+C_2P_2+\ldots$; and, conversely, any identical equations which hold for $C_1P_1+C_2P_2+\ldots$ must hold separately for $C_1P_1, C_2P_2, \ldots$, since all except any one of the power series $P_1, P_2, \ldots$ may be chosen zero.

9. Section II. of the paper contains the proofs of three general theorems. Theorem I. states that the degree of the whole base-point common to two given curves C_1, C_2 at the origin is the same as the degree of the intersection of C_1, C_2 at the origin. Theorem III. states that this base-point is a one-set point; and the converse Theorem II. states that any given one-set point is the whole intersection of two fixed curves at the point.

Although the enunciations of these theorems are capable of concise and unambiguous expression, I have not been able to find any simple proofs for them. The length and intricacy of the proofs may be partly accounted for by the absolute generality of the theorems.

10. Sections III. and IV. contain applications of the theorems proved in Section II. In §§ 32, 33 it is proved that a composite curve SS' passes through a given one-set point if S passes through any t-set point contained in the one-set point, and S' passes through a certain residual t'-set point, also contained in the one-set point, the sum of the degrees of the two residual base-points being equal to the degree of the given one-set point. If S is any curve of a given system possessing a multiple point at the origin, where the one-set point is situated, then the base-point of highest degree on S which is contained in the one-set point will in general vary with S. In order that SS' may pass through the one-set point S' has only to pass through the corresponding residual base-point, which is also variable. In this case the variation in S' is dependent to some extent on the variation of S; and, if the given system S is linear, the system S' will not be linear in the parameters of S. (*Cf. Proc. Lond. Math. Soc.*, Vol. XXXI., p. 17, footnote.) In the more easily conceivable case in which the systems S, S' are independent of one another, as far as their variability is concerned, S and S' must pass through two *fixed*

base-points respectively, which are residual with respect to the one-set point.

In § 34 a theorem of residuation for base-points is proved, and in § 36 a proof of the general theorem of residuation for point-bases is given. The results of some other applications are collected in § 38; and § 40 gives the solution of the problem mentioned in § 1.

11. The paper deals almost exclusively with one-set points and their division into pairs of residual base-points. It does not contain any general discussion of t-set points. The following theorems, however, which are extensions of those mentioned in § 9, are proved in §§ 41–45.

(i.) Through any given t-set point at the origin $t+1$ fixed curves can be drawn, and no less number, which have no base-point in common at the origin beyond the given t-set point.

(ii.) The whole common base-point of any given $t+1$ curves $C_0, C_1, C_2, \ldots, C_t$ at the origin is a t-set point provided no one of the given curves passes through the whole base-point common to the rest at the origin, *i.e.*, provided the identity

$$C_0 S_0 + C_1 S_1 + C_2 S_2 + \ldots + C_t S_t \equiv 0$$

cannot exist without $S_0, S_1, S_2, \ldots, S_t$ all vanishing at the origin.

(iii.) Any two curves drawn through a given t-set point at the origin, which both belong to a set of $t+1$ curves such as that described in (i.), will intersect for the rest at the origin in a $(t-1)$-set point.

Also the remaining intersection of the two curves at the origin is a t-set point if one, but not both, of the curves belongs to such a set of $t+1$ curves, and a $(t+1)$-set point if neither of the curves belongs to such a set. Thus a t-set and a t'-set point cannot be residual unless t and t' are equal or differ by unity. Also, from any given t-set point a chain of residual base-points can be derived, consisting of the t-set point, a $(t-1)$-set point, a $(t-2)$-set point, and so on to a one-set point; and, by reversing the chain, the t-set point can be built up from the one-set point.

One-set points are by no means the simplest kind of base-points. They form rather an extreme case, the other extreme being a t-set point whose order is t. The simplest t-set point of order t is that

which has for its prime equations $z_0^{t-1} = 0,\ z_1^{t-1} = 0,\ \ldots,\ z_{t-1}^{t-1} = 0$, giving $z_q^p = 0$, when $t > p$, as the general derivative. To this t-set point we have previously given the name of an *ordinary t-point* (*l.c.*, Vol. XXXI., p. 19, footnote).

II. General Theorems.

12. Theorem I.—*If C_1, C_2 are two given polynomials, having no common factor, and P_1, P_2 arbitrary power series, then the number of terminating independent linear equations identically satisfied by the coefficients of S, where $S \equiv C_1P_1 + C_2P_2$, is equal to the degree of the intersection of the curves $C_1 = 0$, $C_2 = 0$ at the origin.*

In other words:—*The degree of the whole base-point common to C_1, C_2 at the origin is equal to the degree of the intersection of C_1, C_2 at the origin.* For we have already seen (§ 8) that the equations which are identically satisfied by the coefficients of S are the equations to the base-point common to C_1, C_2 at the origin.

By *terminating* equations we mean those equations which do not include any coefficients of terms beyond a certain unknown, but finite, order. The theorem states that there is a *definite* number of independent terminating equations for the coefficients of S, depending on the given polynomials C_1, C_2. If the coefficients of any given polynomial or power series S_1 satisfy all the terminating equations which exist for the coefficients of $C_1P_1 + C_2P_2$, P_1, P_2 being arbitrary, then S_1 is of the form $C_1P_1 + C_2P_2$ as far as terms of any finite order,* *i.e.*, P_1, P_2 can be so chosen that $S_1 \doteq C_1P_1 + C_2P_2$.

13. Let
$$C_1 \equiv \Sigma a_q^p x^p \eta^q \equiv a_0 + a_1 y + a_2 y^2 + \ldots + a_l y^l,$$
$$C_2 \equiv \Sigma b_q^p x^p \eta^q = b_0 + b_1 y + b_2 y^2 + \ldots + b_m y^m,$$
$$S \equiv \Sigma z_q^p x^p \eta^q \equiv z_0 + z_1 y + z_2 y^2 + \ldots;$$

---

* Equating coefficients in $C_1P_1 + C_2P_2 \equiv S_1$, as far as terms of any finite order, we have equations of the kind $z_q^p = a_q^p$, where the z's are the coefficients of the terms in $C_1P_1 + C_2P_2$, *i.e.*, are linear functions of the unknown coefficients of P_1, P_2, and the a's are the given coefficients of S_1. The only case in which these equations are not consistent, *i.e.*, do not admit of solution, is when certain constants λ exist, such that $\Sigma\lambda_q^p z_q^p$ is identically zero, while $\Sigma\lambda_q^p a_q^p$ is not zero. This case is excluded by hypothesis.

From the fact that, if S_1 passes through the base-point common to C_1, C_2 at the origin, we have $S_1 \equiv C_1P_1 + C_2P_2$, we may state Noether's theorem in the geometrical form:—*A curve S_1 which passes through all the base-points (including all the ordinary points) common to two given curves C_1, C_2 is of the form $S_1 \equiv C_1S' + C_2S'' = 0$, where S', S'' are polynomials.*

so that
$$a_n \equiv a_n^n + a_n^{n+1}x + a_n^{n+2}x^2 + \dots + a_n^l x^{l-n},$$
$$b_n \equiv b_n^n + b_n^{n+1}x + b_n^{n+2}x^2 + \dots + b_n^m x^{m-n},$$
$$z_n \equiv z_n^n + z_n^{n+1}x + z_n^{n+2}x^2 + \dots .$$

If the directions of the axes are changed, without altering the origin, and if S, C_1, C_2, P_1, P_2 in consequence change to S', C_1', C_2', P_1', P_2', we shall have $S' \equiv C_1'P_1' + C_2'P_2'$, where P_1', P_2' are arbitrary. The independent identical equations satisfied by the coefficients of S and S' correspond one-to-one, and are therefore equal in number. Also the degree of the intersection of C_1', C_2' at the origin is the same as that of C_1, C_2. Hence, if the theorem is true for S', C_1', C_2', it will also be true for S, C_1, C_2. It will be sufficient then to prove the theorem when the directions of the axes are general. We may therefore assume that the axes do not touch C_1 or C_2 at the origin, and do not pass through any common point of C_1, C_2 except the origin. Hence, if C_1 has an i-fold point, and C_2 not less than an i-fold point, at the origin, then a_0, a_1, ..., a_{i-1} are divisible by x^i, x^{i-1}, ..., x respectively, while a_i, a_{i+1}, a_{i+2}, ... are not divisible by x; and the same is true for z_0, z_1, z_2, ... Also a_l or a_l^l, and b_m or b_m^m, are non-vanishing constants.

Owing to the directions of the axes being chosen generally, the degree of the intersection of C_1, C_2 at the origin is the index of the power of x that can be divided out of the y-resultant of C_1, C_2, that is, out of the determinant formed by m rows of a's, and l rows of b's, viz.,

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 & \dots & 0 \\ 0 & a_0 & a_1 & a_2 & \dots & 0 \\ 0 & 0 & a_0 & a_1 & \dots & 0 \\ \dots & & \dots & & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & a_l \\ b_0 & b_1 & b_2 & b_3 & \dots & 0 \\ 0 & b_0 & b_1 & b_2 & \dots & 0 \\ 0 & 0 & b_0 & b_1 & \dots & 0 \\ \dots & & \dots & & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & b_m \end{vmatrix}.$$

We have to prove that this index is equal to the number of inde-

pendent linear equations identically satisfied by the coefficients x_q^r of S.

14. In this determinant, if we choose multipliers, which are polynomials in x, for the $(i+1)^{\text{th}}$ and succeeding columns, such that the sum of the products in each row may be divisible by x, then each of the multipliers must be divisible by x. For, if the constant terms of the multipliers are not all zero, it can be proved, by a method similar to that of § 15, that the two curves C_1, C_2 must have a common point $x = 0$, $y \neq 0$; and this case is excluded (§ 13).

15. Multiply the columns of the y-resultant after the $(k+1)^{\text{th}}$ $(k < i)$ by polynomials in x with constant coefficients, viz., $\theta_{k+1}, \theta_{k+2}, \ldots, \theta_{l+m-1}$ and add to the $(k+1)^{\text{th}}$ column. Let the θ's be so chosen that as high a power of x as possible, say x^r, shall be a factor of all the elements of the new $(k+1)^{\text{th}}$ column. We shall then have the following expressions all divisible by x^r:—

$$a_k + a_{k+1}\theta_{k+1} + \ldots + a_l\theta_l$$

$$a_{k-1} + a_k\theta_{k+1} + \ldots + a_{l-1}\theta_l + a_l\theta_{l+1}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_0\theta_{m-1} + \ldots\ldots\ldots\ldots + a_l\theta_{l-m-1}$$

A ——————————————————————————

$$a_0\theta_m + \ldots\ldots\ldots\ldots + a_l\theta_{l+m}$$

$$a_0\theta_{m+1} + \ldots\ldots\ldots\ldots + a_l\theta_{l+m+1}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

B ——————————————————————————

$$b_k + b_{k+1}\theta_{k+1} + \ldots\ldots\ldots\ldots + b_m\theta_m$$

$$b_{k-1} + b_k\theta_{k+1} + \ldots\ldots\ldots\ldots + b_{m-1}\theta_m + b_m\theta_{m+1}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$b_0\theta_{l-1} + \ldots\ldots\ldots\ldots + b_m\theta_{l+m-1}$$

A' ——————————————————————————

$$b_0\theta_l + \ldots\ldots\ldots\ldots + b_m\theta_{l+m}$$

$$b_0\theta_{l+1} + \ldots\ldots\ldots\ldots + b_m\theta_{l+m+1}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

——————————————————————————

Only those expressions above the horizontal A, and between the horizontals B and A', appear as elements in the $(k+1)^{\text{th}}$ column of the y-resultant. To these we may clearly add the expressions between the horizontals A and B, since in each a new θ is added with a non-vanishing constant coefficient $a_l\,(= a_l')$. Now multiply the

first $(m+1)$ lines of the a's by $b_0, b_1, \ldots, b_m$ respectively, and the first l lines of the b's by $-a_0, -a_1, \ldots, -a_{l-1}$, and add up. We thus get the next line of the b's (the first below the horizontal A') multiplied by the constant a_l. Again, multiply $(m+1)$ lines of the a's, beginning with the second line, by $b_0, b_1, \ldots, b_m$, and l lines of the b's, beginning with the second, by $-a_0, -a_1, \ldots, -a_{l-1}$; then, adding up, we get the next line of the b's; and so on. We may therefore add as many lines of a's after A, and of b's after A', as we please.

Again, leaving out all the terms in all the lines which are known to be divisible by x, viz., the terms containing $a_0, a_1, \ldots, a_{i-1}$, $b_0, b_1, \ldots, b_{i-1}$, we see that

$$\begin{array}{r} a_i\theta_i + a_{i+1}\theta_{i+1} + \ldots, \\ a_i\theta_{i+1} + \ldots, \\ \ldots \quad \ldots \quad \ldots \quad \ldots, \\ b_i\theta_i + b_{i+1}\theta_{i+1} + \ldots, \\ b_i\theta_{i+1} + \ldots, \\ \ldots \quad \ldots \quad \ldots \quad \ldots \end{array}$$

all divide by x. Hence $\theta_i, \theta_{i+1}, \theta_{i+2}, \ldots$ all divide by x (§ 14). Again beginning with the second line of a's and second line of b's, and leaving out all terms now known to divide by x^2, we have

$$\begin{array}{r} a_i\theta_{i+1} + a_{i+1}\theta_{i+2} + \ldots, \\ a_i\theta_{i+2} + \ldots, \\ \ldots \quad \ldots \quad \ldots \quad \ldots, \\ b_i\theta_{i+1} + b_{i+1}\theta_{i+2} + \ldots, \\ b_i\theta_{i+2} + \ldots, \\ \ldots \quad \ldots \quad \ldots \quad \ldots \end{array}$$

all divisible by x^2. Hence $\theta_{i+1}, \theta_{i+2}, \ldots$ all divide by x^2. Hence we see that $\theta_i, \theta_{i+1}, \ldots, \theta_{i+r-1}$ divide by $x, x^2, \ldots, x^r$ respectively, and that every θ after θ_{i+r-1} divides by x^r. Hence we may put $\theta_{i+r-1}, \theta_{i+r}, \theta_{i+r+1}, \ldots$ each equal to zero; and the lines of a's and b's will terminate with $a_0\theta_{i+r-2}$ and $b_0\theta_{i+r-2}$ respectively. These last, in fact, divide by x^{i+r-1}.

One other property remains to be noticed. By leaving out the first line of a's, and first line of b's, we see that we can begin with the k^{th} column of the y-resultant, and choose multipliers for the

succeeding columns, such that, on adding to the k^{th} column, at least as high a power of x as x^r will divide out of each element in the new k^{th} column.

16. We now turn to the coefficients of S. Let polynomials in x, viz., $\phi_{k+1}, \phi_{k+2}, \dots, \phi_n$, be so chosen that the coefficient of x^{s-1} in $z_k + z_{k+1}\phi_{k+1} + \dots + z_n\phi_n$ shall vanish identically. Let the coefficient of x^{s-1} in this expression be identically $\Sigma\lambda_q^p z_q^p$. Then the coefficient of x^{s-l-1} ($l \geq 0$) is identically $\Sigma\lambda_q^p z_q^{p-l}$, and this must vanish (§ 6). Thus x^s must divide out of $z_k + z_{k+1}\phi_{k+1} + \dots + z_n\phi_n$. Again, the coefficient of x^{s-1} in $z_{k-1} + z_k\phi_{k+1} + \dots + z_{n-1}\phi_n$ is $\Sigma\lambda_q^p z_{q-1}^{p-1}$, which also must vanish (§ 6), &c., &c. Thus x^s must divide out of each of the following expressions :—

$$
\begin{array}{llll}
z_k & +z_{k+1}\phi_{k+1} & +\dots & +z_n\phi_n, \\
z_{k-1}+ & z_k\phi_{k+1} & +\dots & +z_{n-1}\phi_n, \\
\dots & \dots & \dots & \dots, \\
z_0 \; + & z_1\phi_{k+1} & +\dots & +z_{n-k}\phi_n, \\
 & z_0\phi_{k+1} & +\dots & +z_{n-k-1}\phi_n, \\
\dots & \dots & \dots & \dots, \\
 & & & z_0\phi_n.
\end{array}
$$

The same must be true when for z_n we write either a_n or b_n ($n = 0, 1, 2, \dots, n$). We find then that s cannot be greater than the value of r in § 15; and it can be easily verified that x^r does actually divide out of $z_k + z_{k+1}\theta_{k+1} + \dots + z_{i+r-2}\theta_{i+r-2}$.

17. Let $\theta_1, \theta_2, \theta_3, \dots$ be so chosen that $z_0 + z_1\theta_1 + z_2\theta_2 + \dots$ may divide by as high a power of x as possible, say x^r; similarly, let $\phi_2, \phi_3, \dots$ be so chosen that $z_1 + z_2\phi_2 + z_3\phi_3 + \dots$ divides by x^s as highest power, and $\psi_3, \psi_4, \dots$ be so chosen that $z_2 + z_3\psi_3 + z_4\psi_4 + \dots$ divides by x^t as highest power, and so on. We thus account for $r+s+t+\dots$ ($r \geq s \geq t \geq \dots$ by §§ 15, 16) independent* identical equations among the coefficients of S. And these are all that exist; for, if there are any more, there must be polynomials in x, viz., $\zeta_0, \zeta_1, \zeta_2, \dots$,

* The equations are independent from the fact that, since x^r divides out of $z_0 + z_1\theta_1 + z_2\theta_2 + \dots$, the first r coefficients of z_0, viz., $z_0^0, z_0^1, \dots, z_0^{r-1}$ are expressible in terms of coefficients with higher suffixes, &c.

one of which is not divisible by x, such that

$$\frac{\zeta_0(z_0+z_1\theta_1+z_2\theta_2+\dots)}{x^r}+\frac{\zeta_1(z_1+z_2\phi_2+\dots)}{x^s}+\frac{\zeta_2(z_2+z_3\psi_3+\dots)}{x^t}+\dots$$

divides by some power of x (§ 16). Hence ζ_0 must divide by x; otherwise, by multiplying up by x^r and dividing by ζ_0, we should have $z_0+z_1\chi_1+z_2\chi_2+\dots$ divisible by x^{r+1}, which is impossible (§ 16). Similarly, by leaving out the term containing ζ_0, and multiplying up by x^s, we see that ζ_1 must divide by x; and so on for all the ζ's. Thus there are no more than the $r+s+t+\dots$ independent equations satisfied identically by the coefficients of S. By a similar proof it follows that, while $x^{r+s+t+\dots}$ divides out of the y-resultant, no higher power of x will divide out. This proves the theorem.

18. Corollary 1. — The theorem can be at once extended to any two given polynomials C_1, C_2 whose highest common factor does not vanish at the origin, and also to any two given power series, convergent in a small finite region round the origin, which are not both divisible by any third power series vanishing at the origin. For, if the whole base-point at the origin common to two given convergent power series is not of finite degree, the reason can only be that some branch of one has infinite contact with some branch of the other; that is, the two power series must have a common branch through the origin. This assumes that the branches of the given power series, through the origin, can only be convergent power series.

Corollary 2.—If two given curves, or power series, C_1, C_2, both pass through a given base-point at the origin, and if the general curve, or power series, S, which passes through the base-point is necessarily of the form $S\equiv C_1P_1+C_2P_2$, then the given base-point is the whole intersection of C_1, C_2 at the origin.

19. Theorem II.—*Any polynomial S, or power series, whose coefficients satisfy a given finite one-set system of equations (§ 7) can be written in the form $S\equiv C_1P_1+C_2P_2$, where C_1, C_2 are two properly chosen fixed polynomials belonging to the system S, and P_1, P_2 are power series.*

In other words:—*Any one-set point is the whole intersection at the point of two fixed curves.*

Let $\Sigma^n a_q^p z_q^p = 0$, where $n \geqslant p \geqslant q$, be the prime equation of the given one-set system. Any equation of the one-set is $\Sigma a_q^p z_{q-m}^{p-l} = 0$,

where l, m are any two positive integers (fixed for one equation) such that $l \geqslant m$ (§ 7). It is to be remembered that z_{q-m}^{p-l} is zero unless $p-l \geqslant q-m \geqslant 0$.

It will be sufficient to prove that the theorem holds for the given one-set $\Sigma a_q^p z_q^p = 0$ provided it holds for the one-set $\Sigma a_q^p z_{q-1}^{p-1} = 0$.

20. The equations of the one-set $\Sigma^n a_q^p z_q^p = 0$ may be divided into two sets of equations, the first of these including all the equations in which $m \geqslant 1$ (whence also $l \geqslant 1$, since $l \geqslant m$), and the second including all the equations in which $m = 0$. The first of the two sets forms the one-set $\Sigma a_q^p z_{q-1}^{p-1} = 0$, and the second consists of the supplementary equations

$$\Sigma a_q^p z_q^p = 0, \quad \Sigma a_q^p z_q^{p-1} = 0, \quad \ldots, \quad \Sigma a_q^p z_q^{p-n} = 0.$$

Now assume that the theorem holds for any polynomial S', or power series, whose coefficients satisfy the one-set $\Sigma a_q^p z_{q-1}^{p-1} = 0$, *i.e.*, assume that any such S' is of the form $C_1' P_1' + C_2' P_2'$, where C_1', C_2' are two fixed polynomials which belong to the system S'.

By making the system $S' \equiv C_1' P_1' + C_2' P_2'$ satisfy the supplementary equations $\Sigma a_q^p z_q^{p-l} = 0$ $(l = 0, 1, 2, \ldots, n)$ we obtain certain equations which must be satisfied by the coefficients of P_1', P_2'; and we thus find the system S which satisfies the one-set $\Sigma a_q^p z_q^p = 0$.

21. We shall first show that one of the two polynomials C_1', C_2' may be supposed to belong to the system S. For this purpose we write

$$S' \equiv C_1' (P_1' - PP_2') + (C_2' + C_1' P) P_2',$$

and regard C_1' and $C_2' + C_1' P$ as the two fixed polynomials which determine the system S', P being chosen in any way we please. We have to prove that P can be so chosen that $C_2' + C_1' P$ belongs to the system S.

Let $C_1' \equiv \Sigma a_q^p x^p \eta^q$, $C_2' \equiv \Sigma b_q^p x^p \eta^q$, $P \equiv \Sigma \rho_q^p x^p \eta^q$, where $\eta \equiv y/x$. Now yC_1' belongs to the system S, since

$$yC_1' \equiv x\eta \Sigma a_q^p x^p \eta^q \equiv \Sigma a_{q-1}^{p-1} x^p \eta^q;$$

and hence, in order that $C_2' + C_1' P$ may belong to the system S, it is necessary and sufficient that

$$C_2' + C_1' (\rho_0^0 + \rho_0^1 x + \rho_0^2 x^2 + \ldots)$$

should belong to the system S. The coefficient of $x^p \eta^q$ in this is

$$z_q^p \equiv b_q^p + \rho_0^0 a_q^p + \rho_0^1 a_q^{p-1} + \rho_0^2 a_q^{p-2} + \dots;$$

therefore $\Sigma a_q^p z_q^p \equiv \Sigma a_q^p b_q^p + \rho_0^0 \Sigma a_q^p a_q^p + \rho_0^1 \Sigma a_q^p a_q^{p-1} + \rho_0^2 \Sigma a_q^p a_q^{p-2} + \dots,$

and $\Sigma a_q^p z_q^{p-l} \equiv \Sigma a_q^p b_q^{p-l} + \rho_0^0 \Sigma a_q^p a_q^{p-l} + \rho_0^1 \Sigma a_q^p a_q^{p-l-1} + \rho_0^2 \Sigma a_q^p a_q^{p-l-2} + \dots .$

Hence, in order that $C_2' + C_1' P$ may belong to the system S, we have (§ 20) only to choose $\rho_0^0, \rho_0^1, \rho_0^2, \dots$ so as to satisfy the equations

$$\Sigma a_q^p b_q^{p-l} + \rho_0^0 \Sigma a_q^p a_q^{p-l} + \rho_0^1 \Sigma a_q^p a_q^{p-l-1} + \rho_0^2 \Sigma a_q^p a_q^{p-l-2} + \dots = 0,$$

when $$l = 0, 1, 2, 3, \dots .$$

Suppose that $\Sigma a_q^p a_q^{p-l}$ and $\Sigma a_q^p b_q^{p-l}$ are both zero so long as $l > l_0$, but that they are not both zero when $l = l_0$. Then $\Sigma a_q^p a_q^{p-l_0}$ does not vanish of necessity, and we may therefore assume that it is not zero. Now when $l > l_0$ the above equations for the ρ's are identically satisfied; and on putting $l = l_0, l_0 - 1, \dots, 0$, it is seen that the equations determine the values of $\rho_0^0, \rho_0^1, \dots, \rho_0^{l_0}$ in succession, each of these having in turn the non-vanishing coefficient $\Sigma a_q^p a_q^{p-l_0}$.

22. We may assume that C_1' and C_2' are *general* fixed polynomials belonging to the system S', and consequently that the $C_2' + C_1' P$ of § 21 is a general fixed polynomial belonging to the system S; for $C_2' + C_1' P$ has only been made to satisfy those (supplementary) conditions which convert a polynomial of the system S' into one of the system S. Hence we see that any S' is of the form $C_1' P_1' + C_2 P_2'$, where C_1' and C_2 are any two general fixed polynomials belonging to the systems S' and S respectively.

Now to any S corresponds a power series p in x such that

$$S + pC_2 \equiv yP;$$

for C_2 is general, and therefore does not begin with a higher power of x than S in the terms which do not contain y. Here yP belongs to the system S; hence P belongs to the system S', *i.e.*,

$$P \equiv C_1' P_1' + C_2 P_2';$$

therefore $$S \equiv y C_1' P_1' + C_2 P_2,$$

where yC_1' and C_2 are two fixed polynomials belonging to the system S. This proves the theorem.

23. Corollary.—If $z_q^p = a_q^p$, $z_q^p = b_q^p$ are two properly chosen solutions of the one-set system of equations

$$\Sigma a_q^p z_q^p = 0, \; \ldots, \; \Sigma a_q^p z_{q-m}^{p-l} = 0, \; \ldots,$$

then the general solution for z_q^p is the coefficient of $x^p \eta^q$ in

$$P_1 \Sigma a_q^p x^p \eta^q + P_2 \Sigma b_q^p x^p \eta^q,$$

i.e.,

$$z_q^p = \Sigma \lambda_s^r a_{q-s}^{p-r} + \Sigma \mu_s^r b_{q-s}^{p-r},$$

the λ's and μ's being arbitrary parameters, and the summations extending over all positive integral values of r, s such that $r \geqslant s$. The parameters λ, μ are not in general, however, the least number of independent parameters in terms of which the most general solution of the one-set system can be expressed.

24. Theorem III.—*The whole intersection at the origin of any two given curves C_1, C_2, or power series, having no common branch through the origin, is a one-set point.*

To prove, in the first place, that if the whole intersection at the origin of two given curves C_1, C_2 is a one-set point, then the same property is true of the curves C_1', C_2' into which C_1, C_2 are transformed by means of the transformation equations

$$x = x', \quad y = y' + ax' + bx'^2 + cx'^3 + \ldots .$$

Let $E \equiv \Sigma a_q^p z_q^p = 0$ be the prime equation of the one-set point of intersection of C_1, C_2 at the origin. Let $S \equiv \Sigma z_q^p x^p \eta^q = 0$ be the general curve or power series which passes through the one-set point; and let S transform into $S' \equiv \Sigma z_q'^p x'^p \eta'^q$ when we write

$$x = x', \quad \eta = \eta' + a + bx' + cx'^2 + \ldots .$$

Then S' transforms into S when we write

$$x' = x, \quad \eta' = \eta - a - bx - cx^2 - \ldots ;$$

therefore z_q^p = coefficient of $x^p \eta^q$ in the curve to which S' transforms

= linear function of the coefficients $z_q'^p$ of S'.

Hence the equation $E \equiv \Sigma a_q^p z_q^p = 0$ for S transforms into an equation $E' \equiv \Sigma a_q'^p z_q'^p = 0$ for S'. Also the equation $E' = 0$ holds for $x'^l \eta'^m S'$, since the equation $E = 0$ holds for $x^l (\eta - a - bx - \ldots)^m S$; hence the coefficients of S' satisfy all the equations of the one-set whose prime equation is $E' = 0$, and clearly are not required to satisfy any more.

Again, any polynomial S, or power series, whose coefficients satisfy the one-set $E = 0$ is of the form $C_1 P_1 + C_2 P_2$, and conversely; therefore any polynomial S', or power series, whose coefficients satisfy the one-set $E' = 0$ is of the form $C_1' P_1' + C_2' P_2'$, and conversely; while the coefficients of both C_1' and C_2' satisfy the one-set $E' = 0$. Hence the equations to the whole intersection at the origin of C_1', C_2' form the one-set $E' = 0$ (§ 18, Cor. 2).

25. To prove, in the second place, that if the equations to the whole intersection of C_1, C_2 at the origin form a one-set, then the equations to the whole intersection of $(y + ax + bx^2 + \ldots)\, C_1$ and C_2 at the origin also form a one-set, $y + ax = 0$ not being a tangent to $C_2 = 0$ at the origin. Having regard to § 24 it is sufficient to prove that the equations to the intersection of yC_1 and C_2 at the origin form a one-set, $y = 0$ not being a tangent to $C_2 = 0$ at the origin.

Let $C_1 \equiv \Sigma a_q^p x^p \eta^q$, $C_2 \equiv \Sigma b_q^p x^p \eta^q$, and let j be the order of the multiple point of C_2 at the origin, so that $b_0^j \neq 0$, from above. Let $E_a \equiv \Sigma^n a_q^p z_q^p = 0$ be the prime equation of the one-set point of intersection of C_1, C_2 at the origin. We shall prove that the equations to the whole intersection of yC_1 and C_2 at the origin form a one-set $E_\beta \equiv \Sigma^{n+1} \beta_q^p z_q^p = 0$. It will be necessary to prove not only that the coefficients of yC_1 and of C_2 satisfy the one-set $E_\beta = 0$, but that any S for which the one-set $E_\beta = 0$ is satisfied is of the form $yC_1 P_1 + C_2 P_2$. (*Cf.* § 18, Cor. 2.)

If we take $\beta_q^p = a_{q-1}^{p-1}$ when $q \geqslant 1$, the one-set $E_\beta = 0$ will be satisfied for yC_1, still leaving the values of $\beta_0^0, \beta_0^1, \ldots, \beta_0^{n+1}$ at disposal. Again, the one-set $E_\beta = 0$ will be satisfied for C_2 provided $\Sigma \beta_q^p z_q^p$ vanishes for $x^l C_2$ when $l = 0, 1, 2, \ldots$; for $\Sigma \beta_q^p z_q^p$ vanishes for $x^l \eta^m C_2$ when $m \geqslant 1$, since $\beta_q^p = a_{q-1}^{p-1}$ when $q \geqslant 1$. Hence the only equations which have to be satisfied by $\beta_0^0, \beta_0^1, \ldots, \beta_0^{n+1}$ are $\Sigma \beta_q^p b_q^{p-l} = 0$, when $l = 0, 1, 2, \ldots$. Now if $l > n - j + 1$, then $p - l \leqslant n + 1 - l < j$, and the equation $\Sigma \beta_q^p b_q^{p-l} = 0$ is identically satisfied. If $l = n - j + 1$, the only terms appearing in $\Sigma \beta_q^p b_q^{p-l} = 0$ are those for which $p = n + 1$; thus, of the unknowns $\beta_0^0, \beta_0^1, \ldots, \beta_0^{n+1}$, only β_0^{n+1} comes in, and with a non-vanishing coefficient b_0^j; and so the equation determines β_0^{n+1}. Also, putting $l = n - j,\ n - j - 1,\ \ldots, 1, 0$, the equations determine $\beta_0^n, \beta_0^{n-1}, \ldots, \beta_0^j$ in succession. Therefore the values of $\beta_0^j, \beta_0^{j+1}, \ldots, \beta_0^{n+1}$ can be so chosen that the one-set $E_\beta = 0$ is satisfied for C_2 as well as for yC_1.

26. It remains to prove that any S for which the one-set $E_s = 0$ is satisfied must be of the form $yC_1P_1 + C_2P_2$. Any such S is of the form $C_1P_1' + C_2P_2'$, where the coefficients of C_1P_1' must satisfy the one-set $E_s = 0$. From this it can be proved that the terms in C_1P_1', and therefore also in S, which do not contain y are at least divisible by x^j. For, if not, we can, as in § 22, choose a power series p in x such that

$$C_2 + pC_1P_1' \equiv yP \equiv y\ (C_1P_1 + C_2P_2),$$

or

$$C_2\,(1 - yP_2) \equiv C_1\ (yP_1 - pP_1').$$

Hence $1 - yP_2$ is divisible by C_1,* which is impossible, assuming that C_1 vanishes at the origin.† It follows that to any S corresponds a power series p in x such that

$$S + pC_2 \equiv yP \equiv y\,(C_1P_1 + C_2P_2).$$

Hence any S for which the one-set $E_s = 0$ is satisfied is of the form $yC_1P_1 + C_2P_2$.

27. In order now to prove the theorem generally that the whole intersection of C_1, C_2 at the origin is a one-set point we make use of a process analogous to that of finding the highest common factor of two expressions. Arrange C_1, C_2 in ascending powers of the variables x, y, say $C_1 \equiv u_i + u_{i+1} + \ldots$, $C_2 \equiv v_j + v_{j+1} + \ldots$, i being not less than j. Then divide C_2 into C_1 until we get a remainder $C_3 \equiv w_k + w_{k+1} + \ldots$. The process of division can be carried so far that the following conditions shall be satisfied by the remainder C_3:—(i.) w_k is not divisible by v_j; (ii.) the factors of w_k, other than those which form the H.C.F. of v_j and w_k, are all different from one another, and from any factor of v_j. To each of these factors will correspond a linear branch $y + ax + bx^2 + \ldots$ of C_3; and, these linear branches being all divided out of C_3, we shall have left a power series C_3' whose terms of lowest order divide those of C_2. Now the equations to the whole intersection of C_1, C_2 at the origin are the same as those for C_2, C_3, and these

* If the product P_1P_2 is divisible by P_3, and if P_1, P_3 have no branch in common through the origin, then P_2 is divisible by P_3. This theorem is used again in § 34 and in § 37. It is proved by Berry, "Note on a Case of Divisibility, &c." (*Proc. Lond. Math. Soc.*, Vol. xxx., pp. 271–276).

† The proof given does not apply when C_1 does not vanish at the origin. In this case the coefficients of both yC_1 and C_2 satisfy the equations $z_0^0 = z_0^1 = \ldots = z_0^{j-1} = 0$, which form the one-set $z_0^{j-1} = 0$; and it is easily proved that any S for which the one-set $z_0^{j-1} = 0$ is satisfied can be written in the form $yC_1P_1 + C_2P_2$.

[redacted]

will form a one-set if the equations to the whole intersection of C_2, C'_3 at the origin form a one-set (§ 25). We can now deal with C_2, C'_3 in the same way as with C_1, C_2, and continue the process until we arrive at two power series having no contact of branches at the origin, for which the theorem is true (§ 25). The theorem is then true for any two given curves, or power series, having no common branch through the origin.

28. Corollary 1.—If the process described in § 27 be carried out, we can deduce from it the superior limit, n, of the value of p in the one-set system of equations $\Sigma a_q^p z_q^p = 0$ to the whole intersection of C_1, C_2 at the origin. In the "simple" case, when C_1, C_2 have no contact at the origin, n is less by 2 than the sum of the orders of the multiple points of C_1, C_2 at the origin. In the "general" case n is 2 less than the sum of the orders of the lowest terms in the last divisor and remainder increased by the total number of linear branches divided out of the several remainders. A curve having a multiple point at the origin of order $n+1$ passes, *ipso facto*, through the whole intersection of C_1, C_2 at the origin.*

29. Corollary 2. — Any equation identically satisfied by the coefficients of $C_1P_1+C_2P_2$ may be taken for the prime equation of the one-set for C_1, C_2, provided it includes coefficients z_q^p with index n. For any such equation is

$$\rho_0^0 \Sigma a_q^p z_q^p + \ldots + \rho_m^l \Sigma a_q^p z_{q-m}^{p-l} + \ldots = 0, \quad \text{where} \quad \rho_0^0 \neq 0.$$

From this as prime equation, by first diminishing the index of the coefficients z_q^p by n, and the suffix by $n, n-1, n-2, \ldots, 0$; then diminishing the index by $n-1$ and the suffix by $n-1, n-2, \ldots, 0$, and so on, we obtain the original equations $\Sigma a_q^p z_{q-m}^{p-l} = 0$, irrespective of the values of the ρ's.

This result may be expressed algebraically as follows (§ 8) :—If the system of equations for the λ's

$$\Sigma \lambda_q^p a_q^p = 0, \;\ldots,\; \Sigma \lambda_q^p a_{q-m}^{p-l} = 0, \;\ldots,\; \Sigma \lambda_q^p b_q^p = 0, \;\ldots,\; \Sigma \lambda_q^p b_{q-m}^{p-l} = 0, \;\ldots,$$

$$\text{or } \Sigma a_q^p \lambda_q^p = 0, \;\ldots,\; \Sigma a_q^p \lambda_{q+m}^{p+l} = 0, \;\ldots,\; \Sigma b_q^p \lambda_q^p = 0, \;\ldots,\; \Sigma b_q^p \lambda_{q+m}^{p+l} = 0, \;\ldots,$$

is such that λ_q^p is necessarily zero when $p > n$, but not when $p = n$,

* *Cf.* references given in *Proc. Lond. Math. Soc.*, Vol. xxxi., p. 23.

and if one solution is $\lambda_q^p = a_q^p$, where $a_0^n, a_1^n, \dots, a_n^n$ are not all zero, then the general solution is

$$\lambda_q^p = \Sigma \rho_s^r a_{q+s}^{p+r},$$

the ρ's being chosen arbitrarily, and the summation extending over all positive integral values of r, s such that $r \geq s$. For, by the first statement of the corollary, $\Sigma a_q^p z_q^p = 0$ may be taken as the prime equation of the one-set for C_1, C_2. Hence the equation $\Sigma \lambda_q^p z_q^p = 0$ is deducible from the one-set $\Sigma a_q^p z_q^p = 0$; *i.e.*,

$$\Sigma \lambda_q^p z_q^p \equiv \rho_0^0 \Sigma a_q^p z_q^p + \dots + \rho_s^r \Sigma a_q^p z_{q-s}^{p-r} + \dots;$$

therefore $$\lambda_q^p = \Sigma \rho_s^r a_{q+s}^{p+r}.$$

The complementary result is stated in Theorem II., Corollary.

30. Corollary 3.—Two curves which pass through a given t-set point at the origin $(t > 1)$ intersect at the origin in a one-set point which contains, and is of higher degree than, the t-set point.

III. Applications.

31. Let C', C'' be any two given curves, having no common constituent, and C any curve belonging to a given linear* system, to which C' belongs. To find the nature of the conditions which must be satisfied by S in order that the identity

$$CS \equiv C'S' + C''S''$$

may hold, S, S', S'' being polynomials, and S not involving the parameters of C.

Noether's theorem gives us the necessary and sufficient conditions which CS has to satisfy, viz., that, on transferring the origin to *any* point A of intersection of C', C'', we should be able to find two ordinary power series P', P'', such that

$$CS \equiv C'P' + C''P''.$$

From this we can state explicitly the nature of the conditions for S.

* There would be no increase of generality in the question proposed if C were any given non-linear system, since the solution of the question depends solely on the linear identities which exist among the coefficients of C.

As far as concerns the ordinary points of intersection of C', C'', that is, the points of intersection of degree 1, the conditions are that S has simply to pass through those which do not lie on the general curve of the cutting system C. For a point of intersection of C', C'' at A of degree $a > 1$ the conditions are that S should pass through a certain base-point at A; for whatever equations have to be satisfied by the coefficients of S must also be satisfied by the coefficients of $x^l \eta^m S$ $(l \geqslant m)$, since

$$Cx^l \eta^m S \equiv C'x^l \eta^m P' + C''x^l \eta^m P''.$$

Also, the base-point at A through which S has to pass is contained in, *i.e.*, forms a part of, the whole one-set point of intersection of C', C'' at A, since the identity $CS \equiv C'P' + C''P''$ is evidently satisfied by putting $S \equiv C'$, or $S \equiv C''$. And if the general curve of the system C does not pass through A, the base-point through which S has to pass at A is evidently the whole intersection of C', C'' at A, since S is in this case itself of the form $C'P' + C''P''$.

32. Suppose that the general curve of the system S has been found, and consider the conditions which a curve K, not involving the parameters of S, must satisfy, in order that we may have

$$KS \equiv C'S' + C''S''.$$

As before, the conditions for K are that it should simply pass through certain base-points, situated at the points of intersection of C', C''.

Taking for origin the point A, at which the intersection of C', C'' is of degree a, we must have

$$KS \equiv C'P' + C''P''.$$

We shall now prove that if the base-point at A through which S has to pass is of degree q, then the base-point at A through which K must pass is of degree $r = a - q$. Let the independent equations to the base-point q be $E_1 = 0$, $E_2 = 0$, ..., $E_q = 0$, and the independent equations to the one-set point a (Theorem III.) be $E_1 = 0$, ..., $E_a = 0$, the first q of these being the equations to the base-point q, and the last $E_a = 0$ being the prime equation of the one-set. Denote by E_n^P the value that E_n assumes when the coefficients of any assigned polynomial or power series P are substituted in it for the general coefficients z_q^p.

[redacted]

Then E_a^{KS} is linear in the coefficients of S, and would moreover vanish if $E_1^S, E_2^S, \ldots, E_a^S$ were all zero. Therefore

$$E_a^{KS} \equiv F_a^K E_1^S + F_{a-1}^K E_2^S + \ldots + F_1^K E_a^S,$$

where $F_1^K, F^K, \ldots, F_a^K$ are linear homogeneous functions of the coefficients of K. This is true whatever K and S are; but, in the given case, $E_1^S, \ldots, E_q^S$ all vanish; hence

$$E_a^{KS} \equiv F_r^K E_{q+1}^S + \ldots + F_1^K E_a^S = 0.$$

Also $E_{q+1}^S, \ldots, E_a^S$ are linearly independent, since by making them all vanish we should impose $a-q$ independent conditions on S in addition to the Σq conditions which it already satisfies, S being assumed of sufficiently high order. Hence

$$F_1^K = 0, \quad F_2^K = 0, \quad \ldots, \quad F_r^K = 0.$$

These linear equations for the ratios of the coefficients of K are all independent, since they form but a small part of the independent system of equations to the whole intersection of C', C'' in the plane. It remains to prove that all the equations to the one-set point a (not the prime equation only) are now satisfied for KS, or that the equation $E_a = 0$ is satisfied for $x^l \eta^m KS$, or $K . x^l \eta^m S$. This new equation is

$$F_a^K (E_1^S)_{-m}^{-l} + F_{a-1}^K (E_2^S)_{-m}^{-l} + \ldots + F_1^K (E_a^S)_{-m}^{-l} = 0,$$

where $(E_n^S)_{-m}^{-l}$ denotes the value which E_n^S assumes when the index of every coefficient of S in E_n^S is diminished by l, and the suffix by m. This equation is certainly satisfied, since $(E_1^S)_{-m}^{-l}, \ldots, (E_q^S)_{-m}^{-l}, F_r^K, F_{r-1}^K, \ldots, F_1^K$ all vanish.

Hence, in order that the identity $KS \equiv C'S' + C''S''$ may hold, it is necessary and sufficient that the curve K should pass through a certain base-point r at A, whose equations are $F_1 = 0, \ldots, F_r = 0$,*

* To find $F_1, F_2, \ldots, F_a$ when $E_1, E_2, \ldots, E_a$ are known, we may proceed as follows:—If K and S are *any* polynomials, and $K \equiv \Sigma z_q^p x^p \eta^q$, then

$$\begin{aligned} E_a^{KS} &\equiv \text{value of } E_a \text{ for } (z_0^0 + \ldots + z_s^r x^r \eta^s + \ldots) S \\ &\equiv \text{sum of values of } E_a \text{ for } z_0^0 S, \ldots, z_s^r x^r \eta^s S, \ldots \\ &\equiv z_0^0 (E_a^S) + \ldots + z_s^r (E_a^S)_{-s}^{-r} + \ldots . \end{aligned}$$

Hence $\quad z_0^0 (E_a^S) + \ldots + z_s^r (E_a^S)_{-s}^{-r} + \ldots \equiv F_a^K E_1^S + F_{a-1}^K E_2^S + \ldots + F_1^K E_a^S.$

Equating the coefficients of the coefficients of S in this identity we shall have

and such that $q+r = a$, with similar conditions for each point of intersection of C', C''. Thus we have established the correctness of the statements left unproved on pp. 19, 20, 24, 25, 26 of the *Proc. Lond. Math. Soc.*, Vol. XXXI., with the exception of such as refer to the actual finding of the general systems S and K.

The whole base-point at A common to C'' and all the curves of the cutting system C lies on K; for, if r is this base-point, it is sufficient that S passes through the corresponding base-point q at A, and it is therefore necessary that K passes through r. Again, any base-point r through which K has to pass lies on C'' and all the curves of C. *The system K is therefore the general system which passes through the whole point-base common to C'' and all the curves of C.* Thus K is of the form $C''P'' + \Sigma\, CP$ at every point in the plane (§ 12, footnote); and we should infer, by analogy with Noether's theorem, that K must be of the form $C''S'' + \Sigma\, CS$, where S'', &c., are polynomials. This is proved in § 38 for the case in which the cutting system is a pencil of curves.

33. We may sum up what is proved in § 32 as follows:—

If S, S' are to be independent of one another as regards their variations, the necessary and sufficient conditions that SS' may pass through a given one-set point a are that S passes through any fixed base-point q contained in a, and that S' passes through a corresponding fixed base-point r, also contained in a, where $q+r = a$. The equations to r (or q) are directly determinable from the equations to a and q (or r). Also, if S, S' are of sufficiently high order, there must be altogether a independent linear equations satisfied identically by the coefficients of S and S' separately, q for S and r for S', and there need not be more.

The fact that $q+r = a$ is a special property of one-set points. The total number of independent linear equations that must be satisfied by the coefficients of two independent curves S, S' in order that SS' may pass through a given base-point q, not a one-set point, is less than q. For example, only two equations are required in order that S and S' may both vanish at the origin, but the curve SS' then passes through a two-set point of degree 3, whose prime equations are $z_1^1 = 0$, $z_0^1 = 0$. So, again, the number of equations which have to be satisfied separately by the coefficients of three independent

equations for the values of F_1^K, F_2^K, ..., F_α^K. It has been proved that there are values of F_1^K, F_2^K, ..., F_α^K, independent of the coefficients of S, which satisfy the identity.

curves S, S', S'' in order that $SS'S''$ may pass through a given one-set point a is, in general, less than a.

Definition.—We define two fixed base-points q, r at the origin to be *residual* base-points provided two fixed curves C_1, C_2 exist, whose whole intersection at the origin is of degree $q+r$, and such that

$$\chi\rho \equiv C_1 P_1 + C_2 P_2,$$

where χ, ρ are the general algebraic curves (or power series) through q, r respectively, and P_1, P_2 are power series. We also say that the whole intersection of C_1, C_2 at the origin is the one-set point made up of q, r, or the one-set point $q+r$; and that the base-points q, r are residual on C_2 at the origin, having C_1 for a connecting curve.

Any two fixed curves C_1, C_2 drawn through a given base-point q at the origin intersect again at the origin in a definite base-point r. If χ is the general curve through q, then the equations to r are the equations which must be satisfied by the coefficients of ρ in order that the identity $\chi\rho \equiv C_1 P_1 + C_2 P_2$ may exist.

By a curve which passes through the two residual base-points q, r is to be understood a curve which passes through the one-set point made up of q, r, *i.e.*, a curve of the form $C_1 P_1 + C_2 P_2$. It is not a curve whose coefficients merely satisfy the equations to q and r. These equations are the equations to a single base-point, but not a base-point of degree $q+r$, and not the one-set point made up of q, r.

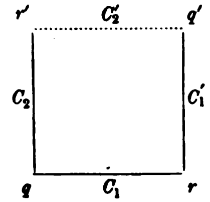
We may have two base-points q, r at a point A on a given base-curve C_2 which are residual on the plane, but not residual on the base-curve. To be residual on the base-curve they must be such that a curve C_1 can be drawn whose *whole* intersection with C_2 at A is the one-set point $q+r$. Also, in calling C_1 the *connecting curve* for q, r, it is specially to be observed that the term is used in a restricted sense, viz., a curve whose *whole* intersection with C_2 at A is the one-set point $q+r$.

34. Theorem of Residuation for Base-points.—The definition (§ 33) of residual base-points at a point A on a given base-curve C_2 leads to an important theorem, analogous to the theorem of residuation for point-groups or point-bases on a base-curve. The theorem is that any two base-points q, q' on C_2 at A which have a common residual base-point r are equivalent or coresidual, *i.e.*, any residual r' of either q or q' on C_2 at A is a residual of both q and q'.

Suppose r' to be any base-point on C_2 at A residual to q. It is required to prove that r' is also residual to q'. Taking A as origin, let

[redacted]

C_1 be the connecting curve for q, r, and C_1' for q', r, and C_2 for q, r'. Also, let χ, ρ, χ', ρ' be the general curves through q, r, q', r' respectively. Then we have, using the symbol P for a power series in general, different in different equations, and P_0, P_1, P_1', Q_1, Q_1', ... for power series which preserve their identity throughout,



Point on base-curve C_3.

$$\chi\rho \equiv C_1 P_1 + C_3 P, \tag{1}$$

$$\chi'\rho \equiv C_1' P_1' + C_3 P, \tag{2}$$

$$\chi\rho' \equiv C_2 P_2 + C_3 P. \tag{3}$$

Also, since C_2, C_1' pass through q, r respectively, we have

$$C_1' C_2 \equiv C_1 P' + C_3 P.$$

This identity shows that the intersection of P' and C_3 at A is fixed, and of finite degree $q'+r'$. Choosing then any fixed curve C_2' whose whole intersection with C_3 at A is the same as that of P', we have

$$P' \equiv C_2' P_0 + C_3 P,$$

where P_0 is a power series with a non-vanishing constant term. Hence we have

$$C_1' C_2 \equiv C_1 C_2' P_0 + C_3 P. \tag{4}$$

Again, since C_1 passes through r and χ' through q',

$$C_1 \chi' \equiv C_1' Q_1' + C_3 P; \tag{5}$$

similarly
$$C_1 \rho' \equiv C_2 Q_2 + C_3 P. \tag{6}$$

From (5) and (2), by multiplying across,

$$C_1 C_1' P_1' \chi' \equiv C_1' Q_1' \chi' \rho + C_3 P;$$

therefore, dividing out $C_1'\chi'$,

$$C_1 P_1' \equiv Q_1' \rho + C_3 P;$$

therefore Q_1' belongs to the system χ, *i.e.*, Q_1' passes through q. Similarly, from (6) and (3),

$$C_1 P_2 \equiv Q_2 \chi + C_3 P;$$

therefore Q_2 belongs to the system ρ. Hence

$$Q_1' Q_2 \equiv C_1 Q_1 + C_3 P. \tag{7}$$

Multiplying (4), (5), (6), (7), and dividing out $C_1^2 C_1' C_2 Q_1' Q_2$, we have

$$\chi'\rho' \equiv C_2' P_0 Q_1 + C_3 P. \tag{8}$$

Hence, since χ', ρ' are the general curves through q', r', and the intersection of C_2', C_3 at the origin is of degree $q'+r'$, it follows that q', r' are residual on C_3, having C_2' for a connecting curve.

35. Corollary.—If q, r, q', r' are four base-points on a base-curve C_3 at the origin, such that each one is residual to the next, and if C_1, C_1', C_2', C_2 are any connecting curves (§ 33) for the pairs (q, r), (r, q'), (q', r'), (r', q) respectively, then

$$C_1'C_2 \equiv C_1C_2'P_0 + C_3P,$$

where P_0 is a power series with a non-vanishing constant term.

Suppose that C_1, C_1', C_2, C_2', C_3, C_3' are any given polynomials, or power series, connected by the identity

$$C_1C_2' - C_1'C_2 \equiv C_3C_3',$$

where the three products C_1C_2', $C_1'C_2$, C_3C_3' are not divisible by any common power series which vanishes at the origin. Then, since $C_1'C_2$ passes through the one-set point of intersection of C_1 and C_3 at the origin, it follows that this one-set point can be divided into a residual pair of base-points q, r, of which q lies on C_2, and r on C_1' (§ 33). Also the remaining base-points q', r' in which C_1', C_2 cut C_3 at the origin lie on C_2', and together make up the one-set point in which C_2' cuts C_3 at the origin (§ 34).

Although the one-set points $q+r$, $r+q'$, $q'+r'$, $r'+q$ of the last paragraph are all unique, it is not true in general that q, r, q', r' are unique. The base-points q, r are *any* pair which satisfy the conditions mentioned in the last paragraph, and these conditions do not determine q, r uniquely. If one of the base-points q, r, q', r' is fixed, all are fixed. For example, by choosing q to be the whole base-point common to C_1, C_2, C_3 at the origin, the four base-points q, r, q', r' become fixed and unique.

36. Suppose that on a base-curve C_3 there are at each of several multiple points two coresidual base-points (q_1, q_1'), (q_2, q_2'), &c. Let a curve C_1 be drawn through $q_1, q_2, \ldots$, cutting C_3 again at the multiple points in $r_1, r_2, \ldots$, and in $Q+R$ ordinary points. Then through $q_1', q_2', \ldots$ a curve C_1' can be drawn, cutting C_3 again at the multiple points in $r_1, r_2, \ldots$ (§ 34), and in $Q'+R$ ordinary points, the R points of $Q+R$ and $Q'+R$ being the same. The point-bases $Q+\Sigma q$, $Q'+\Sigma q'$ have then a common residual point-base $R+\Sigma r$, and are coresidual in the strict sense of the term, *i.e.*, any point-base

$R'+\Sigma r'$ on C_3 which is residual to either $Q+\Sigma q$ or $Q'+\Sigma q'$ is residual to both.

To prove this draw any curve C_2 through $Q+\Sigma q$, cutting C_3 again in $R'+\Sigma r'$. Transfer the origin to any multiple point on C_3, say the point at which q, r, q', r' are situated. Since C_2 passes through $q+r'$, and C_1' through $q'+r$, $C_1'C_2$ passes through $q+r$, which is the whole intersection of C_1 and C_3 at the origin. Hence

$$C_1'C_2 \equiv C_1P_1+C_3P_3.$$

The same is true at each multiple point on C_3; and also $C_1'C_2$ passes through the $Q+R$ ordinary points in which C_1 cuts C_3. Hence, by Noether's theorem,

$$C_1'C_2 \equiv C_1C_2'+C_3C_3',$$

where C_2', C_3' are polynomials. But $C_1'C_2$ cuts C_3 altogether in

$$Q'+\Sigma q'+R+\Sigma r+Q+\Sigma q+R'+\Sigma r',$$

of which $Q+\Sigma q+R+\Sigma r$ makes up the whole intersection of C_1 and C_3; therefore $Q'+\Sigma q'+R'+\Sigma r'$ makes up the whole intersection of C_2' and C_3 (§ 35), which proves the theorem.

It follows that we may treat any base-point q as precisely equivalent in every respect to q ordinary points in the theorem of residuation. Two residual, or coresidual, base-points on a base-curve are to be treated as having separate identities, notwithstanding that they are situated at the same point, and have several, or it may be all, of their equations the same. In applying the theorem, however, some caution is necessary. It is only on a base-curve that a one-set point can be divided into more than two base-points, and it is only one-set points that can be divided at all, so far as has been shown. Again, although there is a clear sense in which it may be said that any two residual base-points are entirely distinct, the same statement cannot be made in general with respect to two coresidual base-points, and it is far from clear what such a statement would mean in any particular case.

37. We shall now consider the conditions which must be satisfied by three power series P_1, P_2, P_3 in order that the identity

$$C_1P_1+C_2P_2+C_3P_3 \equiv 0 \tag{1}$$

may hold, C_1, C_2, C_3 being three given fixed polynomials, having no common factor, and all vanishing at the origin. We shall prove in particular that in general we shall have

$$C_1 \equiv P_2P_3'-P_2'P_3, \quad C_2 \equiv P_3P_1'-P_3'P_1, \quad C_3 \equiv P_1P_2'-P_1'P_2,$$

where P'_1, P'_2, P'_3 are power series which satisfy (1). We shall assume that no two of the polynomials C_1, C_2, C_3 have a common factor; for if the result is true in this case, it is also true when C_1, C_2, C_3 taken in pairs have common factors, provided all three have no common factor. Thus any common factor of C_2, C_3 must be a factor of P_1 and P'_1, and similarly for C_3, C_1, and for C_1, C_2.

The equations which must hold for the coefficients of P_1 alone form a one-set system. For if $E = 0$ is the prime equation to the whole intersection of C_2, C_3 at the origin, it is necessary and sufficient that the equation $E = 0$ should hold for the coefficients of $x^l\eta^m C_1 P_1$, or $C_1 . x^l\eta^m P_1$; and these equations form a one-set system for the coefficients of P_1, since C_1 is fixed. Let the one-set point through which P_1 has to pass be the whole intersection at the origin of the two fixed polynomials or power series a, a' (§ 19). Then, since C_3 and C_2 pass through this one-set point, we have

$$C_3 \equiv ab' - a'b, \tag{2}$$

$$C_2 \equiv ca' - c'a, \tag{3}$$

$$P_1 \equiv ap + a'p', \tag{4}$$

where b, b', c, c' are fixed, and p, p' are arbitrary, power series.

Taking $P_1 \equiv a$, suppose that $-a_3$, a_2 are corresponding values for P_2, P_3. Then

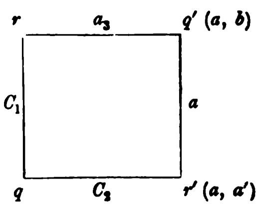
$$C_1 a - C_2 a_3 + C_3 a_2 \equiv 0,$$

i.e., $$C_1 a - (ca' - c'a)\, a_3 + (ab' - a'b)\, a_2 \equiv 0,$$

i.e., $$a\,(C_1 + c'a_3 + b'a_2) \equiv a'\,(ca_3 + ba_2)\,;$$

therefore $$C_1 \equiv -\,(a'a_1 + b'a_2 + c'a_3), \tag{5}$$

and $$aa_1 + ba_2 + ca_3 \equiv 0. \tag{6}$$



Point on base-curve C_3.

Again, $C_1 a$ and $C_2 a_3$ have the same intersection with C_3 at the origin. But C_1 passes through q (the whole base-point common to C_1, C_2, C_3), and a through the one-set point (a, a'), and these make up the whole intersection of C_2 and C_3 at the origin. Therefore the remaining base-points in which C_1 and a cut C_3 at the origin lie on a_3 (§ 35). Now a cuts C_3 again in the one-set point (a, b); for the intersection of a and C_3 at the origin is the same as that of a and $C_3 - ab'$, *i.e.*, of a and $a'b$, and

this may clearly be divided into the two residual one-set points (a, a') and (a, b), in accordance with the definition of § 33. Hence a_3 passes through the one-set point (a, b), *i.e.*,

$$a_3 \equiv ab'_1 - ba'_1. \tag{7}$$

Substituting from (7) in (6), we have

$$aa_1 + ba_2 + c\,(ab'_1 - ba'_1) \equiv 0,$$

i.e., $$a\,(a_1 + cb'_1) \equiv b\,(ca'_1 - a_2)\,;$$

therefore $$a_1 \equiv bc'_1 - cb'_1, \tag{8}$$

and $$a_2 \equiv ca'_1 - ac'_1. \tag{9}$$

Substituting from (7), (8), (9) in (5), we have

$$C_1 \equiv -\,a'\,(bc'_1 - cb'_1) - b'\,(ca'_1 - ac'_1) - c'\,(ab'_1 - ba'_1)$$
$$\equiv a'_1\,(bc' - b'c) + b'_1\,(ca' - c'a) + c'_1\,(ab' - a'b),$$

i.e., $$C_1 - b'_1 C_2 - c'_1 C_3 \equiv a'_1\,(bc' - b'c). \tag{10}$$

From (2), (3), (10), we have

$$(C_1 - b'_1 C_2 - c'_1 C_3)(ap + a'p') + a'_1 C_2\,(bp + b'p') + a'_1 C_3\,(cp + c'p') \equiv 0, \tag{11}$$

where p, p' are arbitrary power series.

Choose any power series p_1 such that

$$a'_1 p_1 \equiv ap + a'p'. \tag{12}$$

Substitute from (12) in (11), and divide out a'_1. Then

$$C_1 p_1 + C_2 p_2 + C_3 p_3 \equiv 0. \tag{13}$$

From (13) it follows that p_1 must belong to the system P_1, *i.e.*, p_1 must pass through the one-set point (a, a'). But in order that (13) may be satisfied it is only necessary that $a'_1 p_1$ should pass through the one-set point (a, a'). Hence a'_1 does not pass through any part of the one-set point (a, a'), *i.e.*, a'_1 does not vanish at the origin. But

$$a'_1 C_3 \equiv a'_1\,(ab' - a'b), \text{ by (2)},$$
$$\equiv aa'_1 b' - a'\,(ab'_1 - a_3), \text{ by (7)},$$
$$\equiv a\,(a'_1 b' - a'b'_1) + a_3 a'.$$

Hence, since a'_1 does not vanish at the origin, C_3 passes through the whole intersection of a and a_3 at the origin. Thus values can be found for P_1, P_2, viz., a, $-a_3$, which satisfy (1), and are such that C_3 passes through the whole intersection of P_1, P_2 at the origin. Pro-

[redacted]

vided then P_1, P_2 are *any* power series satisfying these conditions, we have

$$C_3 \equiv P_1 P_2' - P_1' P_2.$$

Substituting in (1), we have

$$C_1 P_1 + C_2 P_2 + (P_1 P_2' - P_1' P_2)\, P_3 \equiv 0,$$

i.e.,

$$P_1\,(C_1 + P_2' P_3) \equiv P_2\,(P_1' P_3 - C_2);$$

therefore

$$\left.\begin{aligned} C_1 &\equiv P_2 P_3' - P_2' P_3 \\ C_2 &\equiv P_3 P_1' - P_3' P_1 \\ C_3 &\equiv P_1 P_2' - P_1' P_2 \end{aligned}\right\}. \qquad \text{I.}$$

If A_1, A_2, A_3, A_1', A_2', A_3' are fixed values of P_1, P_2, P_3, P_1', P_2', P_3' satisfying the identities I., then the general values of P_1, P_2, P_3 which satisfy (1) are

$$P_1 \equiv A_1 P + A_1' P', \quad P_2 \equiv A_2 P + A_2' P', \quad P_3 \equiv A_3 P + A_3' P', \qquad \text{I}'.$$

where P, P' are arbitrary power series. Also, if either P or P' does not vanish at the origin, the corresponding values of P_1, P_2, P_3 can be used in I.; but if both P and P' vanish at the origin, the corresponding values of P_1, P_2, P_3 cannot in general be used in I.; for, in such a case, C_3 will not in general pass through the whole intersection of P_1, P_2 at the origin.

38. Let S_1, S_2, S_3 be three polynomials satisfying the identity

$$C_1 S_1 + C_2 S_2 + C_3 S_3 \equiv 0.$$

Then it follows by § 37 that S_1, S_2, S_3, if of sufficiently high order, can be so chosen that at any point common to C_1, C_2, C_3 the curve C_1 passes through the whole intersection of S_2, S_3. Let S be a curve which passes through the rest of the intersection of S_2, S_3 in the plane. Then $C_1 S$ passes through the whole intersection of S_2, S_3 in the plane. Hence we have

$$\left.\begin{aligned} C_1 S &\equiv S_2 S_3' - S_2' S_3 \\ C_2 S &\equiv S_3 S_1' - S_3' S_1 \\ C_3 S &\equiv S_1 S_2' - S_1' S_2 \end{aligned}\right\} \qquad \text{II.}$$

where S need not pass through any point common to C_1, C_2, C_3.

If we regard C_3 as a base-curve cut by C_1, C_2, and if C_1, C_2 are two general fixed curves of the pencil $\lambda_1 C_1 + \lambda_2 C_2$, then C_1, C_2 will have multiple points of the same order, and will cut C_3 in base-points of

the same degree, at each of the multiple points of C_3. The same will also be true of S_1, S_2. If S_1, S_2 have a j-fold point at any i-fold point of C_3, it follows from the identity $C_3 S \equiv S_1 S_2' - S_1' S_2$ that $2j \leqslant i$. If, further, S_1, S_2 have no contact either with one another or with C_3,* it follows that $S_1' S_2$ cuts S_1 in a base-point of degree ij, and that S_1' cuts S_1 in a base-point of degree $j\,(i-j)$. Hence S_1, S_1' have contact of order $i-2j$ along each of their branches, and the whole intersection of S_1, S_1' at the point is the same as that of a i-fold and an $(i-j)$-fold point without contact. In any case, if C_1, C_2, C_3, S_1, S_2, S_3 have multiple points of order i_1, i_2, i_3, j_1, j_2, j_3 at a common point, we have $j_2+j_3 \leqslant i_1$, $j_3+j_1 \leqslant i_2$, $j_1+j_2 \leqslant i_3$.

If A_1, A_1', A_1'' are three suitably chosen fixed values of S_1, the general value of S_1 is

$$S_1 \equiv A_1 S + A_1' S' + A_1'' S'', \qquad \text{II.}'$$

where S, S', S'' are arbitrary polynomials. To prove this, let N_1 be the degree of the point-base through which S_1 has, of necessity, to pass. Choose A_1, A_1', A_1'' of sufficiently high order n that they need not, and do not, possess any common intersection beyond the point-base N_1. This condition can be satisfied by A_1, A_1', A_1'', since the base-points of N_1 are all one-set points. Also choose S, S', S'' to be of such order $n' \geqslant n-2$ that the remaining point-group $N_1' = n^2 - N_1$ in which A_1', A_1'' intersect may have no n'-ic excess. Let D be the greatest number of general points chosen arbitrarily in the plane through which the curve $S_1 \equiv A_1 S + A_1' S' + A_1'' S''$ can be made to pass by a suitable choice of the coefficients of S, S', S''. Then, by making S_1 to vanish for $D+1$ arbitrarily chosen general points, S_1 must vanish identically. Moreover, the $D+1$ equations are independent, since no one is a consequence of the rest. Reciprocally, if S_1 vanishes identically, the $D+1$ equations are satisfied. Hence $D+1$ is the number of independent linear equations that must exist among the coefficients of S, S', S'' in order that S_1 may vanish identically. From this we can find the value of D.

Since $A_1 S + A_1' S' + A_1'' S''$ vanishes identically, S must vanish at all the N_1' points, giving N_1' independent equations for the coefficients of S alone, since the n'-ic excess of N_1' is zero. Supposing these equa-

* It may be suggested as probable that S_1, S_2 have never of necessity any contact either with one another or with C_3 at any point at which C_1, C_2, C_3 have not a common higher singularity.

tions only to be satisfied for the present, and S to be the general n'-ic through N_1', we have

$$A_1 S \equiv A_1' S_0' + A_1'' S_0'',$$

where the coefficients of S_0', S_0'' may be assumed to be known linear functions of the coefficients of S. Hence we have

$$S_1 \equiv A_1 S + A_1' S' + A_1'' S'' \equiv A_1' (S' + S_0') + A_1'' (S'' + S_0''),$$

where, as yet, S', S'' are quite arbitrary. In order now that this may vanish identically $S' + S_0'$ must be divisible by A_1'', giving $\frac{1}{2}(n'+1)(n'+2) - \frac{1}{2}(n'-n+1)(n'-n+2)$ independent equations for the coefficients of S' in terms of the coefficients of S_0', *i.e.*, of S. Finally the coefficients of S'' are all known in terms of the coefficients of S and S', giving $\frac{1}{2}(n'+1)(n'+2)$ more independent equations. Hence

$$\begin{aligned} D+1 &= N_1' + \tfrac{1}{2}(n'+1)(n'+2) - \tfrac{1}{2}(n'-n+1)(n'-n+2) + \tfrac{1}{2}(n'+1)(n'+2) \\ &= \tfrac{1}{2}(n'+n+1)(n'+n+2) - N_1. \end{aligned}$$

Hence the form S_1 includes the general $(n'+n)$-ic through N_1; for $N_1 + N_1'$, and therefore N_1, has no $(n'+n)$-ic excess, since $n' \geqslant n-2$. Consequently S_1 includes the general curve of any order $\leqslant n'+n$ through N_1. When the order of the curve through N_1 is less than a certain number we can choose S, S', S'' so that $A_1 S + A_1' S' + A_1'' S''$ is the general curve of the assigned order through N_1, notwithstanding that the separate terms $A_1 S$, $A_1' S'$, $A_1'' S''$ may necessarily have to be of a higher than the assigned order.

Precisely similar reasoning shows that $C_1 S' + C_2 S'' + C_3 S'''$ is the general curve which passes through the whole point-base common to C_1, C_2, C_3. Hence (§ 32, last paragraph)

$$K \equiv C_1 S' + C_2 S'' + C_3 S'''.$$

IV. Geometrical Application.

39. Lemma.—*If a proper base-curve C_m is cut by any linear system of curves, not possessing a fixed constituent in common, and if C_l is a fixed curve of the cutting system of the same order l as the general curve C of the system, and intersecting C_m in a base-point of the same degree as C at each multiple point of C_m, and if C_{m-3} is any curve of order $m-3$ adjoined to C_m at all the multiple points of C_m through which C_l passes, then the composite curve CC_{m-3} passes through the whole intersection of C_l, C_m at all the multiple points of C_m.*

It is evident that any general fixed curve of the cutting system satisfies the conditions mentioned for C_l. It is also to be understood that any multiple point of higher singularity on C_m is to be resolved into its component ordinary multiple points and cusps; and if C_l passes through more than one of the components, then C_{m-3} is to be adjoined to C_m at all such components. The chief importance of the lemma lies in the fact that the $\Sigma\frac{1}{2}i(i-1)$ conditional linear equations for the coefficients of C_{m-3} are known to be all independent, the summation in $\Sigma\frac{1}{2}i(i-1)$ extending to all the i-fold points of C_m, and to all the i-fold components of the multiple points of C_m, through which C_l passes. In the theorem of residuation we have to consider the general curve S which satisfies the condition that CS shall pass through the whole intersection of C_l, C_m at the multiple points of C_m. From what has been said concerning C_{m-3}, it follows that the number of independent conditional equations supplied for the coefficients of S is fixed, at any rate so long as the order of S is not lower than $m-3$. The value of this fixed number depends on the character of the given cutting system C; it may be anything from zero up to $\Sigma\frac{1}{2}i(i-1)$, but cannot exceed $\Sigma\frac{1}{2}i(i-1)$, if C_l is chosen in the manner explained.

The lemma is merely a generalization of a well known property, deducible in the ordinary way by quadric transformation. In the "simple" case, where C_l, C_m have a j-fold and i-fold point respectively at O, and no common tangent, the curve CC_{m-3} has an $(i+j-1)$-fold point at O, and therefore passes through the whole intersection of C_l, C_m at O (Brill-Noether, *Math. Ann.*, Vol. VII.). In the general case let C_l, C_m have multiple points at O of any kind of singularity, with any kind of contact. Take a curve S_n which does not pass through O, nor through any of the multiple points on C_m through which C_l passes, but which does pass through all the ordinary points of intersection of C_l, C_m which are not on C. Choose two fixed general points A, B on C_m, and take OAB as the coordinate triangle, OB being $x=0$, OA being $y=0$, and AB being $z=0$. Since A, B are general points on C_m, we assume that C_l, C_m, C_{m-3}, C, S_n cut the sides of OAB in points which, with the exception of O, are all finitely separated, and that they do not touch the sides of OAB at O, A, or B. By changing x, y, z to $\frac{1}{x'}$, $\frac{1}{y'}$, $\frac{1}{z'}$, let the curves C_l, C_m, C_{m-3}, C, S_n be transformed to $C'_{l'}$, $C'_{m'}$, $C'_{m'-3}$, C', $S'_{n'}$. The transformed curves have precisely the same kind of mutual relations as the originals; viz., the intersection of $C'_{l'}$, $C'_{m'}$ at any multiple point of

[redacted]

$C'_{m'}$ (and also at any non-multiple point of $C'_{m'}$ on $A'B'$) is of the same degree as that of C', $C'_{m'}$; $C'_{m'-3}$ is adjoined to $C'_{m'}$ at all the multiple points, and the components of multiple points, on $C'_{m'}$ through which $C'_{l'}$ passes, and in particular at any such point on $A'B'$; and $S'_{n'}$ passes through all the ordinary points of intersection of $C'_{l'}$, $C'_{m'}$ which are not on C'. Hence, assuming that the theorem which has to be proved is true for $C'_{l'}$, $C'_{m'}$, $C'_{m'-3}$, C', we have the identity

$$C'C'_{m'-3}S'_{n'} \equiv C'_{l'}S'_{m'+n'-3} + C'_{m'}S'_{n'+l'-3}.$$

From the fact that the tangents at O', A', B' to $C'_{l'}$, $C'_{m'}$, $C'_{m'-3}$, C', $S'_{n'}$ are all distinct from one another, and from some other considerations,* it can be proved that $S'_{m'+n'-3}$ and $S'_{n'+l'-3}$ may be so chosen in this identity that $C'_{l'}S'_{m'+n'-3}$ and $C'_{m'}S'_{n'+l'-3}$ have each the same number of tangents at O', A', B' respectively as $C'C'_{m'-3}S'_{n'}$. The numbers of the tangents are given in the table below, from which it will be seen that $S'_{n'+l'-3}$ must have $x'y'$ for a factor. Hence, by changing x', y', z' back to $\frac{1}{x}$, $\frac{1}{y}$, $\frac{1}{z}$, we have

$$CC_{m-3}S_n \equiv C_lS_{m+n-3} + C_mS_{n+l-3}.$$

Here S_n does not pass through O; therefore CC_{m-3} passes through the whole intersection of C_l, C_m at O. Hence the theorem holds for C_l, C_m, C_{m-3}, C, if it holds for $C'_{l'}$, $C'_{m'}$, $C'_{m'-3}$, C'. But, by repeated quadric transformations, the most general case can be brought to the "simple" case. Hence the theorem holds generally.

No. of tangents at	O	A	B	No. of tangents at	O'	A'	B'
C_m	i	1	1	$C'_{m'(=2m-i-2)}$	$m-2$	$m-i-1$	$m-i-1$
C_l	j	0	0	$C'_{l'(=2l-j)}$	l	$l-j$	$l-j$
C	j	0	0	C'	l	$l-j$	$l-j$
C_{m-3}	$i-1$	0	0	$C'_{m'-3(=2m-i-5)}$	$m-3$	$m-i-2$	$m-i-2$
S_n	0	0	0	$S'_{n'(=2n)}$	n	n	n
				$S'_{2m+2n-i-5}$	$m+n-3$	$m+n-i-2$	$m+n-i-2$
				$S'_{2n+2l-j-3}$	$n+l-1$	$n+l-j-1$	$n+l-j-1$

* The other conditions, in addition to the fact that the tangents to $C'_{m'}$, $C'_{l'}$, and $C'_{l'}S'_{m'+n'-3} + C'_{m'}S'_{n'+l'-3}$ at A', B', O' are all distinct from one another, are as follows:—If (i_1, i_2, i_3), (j_1, j_2, j_3), (k_1, k_2, k_3) are the numbers of the tangents at

40. When a given proper base-curve C_m is cut by any given linear system of curves, which all have both ordinary and multiple points in common with C_m, there will be a fixed point-base of (maximum) degree N on C_m through which all the members of the linear system pass, and a series of point-bases on C_m forming the rest of their several intersections with C_m. This series forms a coresidual set (§ 36), each of the set being residual to the fixed point-base N; and the theorem of residuation states that any point-base on C_m which is residual to one of the set is residual to each one of the set. If N' is one of the set, cut out by the curve C_l (say), the most general problem* of residuation is to determine the general algebraic curve through N'; for this will cut C_m again in the most general point-base which is coresidual to N, and residual to each one of the coresidual set to which N' belongs. The point-base N' consists of certain ordinary points on C_m, which present no difficulty, and certain base-points at the multiple points of C_m. Geometrically considered, the problem is to determine, in some visible manner, the effect of these base-points for any algebraic curve. The only way of doing this would appear to be to find other conditions, of a visible kind, which, on being applied to a curve, would necessitate its passing through the base-points of N'. By preference we should choose for C_l a curve of the cutting system whose intersection with C_m at any multiple point is of the same degree as that of the general curve of the system; for this would not only make the lemma of § 39 applicable, but would have the effect of bringing the degrees of the base-points of N' to their lowest possible values.

The problem of determining the visible effect of the base-points of N', as explained in the last paragraph, is naturally complicated and intricate; but it should not be impossible to find the solution. We proceed to give a solution for the case in which the cutting system, instead of being any given linear system, is any given pencil of curves.

(A', B', C') to the three curves respectively, then the inequality

$$(k_1-i_1-j_1)+(k_2-i_2-j_2) \leqslant n'-2$$

must hold, together with the two similar inequalities. These conditions are satisfied in the present case, for it may be seen from the table that

$$k_1-i_1-j_1 = k_2-i_2-j_2 = k_3-i_3-j_3 = n-1 = \tfrac{1}{2}(n'-2).$$

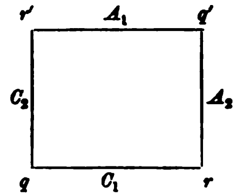
* Noether, "Ueber die Schnittpunktsysteme einer algebraischen Curve mit nicht-adjungirten Curven" (*Math. Ann.*, Vol. xv., 1879, pp. 507–528), gives the solution of the problem for the case in which one of the point-bases N, N' consists entirely of ordinary σ-points, that is, base-points of order σ and degree $\frac{1}{2}\sigma(\sigma+1)$.

Even in this case the problem is very intricate; so we only enter into such details of explanation as are absolutely necessary. Let C_1, C_2, of order l, be two general fixed curves of the cutting pencil, and C_3, of order m, the base-curve. Let $Q+\Sigma q$ be the whole point-base common to C_1, C_2, C_3, and $R+\Sigma r$, $R'+\Sigma r'$ the remaining coresidual point-bases in which C_1, C_2 cut C_3; R, r being equal to R', r' numerically. The base-points r, r' are all one-set points (§ 37); but the base-points q are in general two-set points.

We shall show how to find two fixed corresponding curves A_1, A_2 of the systems S_1, S_2 which satisfy the identity

$$C_1S_1+C_2S_2+C_3S_3 \equiv 0.$$

A_2 must pass through $R+\Sigma r$, and A_1 through $R'+\Sigma r'$, and A_1, A_2 cut C_3 again in one and the same point-base $Q'+\Sigma q'$, which is coresidual to $Q+\Sigma q$. We shall also choose A_1, A_2 so that q' is their whole intersection at the point where q, r, q', r' are situated (§ 37). The curves A_1, A_2 can then be used as a base for finding the general curve through $Q'+\Sigma q'$; and, similarly, two fixed curves A_3, A_3' of the system S_3 can be used as a base for finding the general curve through $R+\Sigma r$. In this way we can determine any point-base of the coresidual set to which $R+\Sigma r$ and $R'+\Sigma r'$ belong, and any point-base of the coresidual set to which $Q+\Sigma q$ and $Q'+\Sigma q'$ belong. Any point-base of the one set is residual to any point-base of the other set.

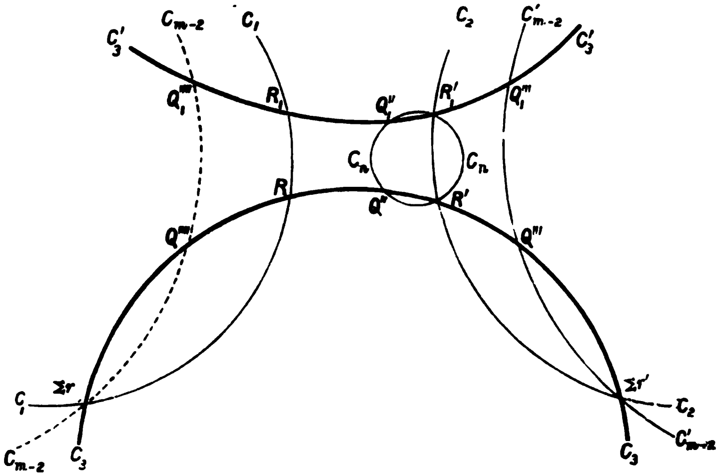


Point on C_3.

In order to effect our purpose of finding A_1, A_2, A_3' we make use of a subsidiary curve* C_3', of order $m-1$, and treat C_3C_3' for a time as base-curve. C_3' is chosen so as not to pass through any part of $Q+\Sigma q$, or R, or R', but it may be drawn through any Q_1 ordinary points of intersection of C_1, C_2 which are not on C_3. In the accompanying diagram of intersections only the two coresidual point-bases $R+R_1+\Sigma r$, $R'+R_1'+\Sigma r'$, cut out by C_1, C_2 on C_3C_3', are represented, the point-base $Q+Q_1+\Sigma q$ common to C_1, C_2, C_3C_3' being omitted. In reality the base-points r are situated at the same points as the base-points r'; but r, r' are to be treated as having separate

* It would not be necessary to introduce a subsidiary curve of any order provided (i.) two or more curves of order $\leqslant \frac{1}{2}(m-3)$ can be drawn through Σr, (ii.) the degree of the intersection of the two curves at each base-point r is equal to r, (iii.) the base-points Σr supply Σr independent conditions for the curves of order $\leqslant \frac{1}{2}(m-3)$.

identities (§ 36), and are represented in the diagram with separate positions.



Now, when any two point-bases $R+R_1+\Sigma r$, $R'+R'_1+\Sigma r'$ are coresidual, we may take away any part of one and add any residual of it to the other, and the resulting point-bases will be coresidual. Draw then a curve C_n through $R'+R'_1$, cutting $C_3C'_3$ again in the point-group $Q''+Q''_1$. It follows that

$$R+R_1+Q''+Q''_1+\Sigma r \text{ is coresidual to } \Sigma r' \text{ on } C_3C'_3. \qquad (1)$$

We shall prove that *any S_{m+n-2} through $R+R_1+Q''+Q''_1$ passes of necessity through Σr, and is consequently a curve of the system S_2 through $R+\Sigma r$.*

Imagine a curve C_{m-2} drawn through Σr, cutting C_3 again in $Q''''+\Sigma q_1$, and C'_3 in Q''''_1. Then, by the same theorem as before,

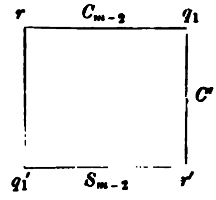
$$R+R_1+Q''+Q''_1 \text{ is coresidual to } Q''''+Q''''_1+\Sigma q_1+\Sigma r' \text{ on } C_3C'_3. \; (2)$$

The point-base $Q''''+Q''''_1+\Sigma q_1+\Sigma r$ is cut out by C_{m-2} on $C_3C'_3$, and Σr supplies independent conditions for curves of order $m-2$ (§ 39). Hence, since $C_3C'_3$ is of order $2m-1$, we know that any S_{2m-4} through $Q''''+Q''''_1+\Sigma q_1$ must have C_{m-2} for a factor.* Draw any S_{2m-4}

* *Cf. Proc. Lond. Math. Soc.*, Vol. XXVI., p. 525, § 19 (iii.); also Castelnuovo, *Mem. della R. Accademia delle Scienze di Torino*, Vol. XLII., 1891, p. 24, end of § 18.; and C. A. Scott, *Bulletin of the American Math. Soc.*, Vol. IV., 1898, p. 269. The theorem, which is a consequence of the Riemann-Roch theorem, has not been proved in the general form in which it is here used, but there is no room for doubt as to its correctness.

through $Q'''' + Q_1'''' + \Sigma q_1 + \Sigma r'$. Then

$$S_{2m-4} \equiv C_{m-2} S_{m-2}.$$



Point O on C_3.

Let C' be a curve whose whole intersection with C_3 at O is $r' + q_1$, and whose whole intersection with C_{m-2} at O is q_1* (§ 37), and let C'' be a curve whose whole intersection with C' at O is r'. Then, since S_{2m-4} passes through $r' + q_1$, the whole intersection of $C_{m-2} C''$ with C' at O, we have

$$S_{2m-4} \equiv C'P + C_{m-2} C''P'';$$

therefore $$C_{m-2} S_{m-2} \equiv C'P + C_{m-2} C''P'';$$

therefore $$S_{m-2} \equiv C'P' + C''P''.$$

Hence S_{m-2} passes through r'. Thus any S_{2m-4} through $Q'''' + Q_1'''' + \Sigma q_1 + \Sigma r'$ cuts C_3 again at the multiple points in $\Sigma r + \Sigma q_1'$. Also to any S_{2m-4} through $Q'''' + Q_1'''' + \Sigma q_1 + \Sigma r'$ corresponds any S_{m+n-2} through $R + R_1 + Q'' + Q_1''$, by (2) above. Hence any S_{m+n-2} through $R + R_1 + Q'' + Q_1''$ passes of necessity through Σr.

Let A_2 be a fixed curve of the system S_{m+n-2} through $R + R_1 + Q'' + Q_1''$ cutting C_3 altogether in the base-points $\Sigma r + \Sigma q'$, and in the point-group $R + Q'' + Q'''$, and C_3' in the point-group $R_1 + Q_1'' + Q_1'''$. (See diagram of intersections, p. 417.) Since A_2 passes through $R + \Sigma r$, and cuts C_3 again in $Q'' + Q''' + \Sigma q'$, there corresponds to it a curve A_1 through $R' + \Sigma r'$, which cuts C_3 again in the same $Q'' + Q''' + \Sigma q'$. Also, since $R' + Q''$ makes up the whole intersection of C_n with C_3, therefore $Q''' + \Sigma q' + \Sigma r'$ makes up the whole intersection of a curve C_{m-2}' with C_3. We may take A_1 to be $C_n C_{m-2}'$. The curve C_{m-2}' is fully determined by Q_1''', the whole point-group in which it cuts C_3'.

The curves A_1, A_2 have the point-base $Q' + \Sigma q'$ in common (Q' being written for $Q'' + Q'''$), and their remaining intersection is a point-group R_2. Thus $Q' + \Sigma q'$ is residual to R_2 on A_2. Find any point-group or point-base R_2' coresidual to R_2 on A_2, by finding first a fixed point-group Q_2 residual to R_2, and then any R_2' residual to Q_2. Then

* In comparing the diagram with that of § 37, r, r' have the same significance in both; C_{m-2}, C', q_1 correspond to a_3, a, q', while C'' corresponds to a'.

$Q'+\Sigma q'$ is residual to R_2', and the curve whose whole intersection with A_2 is $Q'+R_2'+\Sigma q'$ is the general curve through $Q'+\Sigma q'$, which it was required to find. This curve is fully determined by $Q'+R_2'$ if R_2' is a point-group; and, if R_2' is a point-base, the curve is still fully determined by the visible conditions which it has to satisfy.

If the general curve through $Q'+\Sigma q'$, which has now been found, cuts C_3 again in $R''+\Sigma r''$, then the corresponding curve through $Q+\Sigma q+R''+\Sigma r''$ is the general curve through $Q+\Sigma q$, *i.e.*, the general curve of the system K (§ 32). Although (§ 38)

$$K \equiv C_1 S' + C_2 S'' + C_3 S''',$$

and the point-base in which K cuts C_3 is the same as that in which $C_1 S' + C_2 S''$ cuts C_3, yet this point-base may be of less degree than that in which a curve of the same order as $C_1 S' + C_2 S''$ would in general cut C_3, owing to the fact that many of the points in which $C_1 S' + C_2 S''$ cuts C_3 may be made to disappear at infinity. This is what happens when K is of less order than $C_1 S' + C_2 S''$. Thus there are point-bases on C_3 coresidual to and of the same degree as $R+\Sigma r$ and $R'+\Sigma r'$, which are not cut out by the pencil $C \equiv \lambda_1 C_1 + \lambda_2 C_2$.

If we take two fixed curves A_2, A_2' of the system S_{m+n-2} through $R+R_1+Q''+Q_1''$, which pass of necessity through Σr, and have r for their whole intersection at the point where r is situated, we can find the general curve through $R+\Sigma r$, by dealing with A_2, A_2' in the same way as with A_2, A_1 above.

V. Extension of Theorems II. and III.

[*Added April*, 1900.

41. We now prove the theorems mentioned in § 11 as holding for base-points in general, which include as a particular case the theorems of §§ 19, 24 for one-set points.

We say that a base-point q is *determined* by the $t+1$ fixed curves $C_0, C_1, \ldots, C_t$, if q is the *whole* base-point common to $C_0, C_1, \ldots, C_t$ at the origin, that is, if the equations to q comprise all the independent equations which are identically satisfied by the coefficients of $C_0 P_0 + C_1 P_1 + \ldots + C_t P_t$. Any S which passes through q is then of the form $C_0 P_0 + C_1 P_1 + \ldots + C_t P_t$ (§ 12, footnote).

A t-set point has already been defined (§ 7) as a base-point whose equations cannot be expressed by means of less than t prime equations and their derivatives.

42. *Not more than $t+1$ fixed curves are required to determine any given t-set point.*

It is only necessary to prove that if $t+1$ fixed curves are sufficient to determine the base-point q_0, whose prime equations are $E_1 = 0$, $E_2 = 0$, ..., $E_t = 0$, then also $t+1$ fixed curves are sufficient to determine the base-point q_1, whose prime equations are $(E_1)^{+1}_{+1} = 0$, $E_2 = 0$, ..., $E_t = 0$, where $(E_1)^{+1}_{+1}$ is such that E_1 is derived from it by diminishing the index and suffix of each z in $(E_1)^{+1}_{+1}$ by 1. The coefficients of $\overset{0}{z}_0, \overset{1}{z}_0, \overset{2}{z}_0$, ... in $(E_1)^{+1}_{+1}$, which are absent from E_1, may have any fixed values. It is not asserted that q_0 is actually a t-set point, but only that it may be a t-set point, and that in any case it is a t'-set point where $t' \leqslant t$.

Let $C_0, C_1, \ldots, C_t$ be a set of $t+1$ general fixed curves through q_0, which determine q_0. Let q_n be the base-point whose prime equations are $(E_1)^{+1}_{+1} = 0, \ldots, (E_n)^{+1}_{+1} = 0, E_{n+1} = 0, \ldots, E_t = 0$ ($n = 1, 2, \ldots, t$).

We can choose polynomials in x, viz., $a_1, a_2, \ldots, a_t$, such that

$$C_1 + a_1 C_0, \quad C_2 + a_2 C_0, \quad \ldots, \quad C_t + a_t C_0$$

are a set of t fixed general curves through q_1, having no mutual connexions other than those which are involved in the fact that they all pass through q_1 (§ 22). The base-point determined by these curves and C_0 is q_0. Hence we may substitute these curves for $C_1, C_2, \ldots, C_t$, *i.e.*, we may so choose $C_0, C_1, \ldots, C_t$ that $C_1, C_2, \ldots, C_t$ are t general fixed curves through q_1. In the same way we may choose $C_2, C_3, \ldots, C_t$ so as to pass through q_2, and $C_3, C_4, \ldots, C_t$ so as to pass through q_3, and so on. Thus we may suppose the $t+1$ fixed curves $C_0, C_1, \ldots, C_t$ which determine q_0 to be *general* fixed curves through $q_0, q_1, \ldots, q_t$ respectively.

Let S_1 be any curve through q_1. Consider the curve

$$S_1' \equiv b_0 (S_1 + b_1 C_1 + b_2 C_2 + \ldots + b_{t-1} C_{t-1}) + b_t C_t,$$

where $b_0, b_1, \ldots, b_t$ are polynomials in x. We can choose $b_1, b_2, \ldots, b_{t-1}$ in succession so that $S_1 + b_1 C_1$ passes through q_2, $S_1 + b_1 C_1 + b_2 C_2$ through q_3, and, finally, $S_1 + b_1 C_1 + \ldots + b_{t-1} C_{t-1}$ through q_t. We can then choose b_0 and b_t so that S_1' is divisible by y, and so that b_0 has a non-vanishing constant term, since C_t is a general fixed curve through q_t. We now have $S_1' \equiv y S_0$, where S_0 passes through q_0, since S_1' passes through q_t. Hence, by hypothesis,

$$S_1' \equiv y (C_0 P_0 + C_1 P_1 + \ldots + C_t P_t).$$

Equating the two expressions for S_1', we see that S_1 is of the form $yC_0P_0+C_1P_1+C_2P_2+\ldots+C_tP_t$, where yC_0, C_1, C_2, ..., C_t are $t+1$ fixed curves through q_1. This proves the theorem.

43. *If $t+1$ fixed curves C_0, C_1, ..., C_t are such that no identity of the form*

$$C_0S_0+C_1S_1+\ldots+C_tS_t\equiv 0$$

can exist unless the constant terms of S_0, S_1, ..., S_t all vanish, then the base-point q determined by C_0, C_1, ..., C_t at the origin cannot be determined by any set of fixed curves less in number than $t+1$.

If possible, let t fixed curves C_1', C_2', ..., C_t' determine the base-point q. Then we have

$$C_n\equiv A_{n1}'C_1'+A_{n2}'C_2'+\ldots+A_{nt}'C_t' \quad (n=0,1,2,\ldots,t), \quad (1)$$

and
$$C_n'\equiv A_{0n}C_0+A_{1n}C_1+\ldots+A_{tn}C_t \quad (n=1,2,\ldots,t), \quad (2)$$

where A_{0n}, A_{1n}, ..., A_{n1}', A_{n2}', ... are power series. Substituting the values of C_1', C_2', ..., C_t' from (2) in (1), we have

$$C_n\equiv\sum_{m=0}^{m=t}(A_{m1}A_{n1}'+A_{m2}A_{n2}'+\ldots+A_{mt}A_{nt}')\,C_m \quad (n=0,1,2,\ldots,t).$$

Now, by hypothesis, the constant terms of the power series which appear as the coefficients of C_0, C_1, C_2, ..., C_t in the last identity, after C_n on the left has been transferred to the right, must all vanish. Hence, if a_{mn}, a_{mn}' are the constant terms of A_{mn}, A_{mn}', we have

$$a_{m1}a_{n1}'+a_{m2}a_{n2}'+\ldots+a_{mt}a_{nt}'=0 \quad \begin{pmatrix} m=0,1,2,\ldots,t \\ n=0,1,2,\ldots,t\end{pmatrix}, \quad \text{when } m\neq n,$$

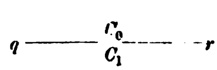
and
$$a_{n1}a_{n1}'+a_{n2}a_{n2}'+\ldots+a_{nt}a_{nt}'=1 \quad (n=0,1,2,\ldots,t).$$

This system of equations is impossible, since it would give unity as the result of multiplying two vanishing determinants, those, namely, whose $(n+1)^{\text{th}}$ rows are $(a_{n1}, a_{n2}, \ldots, a_{nt}, 0)$ and $(a_{n1}', a_{n2}', \ldots, a_{nt}', 0)$, where $n=0,1,2,\ldots,t$.

It follows that q is a t'-set point where $t'\geqslant t$; for, if q were a t'-set point where $t'<t$, then t fixed curves would suffice to determine q (§ 42).

44. *The base-point q of § 43 is a t-set point.*

Let C_0, C_1 through q intersect again at the origin in r, and let χ, ρ be the general algebraic curves or power series through q, r



respectively. Then, by hypothesis,

$$\chi \equiv C_0P_0 + C_1P_1 + \ldots + C_tP_t,$$

where $P_0, P_1, \ldots, P_t$ are arbitrary power series. The r identical equations which the coefficients of ρ have to satisfy are all involved in the condition that $\chi\rho$ should pass through the one-set point $q+r$, *i.e.*, that $\chi\rho$ should be of the form $C_0P + C_1P'$ (§ 33). For this it is necessary and sufficient that $\rho C_2, \rho C_3, \ldots, \rho C_t$ should all pass through $q+r$, which gives $t-1$ one-set systems of equations for the coefficients of ρ (§ 37, 2nd paragraph). Thus r is a t'-set point where $t' \leqslant t-1$; and therefore t fixed curves $C_1', C_2', \ldots, C_t'$ suffice to determine r (§ 42). Hence

$$\rho \equiv C_1'P_1' + C_2'P_2' + \ldots + C_t'P_t',$$

where $P_1', P_2', \ldots, P_t'$ are arbitrary power series. Again, since $\chi\rho$ has only to pass through $q+r$, it is necessary and sufficient for χ that $\chi C_1', \chi C_2', \ldots, \chi C_t'$ should all pass through $q+r$, giving t one-set systems of equations for the coefficients of χ. Hence q is a t'-set point where $t' \leqslant t$. But q is also a t'-set point where $t' \geqslant t$ (§ 43). Therefore q is a t-set point. It follows also that r is a $(t-1)$-set point.

45. *The number of fixed curves required to determine any given t-set point is* $t+1$; for not more than $t+1$ curves are required (§ 42); and not less, since any less number of fixed curves would determine a t'-set point where $t' < t$ (§ 44).

It also follows from § 44 that *if any two fixed curves* C_l, C_m *are drawn through a t-set point q, intersecting again at the point in r, then r is a* $(t-1)$*-set, a t-set, or a* $(t+1)$*-set point.* The different cases may be distinguished as follows:—If C_l, C_m are two of a set of $t+1$ curves which determine q, then r is a $(t-1)$-set point; if one of the two curves C_l, C_m, but not both, belongs to such a set of $t+1$ curves, then r is a t-set point; and if neither C_l nor C_m belongs to such a set of $t+1$ curves, then r is a $(t+1)$-set point. The condition that C_l may be one of a set of $t+1$ curves which determine q is that C_l should be of the form $C_0P_0 + C_1P_1 + \ldots + C_tP_t$, where $C_0, C_1, \ldots, C_t$ are any set of fixed curves which determine q, and $P_0, P_1, \ldots, P_t$ do not all vanish at the origin. This may be proved by the method of § 43. It is clear that the values of the constant terms of $P_0, P_1, \ldots, P_t$ are all unique, as may be seen by equating any two expressions for C_l of the form $C_0P_0 + C_1P_1 + \ldots + C_tP_t$. The condition that the curves

C_l, C_m, ... may be so many of a set of $t+1$ curves which determine q is that C_l, C_m, ... all pass through q, and that no values can be given to the constants λ, μ, ... such that the curve $\lambda C_l + \mu C_m + \ldots$ is not one of a set of $t+1$ curves which determine q.]

A Method for Extending the Accuracy of certain Mathematical Tables. By W. F. SHEPPARD, M.A., LL.M.

Received and read December 14th, 1899. Received, in revised form, April 2nd, 1900.

I. *Introductory.*

1. As an example of the cases to which the following method applies, we may suppose that we have a table of values of

$$u \equiv \tan \tfrac{1}{2}\pi x$$

to seven places of decimals, by intervals of ·01 in x, from $x = {\cdot}00$ to $x = {\cdot}50$, and that we have also a table of values of u to eleven places of decimals, but at larger intervals—say at intervals of ·1. Then our object is to obtain a table which shall have the same intervals as the former, but shall have (approximately) the same accuracy as the latter. For convenience, the table in which the accuracy is less while the intervals are those prescribed may be called the *working table*, while the shorter table, giving the more accurate values of u for a few values of x, may be called the *checking table*.

The method consists in using the working table as the basis for the calculation of the first or second differences in the new table. This latter table is formed by the successive addition of the differences so found; and the values are checked from time to time by means of the more accurate table. The rate at which the accuracy of the table can be extended depends partly on the nature of the function tabulated and partly on the smallness of the successive increments of the argument. Thus, in the case given above, it will be found that the use of first differences, with a certain amount of "smoothing," will extend the table with tolerable accuracy to nine places, while a repetition of the process will extend it to eleven (or certainly to ten)

places; or this latter result may be achieved in one operation by the use of second differences.

Before describing the formulæ employed, some preliminary observations are necessary.

2. There are certain well-known cases in which mathematical tables are (or might be) calculated by a formula of derivation, each value in the table being found from the preceding value or from a limited number of preceding values. Thus, for tabulating the function

$$f(x) \equiv e^x$$

by intervals of h in x, we should obviously use the formula

$$f(x+h) = e^h f(x),$$

each value being found from the preceding by multiplying by a constant factor. Similarly, to tabulate

$$f(x) \equiv \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

to a larger number of places than would be given by ordinary logarithmic tables, we should first write down the successive values of

$$e^{-h(x+\frac{1}{2}h)},$$

and then use the formula

$$f(x+h) = e^{-h(x+\frac{1}{2}h)} f(x).$$

Or, again, for constructing a table of

$$f(x) \equiv \sin \tfrac{1}{2}\pi x,$$

we might use the formula

$$f(x+h) = 2\cos \tfrac{1}{2}\pi h\, f(x) - f(x-h),$$

each value being thus found from the *two* preceding values. Tables formed in this way must be checked at intervals, by direct calculation of particular values, in order to prevent the accumulation of small errors.

3. In the above cases each value in the table is found from the immediately preceding value, or from a finite number of preceding values. This is not always possible. But a formula of derivation can always be obtained, and, *provided the interval h is small enough*, can always be used, in those cases *in which the differential coefficient of the function tabulated can be expressed in terms of the function itself*,

either alone or in conjunction with the argument. For suppose the function to be

$$u \equiv f(x), \tag{1}$$

and that it satisfies the equation

$$du/dx = \phi(x, u), \tag{2}$$

where $\phi(x, u)$ is a function whose value can be calculated for any given value of x if u is known for that value. Then, writing

$$\left.\begin{aligned} u_0 &= f(x_0) \\ u_{\pm n} &= f(x_0 \pm nh) \end{aligned}\right\},$$

where x_0 is any value of x appearing in the table, we have

$$\begin{aligned} u_1 &= f(x_0+h) \\ &= u_0 + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \frac{h^3}{3!}f'''(x_0) + \dots . \end{aligned} \tag{3}$$

Also, if we write

$$v \equiv \phi(x, u), \tag{4}$$

we have

$$\left.\begin{aligned} hv_0 &= hf'(x_0) \\ hv_{-1} &= hf'(x_0-h) \\ &= hf'(x_0) - h^2f''(x_0) + \frac{h^3}{2!}f'''(x_0) - \dots \\ hv_{-2} &= hf'(x_0) - 2h^2f''(x_0) + \frac{4h^3}{2!}f'''(x_0) - \dots \\ \vdots \quad & \quad \vdots \end{aligned}\right\}, \tag{5}$$

and thence

$$\left.\begin{aligned} hv_0 &= hf'(x_0) \\ \Delta\, hv_{-1} &= h^2f''(x_0) - \tfrac{1}{2}h^3f'''(x_0) + \tfrac{1}{6}h^4f^{\text{iv}}(x_0) - \dots \\ \Delta^2\, hv_{-2} &= h^3f'''(x_0) - h^4f^{\text{iv}}(x_0) + \dots \\ \Delta^3\, hv_{-3} &= h^4f^{\text{iv}}(x_0) - \dots \\ \vdots \quad & \quad \vdots \end{aligned}\right\}. \tag{6}$$

If now we eliminate between (3) and (6), we find

$$\begin{aligned} u_1 = u_0 + hv_0 + \tfrac{1}{2}\Delta\, hv_{-1} + \tfrac{5}{12}\Delta^2 hv_{-2} + \tfrac{3}{8}\Delta^3 hv_{-3} + \tfrac{251}{720}\Delta^4 hv_{-4} \\ + \tfrac{95}{288}\Delta^5 hv_{-5} + \dots, \end{aligned} \tag{7}$$

the coefficients in which may be shown to be the same as those in the expansion—

$$\frac{-\theta}{(1-\theta)\log(1-\theta)} = 1 + \tfrac{1}{2}\theta + \tfrac{5}{12}\theta^2 + \tfrac{3}{8}\theta^3 + \tfrac{251}{720}\theta^4 + \tfrac{95}{288}\theta^5 + \dots .$$

The values of hv_0, Δhv_{-1}, $\Delta^2 hv_{-2}$, ... are known if those of u_0, u_{-1}, u_{-2}, ... are known; and thus each value of u can be calculated from those which precede it.

To illustrate this formula, let us take the example given in § 1. We have then

$$u = \tan \tfrac{1}{2}\pi x,$$

$$v = du/dx = \tfrac{1}{2}\pi \sec^2 \tfrac{1}{2}\pi x$$

$$= \tfrac{1}{2}\pi(1+u^2). \qquad (a)$$

Suppose that the values of u have been found to seven places of decimals by intervals of

$$h \equiv \cdot 01$$

up to $x = \cdot 35$. Then, if we calculated the values of hv, and took their differences, we should have a table of the following form:—

x	u	hv	Δhv	$\Delta^2 hv$	$\Delta^3 hv$	$\Delta^4 hv$
			+	+	+	+
·30	·5095254	·0197860	3256	174	8	
·31	·5294727	·0201116	3438	182	11	1
·32	·5497547	·0204554	3631	193	10	
·33	·5703899	·0208185	3834	203	11	
·34	·5913984	·0212019	4048	214		
·35	·6128008	·0216067				

the fourth difference being found from the average of three or four successive values.

To apply (7), we only require the quantities at the end of the table, which may be written thus, Δ' denoting $\Delta/(1+\Delta)$:—

x	u	hv	$\Delta' hv$	$\Delta'^2 hv$	$\Delta'^3 hv$	$\Delta'^4 hv$
·35	·6128008	·0216067	+4048	+214	+11	+1

The formula then gives, for $x = \cdot 36$,

$$u = 10^{-7}(6128008+216067+\tfrac{1}{2}\text{ of }4048+\tfrac{5}{12}\text{ of }214+\tfrac{3}{8}\text{ of }11+\tfrac{251}{720}\text{ of }1)$$

$$= \cdot 6346193,$$

whence, by (a), $\qquad hv = \cdot 0220342.$

Taking the differences, we get the next line:—

x	u	hv	$\Delta' hv$	$\Delta'^2 hv$	$\Delta'^3 hv$	$\Delta'^4 hv$
·36	·6346193	·0220342	+4275	+227	+13	+1

Continuing the process, I get the following table:—

x	u	hv	$\Delta' hv$	$\Delta'^2 hv$	$\Delta'^3 hv$	$\Delta'^4 hv$
·35	·6128008	·0216067	4048	214	11	1
·36	·6346193	·0220342	4275	227	13	1
·37	·6568772	·0224858	4516	241	14	1
·38	·6795994	·0229628	4770	254	13	1
·39	·7028118	·0234668	5040	270	16	1
·40	·7265425					

The values of u so found are all (practically) correct within 1 in the final figure.

4. The above method might often, I think, be found useful, provided the differences of hv diminish fairly rapidly. But, when this is not the case, there are two objections to be met. In the first place, a great many differences have to be taken into account; and this is troublesome, as the coefficients by which these differences have to be multiplied are not very convenient for calculation. In the second place, the coefficients do not diminish at all rapidly. The effect of this is that the necessary errors in u, due to the results being initially only accurate to seven places, become greatly magnified. and the values have to be checked at very short intervals.

This latter difficulty might be almost entirely removed, in the majority of cases, by taking hv to a larger number of decimal places than u. Thus, in the above example, if the initial values of u are correct to seven places, the initial values of u^2 (up to $x = \cdot 50$) will be correct within 1×10^{-7}, and the values of hv will therefore be correct within $\frac{1}{2}\pi \times 10^{-9}$. By keeping in the two extra figures, the first differences may be found very accurately to seven places of decimals.

There will, however, still remain the difficulty as to the number of differences to be taken into account. And it may be added that there is a third objection, which will appeal strongly to any one who has had practical experience in constructing tables. The series of calculations by which a value of u is found is always the same, but each of these series of calculations has to be performed independently. A great saving of labour would be effected if the calculations could be taken in sets of similar processes, performed separately on u, hv, $\Delta' hv$, ... This is not possible when the table is being constructed for the first time. But when we possess a "working table" of u, of a less degree of accuracy than that which we are seeking, it becomes

possible to simplify the work, by using the method explained in the following sections.

II. *Method as applied to First Differences.*

5. In all cases in which we are concerned with the successive values of a tabulated function, the method of central differences provides us with series which converge very rapidly, and therefore are suitable for numerical calculation. Thus—retaining for the moment the ordinary notation, but using the particular differences which enter into the improved formulæ—our formula of derivation (7) is replaced by

$$\begin{aligned} u_1 &= u_0 + hv_0 + \tfrac{1}{2}\Delta hv_0 - \tfrac{1}{12}\Delta^2 hv_{-1} - \tfrac{1}{24}\Delta^3 hv_{-1} + \tfrac{11}{720}\Delta^4 hv_{-2} \\ &\qquad + \tfrac{11}{1440}\Delta^5 hv_{-2} - \dots \\ &= u_0 + \tfrac{1}{2}(hv_0 + hv_1) - \tfrac{1}{24}(\Delta^2 hv_{-1} + \Delta^2 hv_0) \\ &\qquad + \tfrac{11}{1440}(\Delta^4 hv_{-2} + \Delta^4 hv_{-1}) - \dots . \end{aligned} \qquad (8)$$

This is obviously more convenient than (7). The apparent difficulty is that we do not know the values of Δhv_0, $\Delta^2 hv_{-1}$, ..., until the values of u_1 and u_2, and perhaps also those of u_3 and u_4, have been calculated. But the point to be noticed is that the unknown quantities all contain h as a factor, and therefore, if h is sufficiently small, they can be obtained with sufficient accuracy from a shorter table of values of u. Suppose, for instance, that the rate of change of u is less, or at any rate not much greater, than that of x, and that $h = \cdot 01$. Then, if we have a working table of u, correct to seven places of decimals, we can deduce a table of hv, practically correct to nine places of decimals; and thus, calculating the successive differences of u from (8), we can build up a new table of u to nine places of decimals. This, again, can be used as a working table for getting a new table to eleven places; and so on, indefinitely.

6. In the absence of any recognized notation for central-difference formulæ, I find it convenient to use* two operators δ and μ, defined by the following relations:—

$$\left.\begin{aligned} \delta f(x) &= f(x + \tfrac{1}{2}h) - f(x - \tfrac{1}{2}h) \\ \mu f(x) &= \tfrac{1}{2}\{f(x + \tfrac{1}{2}h) + f(x - \tfrac{1}{2}h)\} \end{aligned}\right\}. \qquad (9)$$

* These operators are more fully discussed in a subsequent paper (*post*, pp. 449–488).

[redacted]

Thus, if $u = f(x)$, and if u_0 and u_1 denote any two successive values of u in a table proceeding by intervals of h in x,

$$\left.\begin{aligned} \delta u_{\frac{1}{2}} &= u_1 - u_0 \\ &= \Delta u_0 \\ \mu u_{\frac{1}{2}} &= \tfrac{1}{2}(u_1 + u_0) \end{aligned}\right\}. \tag{10}$$

Repeating the process denoted by δ, we have

$$\left.\begin{aligned} \delta^2 u_0 &= u_1 - 2u_0 + u_{-1} & \delta^3 u_{\frac{1}{2}} &= u_2 - 3u_1 + 3u_0 - u_{-1} \\ &= \Delta^2 u_{-1}, & &= \Delta^3 u_{-1} \\ \delta^4 u_0 &= \Delta^4 u_{-2}, & \delta^5 u_{\frac{1}{2}} &= \Delta^5 u_{-2} \\ &\vdots & &\vdots \end{aligned}\right\};$$

and, taking μ with powers of δ,

$$\left.\begin{aligned} \mu\delta u_0 &= \tfrac{1}{2}(\delta u_{\frac{1}{2}} + \delta u_{-\frac{1}{2}}) & \mu\delta^2 u_{\frac{1}{2}} &= \tfrac{1}{2}(\delta^2 u_1 + \delta^2 u_0) \\ &= \tfrac{1}{2}(\Delta u_0 + \Delta u_{-1}), & &= \tfrac{1}{2}(\Delta^2 u_0 + \Delta^2 u_{-1}) \\ \mu\delta^3 u_0 &= \tfrac{1}{2}(\delta^3 u_{\frac{1}{2}} + \delta^3 u_{-\frac{1}{2}}) & \mu\delta^4 u_{\frac{1}{2}} &= \tfrac{1}{2}(\delta^4 u_1 + \delta^4 u_0) \\ &= \tfrac{1}{2}(\Delta^3 u_{-1} + \Delta^3 u_{-2}), & &= \tfrac{1}{2}(\Delta^4 u_{-1} + \Delta^4 u_{-2}) \\ \mu\delta^5 u_0 &= \tfrac{1}{2}(\delta^5 u_{\frac{1}{2}} + \delta^5 u_{-\frac{1}{2}}) & \mu\delta^6 u_{\frac{1}{2}} &= \tfrac{1}{2}(\delta^6 u_1 + \delta^6 u_0) \\ &= \tfrac{1}{2}(\Delta^5 u_{-2} + \Delta^5 u_{-3}), & &= \tfrac{1}{2}(\Delta^6 u_{-2} + \Delta^6 u_{-3}) \\ &\vdots & &\vdots \end{aligned}\right\}.$$

In this notation our formula (8) becomes*

$$\begin{aligned} u_1 - u_0 &= \delta u_{\frac{1}{2}} \\ &= \delta \int^{x_{\frac{1}{2}}} v\,dx \\ &= \mu\left(1 - \tfrac{1}{12}\delta^2 + \tfrac{11}{720}\delta^4 - \tfrac{191}{60480}\delta^6 + \tfrac{2497}{3628800}\delta^8 - \ldots\right) h v_{\frac{1}{2}}. \end{aligned} \tag{11}$$

The process therefore consists in using the values of u, given by the working table, as the basis for calculating the values of $hv \equiv h\,du/dx$, and then applying the formula (11) to determine the successive first differences of u in the new table. The values so found must be compared from time to time with the values given in the working table, in order to prevent an accumulation of errors.

---

* See p. 480, formula (140).

7. In the preceding section we have supposed that the values of x in the final table are to be the same as the values in the initial or working table. But it will be found simpler, in practice, to take the values in the working table halfway between the values in the final table. Thus, to tabulate u for $x = \cdot 00, \cdot 01, \cdot 02, \ldots$, we should use a working table in which the values of x are $\ldots, \cdot 005, \cdot 015, \cdot 025, \ldots$. Using this table as the basis for calculating the values of $\ldots, hv_{\frac{1}{2}}, hv_{\frac{3}{2}}, \ldots$, we have*

$$\delta u_{\frac{1}{2}} = \left(1 + \tfrac{1}{24}\delta^2 - \tfrac{17}{5760}\delta^4 + \tfrac{367}{967680}\delta^6 - \tfrac{27859}{464486400}\delta^8 + \ldots\right) hv_{\frac{1}{2}}. \quad (12)$$

The coefficients in this formula are a good deal smaller than the coefficients in (11).

8. The principle underlying the method may be made clearer by a geometrical explanation. Let the successive values $\ldots, x_0, x_1, \ldots$ of the argument be represented by abscissæ $\ldots, OM_0, OM_1, \ldots$ measured along a line OX, so that $M_0M_1 = M_1M_2 = \ldots = h$; and at $\ldots, M_0, M_1, \ldots$, let ordinates $\ldots, M_0Q_0, M_1Q_1, \ldots$ be erected, equal to the values of u *given by the working table*. Let the true ordinates of the curve $u = f(x)$ be $\ldots, M_0P_0, M_1P_1, \ldots$; and suppose that each value in the working table is correct within $\pm\frac{1}{2}\rho$. Then, if on M_rQ_r we take

$$q'_rQ_r = Q_rq_r = \tfrac{1}{2}\rho,$$

all that the working table shows us is that P_r lies somewhere between q'_r and q_r; and similarly for P_{r+1}. Now let $\phi_{r+\frac{1}{2}}$ denote the inclination of P_rP_{r+1} to OX, so that

$$M_{r+1}P_{r+1} = M_rP_r + h\tan\phi_{r+\frac{1}{2}}.$$

Then, if we knew the exact position of P_0, and also the exact values of $\phi_{\frac{1}{2}}, \phi_{\frac{3}{2}}, \ldots$, we could (theoretically) determine the exact positions of $P_1, P_2, \ldots$. All that the working table tells us directly about $\tan\phi_{r+\frac{1}{2}}$ is that it lies somewhere between $(M_{r+1}Q_{r+1} - M_rQ_r - \rho)/h$ and $(M_{r+1}Q_{r+1} - M_rQ_r + \rho)/h$; *i.e.*, there is a possible error of $\pm\rho$ in $h\tan\phi_{r+\frac{1}{2}}$. But, if $\tan\phi_{r+\frac{1}{2}}$ can be expressed as a function of M_rP_r and $M_{r+1}P_{r+1}$, and the ordinates immediately preceding and following them, its value can be calculated with a certain degree of accuracy by using $\ldots, M_rQ_r, M_{r+1}Q_{r+1}, \ldots$ in the place of $\ldots, M_rP_r, M_{r+1}P_{r+1}, \ldots$. The limit of the error so introduced will usually be comparable with ρ; and therefore the limit of the error in $h\tan\phi_{r+\frac{1}{2}}$ will be comparable with $h\rho$. If h is so small that this limit is appreciably less than ρ,

* See p. 480, formula (139).

we can substitute the values of $h\tan\phi_{\frac{1}{2}}$, $h\tan\phi_{\frac{3}{2}}$, found in this way, for the values as shown directly by the table; and then, starting from a more accurate position of P_0, we shall arrive at more accurate positions of P_1, P_2, ...

We have supposed, in the above, that the values of x are to be the same in both tables; the formula (11) then gives $h\tan\phi_{r+\frac{1}{2}}$ in terms of ..., $h\tan\psi_r$, $h\tan\psi_{r+1}$, ..., where ψ_r denotes the inclination to OX of the tangent at P_r to the curve $u = f(x)$. But the explanation applies, with the necessary modifications, if the ordinates given in the working table are ..., $M_{-\frac{1}{2}}Q_{-\frac{1}{2}}$, $M_{\frac{1}{2}}Q_{\frac{1}{2}}$, $M_{\frac{3}{2}}Q_{\frac{3}{2}}$, ... ; the value of $h\tan\phi_{r+\frac{1}{2}}$, in terms of ..., $h\tan\psi_{r-\frac{1}{2}}$, $h\tan\psi_{r+\frac{1}{2}}$, $h\tan\psi_{r+\frac{3}{2}}$, ..., is then given by (12).

Let the ordinates whose more accurate values are given by the checking table be M_0P_0, M_nP_n, $M_{2n}P_{2n}$, ... Then, starting from the given position of P_0, and proceeding by the successive steps, we may or may not hit the given position of P_n. If we do not, one or more of the values of $h\tan\phi$ must be altered. But, even then, there is always the possibility that our path may in the interval have steadily diverged, and then steadily come back again. It is therefore necessary that the limit of the error in $h\tan\phi$, as deduced from the ordinates given by the working table, should be *appreciably* less than ρ. Suppose, for instance, that this limit is $\frac{1}{10}\rho$, and that $n = 10$. Then, starting with the accurate value of M_0P_0, the deduced value of M_1P_1 is correct within $\frac{1}{10}\rho$. But the errors in M_0Q_0, M_1Q_1, ..., which give rise to the errors in $\tan\phi$, are independent, and therefore the errors in $\tan\phi_{\frac{1}{2}}$, $\tan\phi_{\frac{3}{2}}$, ... are practically independent. We can therefore only be sure that M_2P_2 is correct within $\frac{1}{5}\rho$; and, similarly, we can only be sure that M_5P_5 is correct within $\frac{1}{2}\rho$, *i.e.*, we cannot be sure that it is more correct than the original value in the working table.

In practice, however, these difficulties do not arise, on account of the tendency of independent errors to balance one another. Suppose, for instance, that by taking $h = \cdot 01$, and checking at intervals of $10h \equiv \cdot 1$, we can extend a seven-place table to nine places. Then, if we took $h = \cdot 001$ (a seven-place table at these intervals being supposed to exist), and only checked at intervals of $100h \equiv \cdot 1$, the possible error at each step would be divided by 10, but the number of steps would be multiplied by 10. The *possible* limit of error at the middle of the checking interval would therefore be unaltered. But the *probable* error would be about $1/\sqrt{10}$ of what it was before, so that, with some care in smoothing, the table would be tolerably

correct to ten places. It might, at any rate, be used to ten places for the purpose of getting a new table to twelve or thirteen places.

9. Suppose that we are using (12), and that u_0 and u_n are two consecutive values in the checking table. Then, taking u_0 and u_n to the same number of places as hv, and calculating $\delta u_{\frac{1}{2}}$, $\delta u_{\frac{3}{2}}$, ... from (12), we obtain successively

$$\left.\begin{aligned} u_1 &= u_0 \;\;\; + \delta u_{\frac{1}{2}} \\ u_2 &= u_1 \;\;\; + \delta u_{\frac{3}{2}} \\ &\;\vdots \qquad\;\; \vdots \\ u_n &= u_{n-1} + \delta u_{n-\frac{1}{2}} \end{aligned}\right\}.$$

If the sum of the calculated values of $\delta u_{\frac{1}{2}}$, $\delta u_{\frac{3}{2}}$, ..., $\delta u_{n-\frac{1}{2}}$ is not equal to $u_n - u_0$, one or more of them will require correction. But it is not always easy to decide whether the correction should be in one of the tabulated values of

$$hv,$$

or in one of the values of

$$\left(\tfrac{1}{24}\delta^2 - \tfrac{17}{5760}\delta^4 + \ldots\right) hv.$$

To avoid this difficulty, and the similar difficulty which arises in using (11), it is better to convert each formula into the corresponding formula for central summation.

In the notation which we shall adopt, the successive values of any function $f(x)$, for values of x proceeding by a constant difference h, are regarded as the first differences (δ) of another function, denoted by

$$\sigma f(x).$$

This function therefore satisfies the relation

$$\delta\,\sigma f(x) = f(x), \tag{13}$$

so that

$$\left.\begin{aligned} &\;\;\vdots & &\;\;\vdots \\ \sigma f(x-\tfrac{1}{2}h) - \sigma f(x-\tfrac{3}{2}h) &= f(x-h) \\ \sigma f(x+\tfrac{1}{2}h) - \sigma f(x-\tfrac{1}{2}h) &= f(x) \\ \sigma f(x+\tfrac{3}{2}h) - \sigma f(x+\tfrac{1}{2}h) &= f(x+h) \\ &\;\;\vdots & &\;\;\vdots \end{aligned}\right\}. \tag{14}$$

Replacing $f(x)$ by u, this gives

$$\left.\begin{aligned} &\vdots && \vdots \\ \sigma u_{-\frac{1}{2}} &= \ldots + u_{-2} + u_{-1} \\ \sigma u_{\frac{1}{2}} &= \ldots + u_{-2} + u_{-1} + u_0 \\ \sigma u_{\frac{3}{2}} &= \ldots + u_{-2} + u_{-1} + u_0 + u_1 \\ &\vdots && \vdots \end{aligned}\right\}. \tag{15}$$

We may adopt (15) as the definition of the operator σ, and we then have also

$$\left.\begin{aligned} &\vdots && \vdots \\ \mu\sigma u_{-1} &= \ldots + u_{-2} + \tfrac{1}{2}u_{-1} \\ \mu\sigma u_0 &= \ldots + u_{-2} + u_{-1} + \tfrac{1}{2}u_0 \\ \mu\sigma u_1 &= \ldots + u_{-2} + u_{-1} + u_0 + \tfrac{1}{2}u_1 \\ &\vdots && \vdots \end{aligned}\right\}. \tag{16}$$

The operation represented by σ involves the introduction of an arbitrary constant. If this constant is properly chosen, we have

$$\begin{aligned} \sigma\delta u_0 &= \ldots + \delta u_{-\frac{3}{2}} + \delta u_{-\frac{1}{2}} \\ &= u_0, \end{aligned}$$

and, similarly,

$$\begin{aligned} \sigma\delta^2 u_{\frac{1}{2}} &= \delta u_{\frac{1}{2}}, \\ \sigma\delta^3 u_0 &= \delta^2 u_0, \\ &\&c. \end{aligned}$$

Comparing these with (13), we see that σ combines with powers of δ according to the laws of algebra in the same way as if

$$\sigma = \delta^{-1}; \tag{17}$$

and it also combines according to these laws with powers of μ.

With this notation, the formulæ (11) and (12) become respectively, by successive additions,

$$u = \mu\left(\sigma - \tfrac{1}{12}\delta + \tfrac{11}{720}\delta^3 - \tfrac{191}{60480}\delta^5 + \tfrac{2497}{3628800}\delta^7 - \ldots\right) hv, \tag{18}$$

$$u = \left(\sigma + \tfrac{1}{24}\delta - \tfrac{17}{5760}\delta^3 + \tfrac{367}{967680}\delta^5 - \tfrac{27859}{464486400}\delta^7 + \ldots\right) hv. \tag{19}$$

To apply these latter formulæ, we write down the values of hv_0, hv_1, ..., or of $hv_{\frac{1}{2}}$, $hv_{\frac{3}{2}}$, ..., as the case may be, and take their differences. We then calculate the value of $\sigma hv_{\frac{1}{2}}$ or of σhv_0 from the value of u_0 given in the checking table, by writing (18) or (19) in the form

$$\sigma hv_{\frac{1}{2}} = u_0 + \tfrac{1}{2}hv_0 + \tfrac{1}{12}\mu\delta\, hv_0 - \tfrac{11}{720}\mu\delta^3\, hv_0 + \ldots \tag{20}$$

or

$$\sigma hv_0 = u_0 - \tfrac{1}{24}\delta\, hv_0 + \tfrac{17}{5760}\delta^3\, hv_0 - \ldots, \tag{21}$$

and perform a similar process for $\sigma hv_{n+\frac{1}{2}}$, $\sigma hv_{2n+\frac{1}{2}}$, ..., or for σhv_n, σhv_{2n}, ... If we use (18), we ought then to have

$$\left.\begin{array}{l} \sigma hv_{n+\frac{1}{2}} - \sigma hv_{\frac{1}{2}} \;\;= hv_1 \;\;+ hv_2 \;\;+ \ldots + hv_n \\ \sigma hv_{2n+\frac{1}{2}} - \sigma hv_{n+\frac{1}{2}} = hv_{n+1} + hv_{n+2} + \ldots + hv_{2n} \\ \quad\vdots \qquad\qquad\qquad\quad \vdots \end{array}\right\} \tag{22}$$

or

$$\left.\begin{array}{l} \sigma hv_n - \sigma hv_0 = hv_{\frac{1}{2}} \;\;+ hv_{\frac{3}{2}} \;\;+ \ldots + hv_{n-\frac{1}{2}} \\ \sigma hv_{2n} - \sigma hv_n = hv_{n+\frac{1}{2}} + hv_{n+\frac{3}{2}} + \ldots + hv_{2n-\frac{1}{2}} \\ \qquad\vdots \qquad\qquad\quad \vdots \end{array}\right\}. \tag{23}$$

If these conditions are not satisfied, one or more of the values of hv must be altered, by inspection of the differences. The necessary alterations having been made, the intermediate values of σhv are found by successive addition of the values of hv, and (18) or (19) is then applied for calculating the values of u.

If any values of hv are altered, it will not usually be necessary to make the corresponding alterations in the first or higher differences, since the coefficients of δhv, $\delta^3 hv$, ... in the formula used are so small that the resulting terms are hardly affected by the alterations.

It should be noticed that the arbitrary constant in σhv has the same value in (18) as in (19). As soon, therefore, as the series of values of σhv has been obtained, we can apply either formula indifferently. Or we may, if we like, apply both formulæ, and thereby obtain a table in which the values of u proceed by intervals of $\frac{1}{2}h$ in x.

Some numerical examples, illustrating the method, will be found in the following sections.

[redacted]

10. For a simple example, let us take, as in § 3,

$$u = \tan \tfrac{1}{2}\pi x,$$

so that
$$v = du/dx = \tfrac{1}{2}\pi\,(1+u^2). \qquad (a)$$

We may suppose that we have a seven-place table of u by intervals of ·01 in x, and a nine-place table by intervals of ·1, and that we require a nine-place table by intervals of ·01. For $x =$ ·30 and $x =$ ·40 our checking (nine-place) table gives

x	u
·30	·50952 5449
·40	·72654 2528

(β)

From the seven-place table, with a little smoothing of differences, I get a table of hv, of which the following is a portion :—

x	u	hv	$\delta\, hv$	$\delta^2\, hv$	$\delta^3\, hv$	$\delta^4\, hv$
		+	+	+	+	+
			30 8176		838	
·30	·5095254	·01978 6005		1 7375		64
			32 5551		902	
·31	·5294727	·02011 1556		1 8277		74
			34 3828		976	
·32	·5497547	·02045 5384		1 9253		80
			36 3081		1056	
·33	·5703899	·02081 8465		2 0309		83
			38 3390		1139	
·34	·5913984	·02120 1855		2 1448		96
			40 4838		1235	
·35	·6128008	·02160 6693		2 2683		103
			42 7521		1338	
·36	·6346193	·02203 4214		2 4021		111
			45 1542		1449	
·37	·6568772	·02248 5756		2 5470		125
			47 7012		1574	
·38	·6795993	·02296 2768		2 7044		134
			50 4056		1708	
·39	·7028118	·02346 6824		2 8752		152
			53 2808		1860	
·40	·7265425	·02399 9632		3 0612		166
			56 3420		2026	

(γ)

From (β) and (γ), by means of (20), we get, for $x =$ ·305,

$$\sigma hv = 10^{-9}\,(50952\ 5449 + \tfrac{1}{2} \text{ of } 1978\ 6005 + \tfrac{1}{24} \text{ of } 63\ 3727 - \tfrac{11}{1440} \text{ of } 1740)$$

$$= \cdot 51944\ 4843\,;$$

and, for $x = \cdot 405$,

$$\sigma hv = \cdot 73858\ 7990.$$

The difference of these is ·21914 3147, which is equal to the sum of the calculated values of hv from $x = \cdot 31$ to $x = \cdot 40$, so that no correction is needed in these values. Finding the successive values of σhv by successive additions of hv, and applying (19), we get the result shown below. The second and fourth differences of hv are omitted, for convenience of printing. All the values of u, as shown in the last column, are correct within 1 in the final figure.

x	σhv	hv	δhv	$\delta^3 hv$	u
·295	·49965 8838		30 8176	838	·49967 1676
		1978 6005			
·305	·51944 4843		32 5551	902	·51945 8405
		2011 1556			
·315	·53955 6399		34 3828	976	·53957 0722
		2045 5384			
·325	·56001 1783		36 3081	1056	·56002 6908
		2081 8465			
·335	·58083 0248		38 3390	1139	·58084 6219
		2120 1855			
·345	·60203 2103		40 4838	1235	·60204 6968
		2160 6693			
·355	·62363 8796		42 7521	1338	·62365 6606
		2203 4214			
·365	·64567 3010		45 1542	1449	·64569 1820
		2248 5756			
·375	·66815 8766		47 7012	1574	·66817 7637
		2296 2768			
·385	·69112 1534		50 4056	1708	·69114 2531
		2346 6824			
·395	·71458 8358		53 2808	1860	·71461 0553
		2399 9632			
·405	·73858 7990		56 3420	2026	·73861 1460

If we wished the values of x in our final table to be ..., ·30, ·31, ·32, ..., we should have to use (18). To do this, we must calculate

$$\left(\sigma - \tfrac{1}{12}\delta + \tfrac{11}{720}\delta^3 - \ldots\right) hv \qquad (\epsilon)$$

for the intermediate values ..., ·295, ·305, ·315, ..., and then take the arithmetic mean of each pair of consecutive values. When, of two consecutive values given by (ϵ), one ends with an odd integer, and the other with an even integer, their arithmetic mean is doubtful. The doubt might be avoided by originally calculating $\frac{1}{2}hv$ instead of hv, and adding consecutive values of

$$\left(\sigma - \tfrac{1}{12}\delta + \tfrac{11}{720}\delta^3 - \ldots\right) \tfrac{1}{2}hv\,;$$

or, having calculated hv, we may keep in one or two extra figures in (ϵ), these figures being dropped after the arithmetic means have

been found. The following shows the process, from $x = \cdot 30$ to $x = \cdot 35$:—

x	$(\sigma - \frac{1}{12}\delta + \frac{11}{720}\delta^3 - \dots)\, hv$	u
·295	·49963 3169 47	....
·300	....	·50952 5449
·305	·51941 7727 61	....
·310	....	·52947 2745
·315	·53952 7761 49	....
·320	....	·54975 4652
·325	·55998 1542 38	....
·330	....	·57038 9929
·335	·58079 8316 15	....
·340	....	·59139 8351
·345	·60199 8385 45	....
·350	....	·61280 0788
·355	·62360 3189 69	....

These values, like the former, are correct within 1 in the final figure.

11. One class of functions to which the method is readily applicable comprises functions of the form

$$u = e^{\int P\,dx},$$

where P is some simple function of x. For we have, then,

$$du/dx = Pe^{\int P\,dx} = Pu,$$

so that hv is very easily calculated. If, for instance, we had constructed by means of logarithmic tables a table of values of

$$u = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2},$$

approximately correct to seven places of decimals, we could easily extend it, the formula being

$$hv = -hxu.$$

12. The method is also specially useful in dealing with the *inversion of integrals*. If x is given in terms of u by

$$x = \int_0^u \phi(u)\, du. \tag{24}$$

and if the values of x in terms of u are tabulated, we can by ordinary methods of approximation construct a table of u in terms of x. The accuracy of this latter table will be limited by the accuracy of the former. But we have, from (24),

$$dx/du = \phi(u);$$

and therefore $$hv = h\,du/dx = h/\phi(u). \qquad (25)$$

Hence, if h is sufficiently small, we can substitute in (25) the values already found for u, and then apply (18) or (19) to get a more accurate table.

The example in § 10 may be regarded as coming under this head. For we have

$$\tan^{-1} u = \int^{u} du/(1+u^2),$$

and our original seven-figure table of $u \equiv \tan \frac{1}{2}\pi x$ may be supposed to have been obtained by inversion from a table of $x \equiv 2/\pi \tan^{-1} u$.

A case of special interest is that in which $\phi(u)$ is an exponential function of u. Suppose that

$$\phi(u) = e^{\int Q\,du}, \qquad (26)$$

where Q is some simple function of u. Then we have, as in § 11,

$$\phi'(u) = Q\,\phi(u). \qquad (27)$$

But, by (24), $$dx/du = \phi(u),$$

whence $$du/dx = 1/\phi(u). \qquad (28)$$

Combining (27) and (28), we find that

$$d\{\phi(u)\}/dx = Q. \qquad (29)$$

Now the working table gives u in terms of x, at intervals of h. Calculating the values of

$$hQ,$$

and applying (18) or (19) to (29), we get the values of $\phi(u)$. Then, calculating the values of $h\,du/dx = h/\phi(u)$, and applying (18) or (19) again, we get back to a table of x, but with more accurate values.

Consider, for instance, the integral

$$a = 2\int_0^x z\,dx,$$

where $$z = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}.$$

Here we have
$$da/dx = 2z,$$
$$dz/dx = -xz,$$
so that
$$d(2z)/da = -x,$$
$$dx/da = 1/(2z).$$

I have used the above method for constructing tables of values of $2z$ and x in terms of a, by intervals of $\theta \equiv \cdot 01$ in a, from $a = \cdot 00$ to $a = \cdot 80$. The two tables given below show a portion of the work. The working table was obtained, to seven places of decimals, from Kramp's tables of $\log_{10} \int_t^\infty e^{-t^2} dt$; and, for the checking table,

a	x	$\sigma(-\theta x)$	$-\theta x$	δ	δ^2	δ^3	$2z$
			−	−	−	−	
·40	·5244005	·6979952 47	52440 05	1432 70	10 85	47	·6979892 78
·41	·5388360	·6927512 42	53883 60	1443 55	11 32	47	·6927452 28
·42	·5533847	·6873628 82	55338 47	1454 87	11 81	49	·6873568 20
·43	·5680515	·6818290 35	56805 15	1466 68	12 32	51	·6818229 24
·44	·5828415	·6761485 20	58284 15	1479 00	12 86	54	·6761423 57
·45	·5977601	·6703201 05	59776 01	1491 86	13 43	57	·6703138 89
·46	·6128130	·6643425 04	61281 30	1505 29	14 01	58	·6643362 31
·47	·6280060	·6582143 74	62800 60	1519 30	14 64	63	·6582080 43
·48	·6433454	·6519343 14	64334 54	1533 94	15 29	65	·6519279 22
·49	·6588377	·6455008 60	65883 77	1549 23	15 98	69	·6454944 04
·50	·6744898	·6389124 83	67448 98	1565 21	16 69	71	·6389059 61
		·6321675 85		1581 90		79	·6321609 94

a	$\sigma(\theta/2z)$	$\theta/2z$	δ	δ^3	δ^5	x
			+	+	+	
·40	·5243959 9438	144353 2138	1084 5374	1 9416	87	·5244005 1271
·41	·5388313 1576	145484 8444	1131 6306	2 0784	127	·5388360 3028
·42	·5533798 0020	146665 6466	1180 8022	2 2279	127	·5533847 1955
·43	·5680463 6486	147897 8485	1232 2019	2 3901	134	·5680514 9833
·44	·5828361 4971	149183 8402	1285 9917	2 5657	168	·5828415 0725
·45	·5977545 3373	150526 1874	1342 3472	2 7581	173	·5977601 2603
·46	·6128071 5247	151927 6482	1401 4608	2 9678	192	·6128129 9102
·47	·6279999 1729	153391 1904	1463 5422	3 1967	220	·6280060 1444
·48	·6433390 3633	154920 0107	1528 8203	3 4476	249	·6433454 0540
·49	·6588310 3740	156517 5567	1597 5460	3 7234	269	·6588376 9274
·50	·6744827 9307		1669 9951	4 0261	306	·6744897 5020

[redacted]

the values of x and z for $a = \cdot 1, \cdot 2, \ldots$ were determined very accurately. Calculating the initial and checking values of $\sigma(-\theta x)$ by means of (20), the application of (19) gives a table of $2z$ to nine places of decimals for $a = \cdot 005, \cdot 015, \ldots$. Thence the values of $\theta/(2z)$ are found (with a little smoothing) correct to eleven or ten places of decimals: and our final table is formed by a second application of (19). The first table on the preceding page shows the calculation of $2z$ for $a = \cdot 395, \cdot 405, \ldots, \cdot 505$, x being taken initially correct to seven places, and the first two and last two values of $\sigma(-\theta x)$ being adjusted so as to give $2z$ correct to nine places for $a = \cdot 40$ and $a = \cdot 50$. In the second table the values of $\theta/(2z)$ are inserted as found from the first table, and $\sigma\theta/(2z)$ is adjusted for $a = \cdot 40$ and $a = \cdot 50$. For convenience of printing, the even differences of $\theta/(2z)$ have been omitted.

13. The last class of cases which we shall consider under this head comprises those in which we are dealing with a definite integral, but, for convenience of interpolation, the quantity tabulated is not the integral itself, but is some function of the integral, or of the integral and the argument. If, for instance, we have

$$a = \sqrt{\frac{2}{\pi}} \int_0^x e^{-\frac{1}{2}x^2} dx,$$

it will be found that when x becomes tolerably great the higher differences of a become (relatively) very great. It is therefore more convenient to tabulate either

$$u_1 = e^{\frac{1}{2}x^2} \int_x^\infty e^{-\frac{1}{2}x^2} dx$$

or

$$u_2 = \log_{10} \sqrt{\frac{2}{\pi}} \int_x^\infty e^{-\frac{1}{2}x^2} dx.$$

The method can then be applied for extending the table either of u_1 or of u_2.

Generally, let $$u = \int_x^\infty z\,dx,$$

where $$z = e^{-\int P dx},$$

and let us write $$u_1 = z^{-1} u,$$

$$u_2 = \log_{10} u.$$

Then it is easily shown that for extending a table of u_1 we have

$$h\,du_1/dx = -(h - hPu_1)\,;$$

while for extending a table of u_2 we have

$$h\,du_2/dx = -\log_{10} e \,.\, hz/u,$$

or

$$\log_{10}(-h\,du_2/dx) = \log_{10}(\log_{10} e \,.\, hz) - u_2.$$

The values of hP in the first case, and of $\log_{10} e \,.\, hz$ or $\log_{10}(\log_{10} e \,.\, hz)$ in the second, are calculated directly from those of x, and may be taken, at once, to any degree of accuracy we require. When they have been found, the process of extension may be repeated indefinitely, with very little trouble.

14. The method fails whenever u is changing so rapidly that, even though h is small, the working table does not give $h\,du/dx$ to a much greater degree of accuracy than that of the first differences of u as actually shown. As a general rule, we should not apply the method to cases in which the first difference of u contains about the same number of significant figures as u itself; but this rule is open to a good many exceptions. In particular, we should notice whether the greater part of the first difference depends on a function of x not involving u; if so, it may still be possible to extend the table.

The failure of the method, however, is only a failure in its practical utility; theoretically we could go on applying it by taking smaller and smaller values of h.

III. *Extension to use of Second Differences.*

15. The formula (19) may be written

$$u = (\sigma + \tfrac{1}{24}\delta - \tfrac{17}{5760}\delta^3 + \dots)\, hD\, u,$$

where, as usual, D denotes differentiation. If we apply this to a table of u for $x = \dots x_0 - h,\ x_0,\ x_0 + h,\ \dots$, so as to get a table for $x = \dots x_0 - \tfrac{1}{2}h,\ x_0 + \tfrac{1}{2}h,\ \dots$, and then repeat the process, the result of the double operation may be denoted by

$$u = (\sigma + \tfrac{1}{24}\delta - \tfrac{17}{5760}\delta^3 + \dots)\, hD\, (\sigma + \tfrac{1}{24}\delta - \tfrac{17}{5760}\delta^3 + \dots)\, hD\, u.$$

The operators σ, δ, and hD combine according to the ordinary laws

of algebra, so that the result is equivalent to

$$u = (\sigma + \tfrac{1}{24}\delta - \tfrac{17}{5760}\delta^3 + \ldots)^2 h^2 D^2 u,$$

or*
$$u = (\sigma^2 + \tfrac{1}{12} - \tfrac{1}{240}\delta^2 + \tfrac{31}{60480}\delta^4 - \tfrac{289}{3628800}\delta^6 + \ldots)\, h^2 D^2 u. \tag{30}$$

This formula is quite general, and it enables us to extend the table of u, when u satisfies an equation of the form

$$d^2u/dx^2 = \psi\,(x, u), \tag{31}$$

even if du/dx cannot be expressed as a simple function of u and x. The symbol σ^2 represents the operation of a double summation, introducing two arbitrary constants. Its relation to the operations represented by σ and by powers of δ is shown by the following table :—

x	$\sigma^2 f(x)$	1st diff.	2nd diff.	3rd diff.	4th diff.	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$x_0 - h$	$\sigma^2 f(x_0 - h)$		$f(x_0 - h)$		$\delta^2 f(x_0 - h)$	.
		$\sigma f(x_0 - \frac{1}{2}h)$		$\delta f(x_0 - \frac{1}{2}h)$		...
x_0	$\sigma^2 f(x_0)$		$f(x_0)$		$\delta^2 f(x_0)$	
		$\sigma f(x_0 + \frac{1}{2}h)$		$\delta f(x_0 + \frac{1}{2}h)$		...
$x_0 + h$	$\sigma^2 f(x_0 + h)$		$f(x_0 + h)$		$\delta^2 f(x_0 + h)$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

To apply the formula, we must first tabulate the values of

$$h^2 w \equiv h^2 d^2u/dx^2,$$

by means of (31), using the values of u given in the working table. We then, by successive additions, construct a table of values of

$$\sigma\, h^2 w,$$

and, repeating the process, we get the values of

$$\sigma^2\, h^2 w,$$

from which u is given by

$$u = (\sigma^2 + \tfrac{1}{12} - \tfrac{1}{240}\delta^2 + \tfrac{31}{60480}\delta^4 - \tfrac{289}{3628800}\delta^6 + \ldots)\, h^2 w. \tag{32}$$

* See p. 483, formula (145).

The determination of the values $\sigma h^2 w_{\frac{1}{2}}$, $\sigma h^2 w_{n+\frac{1}{2}}$, $\sigma h^2 w_{2n+\frac{1}{2}}$, ..., by which the table of $\sigma h^2 w$ has to be checked, involves the knowledge of the accurate values of u_0, u_n, ..., and also of either u_1, u_{n+1}, ... or $(du/dx)_0$, $(du/dx)_n$, ... If the values known are u_1, u_{n+1}, ..., we must calculate $\sigma^2 h^2 w_0$, $\sigma^2 h^2 w_n$, ..., and also $\sigma^2 h^2 w_1$, $\sigma^2 h^2 w_{n+1}$, ..., by means of (32), written in the form

$$\sigma^2 h^2 w_0 = u_0 - \tfrac{1}{12} h^2 w_0 + \tfrac{1}{240} \delta^2 h^2 w_0 - \dots, \tag{33}$$

and then take

$$\left.\begin{aligned} \sigma h^2 w_{\frac{1}{2}} &= \sigma^2 h^2 w_1 - \sigma^2 h^2 w_0 \\ \sigma h^2 w_{n+\frac{1}{2}} &= \sigma^2 h^2 w_{n+1} - \sigma^2 h^2 w_n \\ \vdots \quad & \quad \vdots \end{aligned}\right\}. \tag{34}$$

But in most cases du/dx can be expressed in terms of u and x, so that $(du/dx)_0$, $(du/dx)_n$, ... can be calculated from the checking values of u_0, u_n, ... We have, for these cases,

$$\sigma h^2 w_{\frac{1}{2}} = h\,(du/dx)_0 + \tfrac{1}{2} h^2 w_0 + \left(\tfrac{1}{12}\mu\delta - \tfrac{11}{720}\mu\delta^3 + \dots\right) h^2 w_0, \tag{35}$$

which is obtained from (20) by writing $h\,du/dx$ for u.

16. The process of checking may be performed by first tabulating $\sigma h^2 w$, with any necessary alterations of $h^2 w$, and then making any further alterations which are necessary to make the values of $\sigma^2 h^2 w_n$, ... agree with those found from (33). But it is better to consider the checking values of $\sigma h^2 w$ and of $\sigma^2 h^2 w$ simultaneously. We should have, if the values of $h^2 w$ were exact,

$$\sigma h^2 w_{n+\frac{1}{2}} = \sigma h^2 w_{\frac{1}{2}} + h^2 w_1 + h^2 w_2 + h^2 w_3 + \dots + h^2 w_n, \tag{36}$$

$$\sigma^2 h^2 w_n = \sigma^2 h^2 w_0 + n \sigma h^2 w_{\frac{1}{2}} + (n-1) h^2 w_1 + (n-2) h^2 w_2 + \dots + h^2 w_{n-1}. \tag{37}$$

If these equations are satisfied by the tabulated values of $h^2 w$, no correction is necessary; but, if they are not satisfied, one or more values of $h^2 w$ must be altered accordingly, it being noted that an alteration of ± 1 in $h^2 w_r$ makes a difference of ± 1 in $\sigma h^2 w_{n+\frac{1}{2}}$, and of $\pm (n-r)$ in $\sigma^2 h^2 w_n$.

To illustrate this by a simple example, let us take

$$u = e^x,$$

which gives

$$w = d^2u/dx^2 = u.$$

Taking $h = \cdot 01$, and starting with $x = 1\cdot 00$, we have the following table to seven places of decimals:—

x	u	δ	δ^2	δ^3
		+	+	+
		270473		29
1·00	2·7182818		2719	
		273192		27
1·01	2·7456010		2746	
		275938		26
1·02	2·7731948		2772	
		278710		30
1·03	2·8010658		2802	
		281512		27
1·04	2·8292170		2829	
		284341		29
1·05	2·8576511		2858	
		287199		28
1·06	2·8863710		2886	
		290085		30
1·07	2·9153795		2916	
		293001		28
1·08	2·9446796		2944	
		295945		30
1·09	2·9742741		2974	
		298919		31
1·10	3·0041660		3005	
		301924		29

The values to eleven places for $x = 1\cdot 00$ and $x = 1\cdot 10$ are

x	u
1·00	2·7182818 2846
1·10	3·0041660 2395

Hence, by (35), since $h^2 w = h^2 u$,

$$\sigma h^2 w_{\frac{1}{2}} = 10^{-11}\{271828\ 1828 + 1359\ 1409 + \tfrac{1}{24} \text{ of } 54\ 3665 - \tfrac{11}{1440} \text{ of } 56\}$$

$$= \cdot 0273189\ 5889,$$

$$\sigma h^2 w_{n+\frac{1}{2}} = \cdot 0301921\ 1889.$$

The difference of these (omitting decimal point) is 28731 6000, and the sum of the corresponding values of $h^2 w$ comes to 28731 5999; one value must therefore be increased by 1. Again, by (33), we find

$$\sigma^2 h^2 w_0 = 2\cdot 7182591\ 7622,$$

$$\sigma^2 h^2 w_n = 3\cdot 0041409\ 8936,$$

the difference of the two being

$$2858818\ 1314.$$

If with the above value of $\sigma h^2 w_{\frac{1}{2}}$ we calculate

$$10\,\sigma h^2 w_{\frac{1}{2}} + 9h^2 w_1 + 8h^2 w_2 + \ldots + h^2 w_9,$$

we get 2858818 1303;

so that the sum must be increased by 11. If we take this $= 7+6-2$, so that two values of h^2w are increased by 1, and one is diminished by 1, we get the result shown below. The odd differences of h^2w are omitted, and the last column starts with the figure in the sixth decimal place of u. The altered values of h^2w are indicated by asterisks.

x	$\sigma^2 h^2 w$	$\sigma h^2 w$	$h^2 w$	$\delta^2 h^2 w$	u
1·00	2·7182591 7622		2718 2818	2719	... 18 2846
		273189 5889			
1·01	2·7455781 3511		2745 6010	2746	... 10 1501
		275935 1899			
1·02	2·7731716 5410		2773 1948	2773	... 47 6394
		278708 3847			
1·03	2·8010424 9257		2801 0659*	2801	... 58 3467
		281509 4506			
1·04	2·8291934 3763		2829 2171*	2828	... 70 1432
		284338 6677			
1·05	2·8576273 0440		2857 6511	2859	... 11 1804
		287196 3188			
1·06	2·8863469 3628		2886 3710	2886	... 09 8925
		290082 6898			
1·07	2·9153552 0526		2915 3795	2915	... 94 9997
		292998 0693			
1·08	2·9446550 1219		2944 6795*	2946	... 95 5106
		295942 7488			
1·09	2·9742492 8707		2974 2741	2973	... 40 7256
		298917 0229			
1·10	3·0041409 8936		3004 1660	3005	... 60 2395
		301921 1889			

These values are all correct within 3 (or $3\frac{1}{2}$) in the last figure.

17. This method will be found useful for constructing a table of values of

$$u = \int^x v\,dx,$$

when we already possess a table of values of v at the required intervals, but not of sufficient accuracy to give us u to the number of places we desire. If we have

$$dv/dx = \phi\,(x,\, v),$$

we can apply (19) to determine a more accurate table of v; and a second application of (19) will give u. But, if we have no particular use for the more accurate table of v, we can omit the calculations represented by the formula

$$v = (\sigma + \tfrac{1}{24}\delta - \tfrac{17}{5760}\delta^3 + \dots)\, h\phi\,(x,\, v),$$

and proceed at once to the calculation of u by the formula

$$u = (\sigma^2 + \tfrac{1}{12} - \tfrac{1}{240}\delta^2 + \tfrac{31}{60480}\delta^4 - \dots)\, h^2\phi\,(x,\, v).$$

[redacted]

Suppose, for instance, that

$$v = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2},$$

and that we have tabulated v to seven places by intervals of ·01 in x. Then we have

$$dv/dx = -xv,$$

and, by using the formula

$$u = (\sigma^2 + \tfrac{1}{12} - \tfrac{1}{240}\delta^2 + \ldots)(-h^2 xv),$$

we shall get u practically correct to eleven places, instead of merely to nine places.

18. If the values of x in the original table of u (or, in the case considered in the last section, in the original table of v) are halfway between the values to be adopted in the final table, we must use the formula*

$$u_{r+\frac{1}{2}} = \mu\,(\sigma^2 - \tfrac{1}{24} + \tfrac{17}{1920}\delta^2 - \tfrac{367}{193536}\delta^4 + \tfrac{27859}{66355200}\delta^6 - \ldots)\, h^2 w_{r+\frac{1}{2}}. \quad (38)$$

This formula, written in the form

$$\sigma^2 h^2 w_0 = u_{\frac{1}{2}} - \tfrac{1}{2}\sigma\, h^2 w_{\frac{1}{2}} + (\tfrac{1}{24}\mu - \tfrac{17}{1920}\mu\delta^2 + \ldots)\, h^2 w_{\frac{1}{2}}, \quad (39)$$

may also be used for calculating the checking values $\sigma^2 h^2 w_0$, $\sigma^2 h^2 w_n$, ..., when the values of u in the checking table are $u_{\frac{1}{2}}$, $u_{n+\frac{1}{2}}$, ..., instead of u_0, u_n, ...

IV. *Generalizations.*

19. Suppose that u satisfies a differential equation

$$d^2u/dx^2 - \phi\,(x, u)\, du/dx - \psi\,(x, u) = 0, \quad (40)$$

or, more generally,

$$d^2u/dx^2 = F\,(x, u, du/dx), \quad (41)$$

and that we have a table of u by intervals h in x. For any tabulated value, as x_0, we have†

$$h\,(du/dx)_0 = (\mu\delta - \tfrac{1}{6}\mu\delta^3 + \tfrac{1}{30}\mu\delta^5 - \tfrac{1}{140}\mu\delta^7 + \ldots)\, u_0, \quad (42)$$

and similarly for $x = x_1, x_2, \ldots$. If by substituting from (42) in (41) we get $h^2 d^2u/dx^2$ to a greater degree of accuracy than is given by the original table, we can apply (32) to obtain a more accurate table of u.

In a great many cases, however, the coefficient of du/dx in the

* See p. 484, formula (152). † See p. 465, formulæ (74).

differential equation does not involve u. If, then, we have

$$d^2u/dx^2 - f(x)\, du/dx - \psi(u) = 0, \tag{43}$$

it is simpler to write

$$U = e^{-\frac{1}{2}\int^x f(x)\,dx}\, u, \tag{44}$$

and we have, for the differential equation of U,

$$d^2U/dx^2 = e^{-\frac{1}{2}\int^x f(x)\,dx}\, \psi\left(x,\, e^{\frac{1}{2}\int^x f(x)\,dx}\, U\right) - \left[\tfrac{1}{2}f'(x) - \tfrac{1}{4}\{f(x)\}^2\right] U. \tag{45}$$

By tabulating U instead of u, we are able to proceed at once to the application of the method of §§ 15 and 16.

20. For example, consider Bessel's function of order 1,

$$u = J_1(x),$$

which satisfies the equation

$$x^2\, d^2u/dx^2 + x\, du/dx + (x^2 - 1)\, u = 0.$$

Writing $$U = \sqrt{x}\,.\,u,$$

we have $$d^2U/dx^2 + (1 - 3/4x^2)\, U = 0,$$

or $$h^2 d^2U/dx^2 = -h^2 (1 - 3/4x^2)\, U,$$

so that we can apply (32) for values of x exceeding $\sqrt{3/8} = \cdot 612 \ldots$.

Thus, taking $h = \cdot 1$, Lommel's table* of $J_1(x)$ gives the following values of U:—

x	u	U	x	u	U
·8	·368842	·329902	1·6	·569896	·720868
·9	·405950	·385118	1·7	·577765	·753314
1·0	·440051	·440051	1·8	·581517	·780187
1·1	·470902	·493886	1·9	·581157	·801070
1·2	·498289	·545848	2·0	·576725	·815612
1·3	·522023	·595198	2·1	·568292	·823534
1·4	·541948	·641243	2·2	·555963	·824627
1·5	·557937	·683331			

By direct calculation, I find

x	U	dU/dx
1·0	·44005 05857	·54517 23937
2·0	·81561 20449	·11272 63651

* E. Lommel, *Studien über die Bessel'sche Functionen* (Leipzig, 1868), p. 127.

		[illegible]		59685	[illegible]
[illegible]	[illegible]		455554		556[illegible]
		3758291		54122	
1·6	·72129267		509676		5981
		3248615		48141	
1·7	·75377882		557817		6376
		2690798		41771	
1·8	·78068680		599588		6717
		2091210		35054	
1·9	·80159890		634642		7011
		1456568		28043	
2·0	·81616458		662685		7249

The approximation would of course be more r
intervals of ·01, and calculated some of the
The process can be repeated, the multipliers o
be calculated once.

21. More generally, suppose that u satisfies a

$$d^n u/dx^n = F(x, u, du/dx, d^2u/dx^2, \ldots, d^n$$

Then, if u is tabulated by intervals of h in x, th
$h^2 d^2u/dx^2, \ldots$ are given by (42) and similar for
these values, as found from the table of u, in t
by (46), we get a series of values of $h^n d^n u/dx^n$; a

$$u = (\sigma + \tfrac{1}{24}\delta - \tfrac{17}{5760}\delta^3 + \ldots)^n h^n d^n$$

or $$u = \mu(1 + \tfrac{1}{4}\delta^2)^{-\frac{1}{2}} (\sigma + \tfrac{1}{24}\delta - \tfrac{17}{5760}\delta^3 + \ldots)^n$$

the powers of σ and of δ in the expanded series
accordance with the relation

$$\sigma = \delta^{-1}.$$

Central-Difference Formulæ. By W. F. SHEPPARD, M.A., LL.M.

Received and read December 14th, 1899.

PART I.—*Introductory.*

1. In the preceding paper I have found it necessary to make use of certain formulæ involving central differences. The formulæ are not new, but, in the absence of any complete text-book* on finite differences, they are not very familiar. The value of the method in practical work is well established, and it has received attention in various text-books dealing with special subjects.† But in almost all these cases the investigation of the formulæ is limited to the particular purposes for which they are required, and a discussion of their general relations would be out of place. The formulæ are therefore obtained, usually to a very few terms, from the corresponding formulæ for advancing or receding differences;‡ the notation of the latter is adopted without appreciable alteration; and very little attention is given to the meaning of the *operators* which appear in the formulæ. The only paper I have seen in which these operators are treated independently is one by P. A. Hansen,§ published in 1865; but even in that paper there is no clear statement as to the algebraical identities which lead up to the formulæ. In the absence of such a statement, the manipulation of symbols of operation is apt to lead to distrust of the results obtained.

The object of the present paper is to give the formulæ which are

* In Boole and Moulton's *Calculus of Finite Differences*, some central-difference formulæ are given as "examples," but the central-difference notation is condemned (see note below). In A. Markov's *Differenzen-Rechnung* (Leipzig, 1896), which I believe to be the latest text-book, central differences are entirely ignored.

† See, *e.g.*, *Text-Book of the Institute of Actuaries*, Part II., p. 447; Chauvenet, *Spherical and Practical Astronomy*, Vol. I., ch. ii., pp. 81–91; *Official Text-Book of Gunnery*, pp. 213 *seqq.* For some historical references, see C. W. Merrifield, "On Quadratures and Interpolation," *British Association Report*, 1880, pp. 352 *seqq.*

‡ Prof. Everett, in a recent paper, has combined advancing and receding differences in such a way as to give formulæ practically identical with the central-difference formulæ (*Quarterly Journal of Mathematics*, No. 124, 1900, p. 357). I ought to add that I am indebted to Prof. Everett for some criticisms and suggestions with regard to this and the preceding paper.

§ "Relationen zwischen Summen und Differenzen," *Abhandlungen der kön. sächs. Gesellschaft* (Leipzig), Vol. XI., 1865, pp. 505–583 (Vol. VII. of *Abhandlungen der math.-phys. Classe*). The paper is referred to in Boole and Moulton's treatise (2nd edition, 1872, p. 55), but the notation is banned as "unscientific."

of the most use for practical purposes, treating the theory of central differences as a distinct subject from the theory of advancing differences. The formulæ depend on certain algebraical expansions; for convenience of reference, these are collected in the following sections of Part I. Part II. of the paper (pp. 458–473) deals with central differences, and Part III. (pp. 473–488) with central sums, and their relation to integration.

The notation adopted* is practically identical with that of Hansen, except that, to prevent misunderstanding, I use δ and σ instead of Δ and Σ, the latter symbols being confined to the theory of advancing differences. I have also introduced a new operator, denoted by μ.

2. *Hyperbolic Functions.*—The following are familiar expansions:—

$$\begin{aligned}\cosh\phi &= \tfrac{1}{2}(e^{\phi}+e^{-\phi})\\ &= 1+\phi^2/2!+\phi^4/4!+\phi^6/6!+\dots\end{aligned}\tag{1}$$

$$\begin{aligned}\sinh\phi &= \tfrac{1}{2}(e^{\phi}-e^{-\phi})\\ &= \phi/1!+\phi^3/3!+\phi^5/5!+\phi^7/7!+\dots,\end{aligned}\tag{2}$$

$$\tanh\phi = T_1\phi/1!-T_2\phi^3/3!+T_3\phi^5/5!-T_4\phi^7/7!+\dots,\tag{3}$$

$$\phi\coth\phi = 1+2^2B_1\phi^2/2!-2^4B_2\phi^4/4!+2^6B_3\phi^6/6!-\dots,\tag{4}$$

$$\phi\,\mathrm{cosech}\,\phi = 1-P_1\phi^2/2!+P_2\phi^4/4!-P_3\phi^6/6!+\dots,\tag{5}$$

where B_1, B_2, B_3, ... are Bernoulli's numbers, and

$$T_r = 2^{2r}(2^{2r}-1)\,B_r/2r,\tag{6}$$

$$P_r = (2^{2r}-2)\,B_r.\tag{7}$$

The series (1) and (2) are absolutely convergent. The radii of convergence of (3), (4), and (5) are $\frac{1}{2}\pi$, π, and π respectively.

Multiplying (4) and (5) by $-1/\phi$, and differentiating, we have

$$\mathrm{cosech}^2\phi = \phi^{-2}-2^2B_1/2!+2^4B_2.3\phi^2/4!-2^6B_3.5\phi^4/6!+\dots,\tag{8}$$

$$\mathrm{cosech}\,\phi\coth\phi = \phi^{-2}+P_1/2!-P_2.3\phi^2/4!+P_3.5\phi^4/6!-\dots.\tag{9}$$

The radius of convergence of each of these series is π.

* The notation has already been explained, in the previous paper (pp. 423–448); but the explanations are repeated, in order to make the present paper complete in itself.

3. *Powers of Hyperbolic Sine.*—From (2) we get $\sinh^n\phi$ in ascending powers of ϕ, n being a positive integer. If we write

$$\sinh^n\phi = {}_nA_n\phi^n + {}_nA_{n+2}\phi^{n+2}/(n+1)(n+2)$$
$$+{}_nA_{n+4}\phi^{n+4}/(n+1)\dots(n+4)+\dots, \tag{10}$$

where
$${}_nA_n = 1, \tag{11}$$

we have, by differentiation,

$$\sinh^{n-1}\phi\cosh\phi = {}_nA_n\phi^{n-1} + {}_nA_{n+2}\phi^{n+1}/n(n+1)$$
$$+{}_nA_{n+4}\phi^{n+3}/n\dots(n+3)+\dots. \tag{12}$$

Differentiating again, substituting for $\cosh^2\phi$ in terms of $\sinh^2\phi$, and replacing $\sinh^{n-2}\phi$ by the series given by (10), we find

$${}_nA_{n+2r} = n^2\,{}_nA_{n+2r-2} + {}_{n-2}A_{n+2r-2}, \tag{13}$$

by means of which the successive coefficients may be calculated.

The series (10) is absolutely convergent for all values of n. For we have

$$\sinh^{2m-1}\phi = \left\{\tfrac{1}{2}(e^\phi - e^{-\phi})\right\}^{2m-1}$$
$$= \Big\{\sinh(2m-1)\phi - (2m-1)\sinh(2m-3)\phi$$
$$+\frac{(2m-1)(2m-2)}{2!}\sinh(2m-5)\phi - \dots$$
$$\dots + (-)^{m-1}\frac{(2m-1)!}{(m-1)!\,m!}\sinh\phi\Big\}\Big/2^{2m-2}, \tag{14}$$

$$\sinh^{2m}\phi = \left\{\tfrac{1}{2}(e^\phi - e^{-\phi})\right\}^{2m}$$
$$= \Big\{\cosh 2m\phi - 2m\cosh(2m-2)\phi$$
$$+\frac{2m(2m-1)}{2!}\cosh(2m-4)\phi - \dots$$
$$\dots + (-)^{m-1}\frac{(2m)!}{(m-1)!\,(m+1)!}\cosh 2\phi + (-)^m\tfrac{1}{2}\frac{(2m)!}{m!\,m!}\Big\}\Big/2^{2m-1}, \tag{15}$$

so that $\sinh^n\phi$ consists of the sum of a finite number of series, each of which is absolutely convergent. Similarly, $\sinh^{n-1}\phi\cosh\phi$ might be expressed in terms of $\sinh n\phi$, $\sinh(n-2)\phi$, ..., or of $\cosh n\phi$, $\cosh(n-2)\phi$, ..., so that the series (12) is absolutely convergent.

[redacted]

From (14) and (15), by comparison with (10), we have

$$
{}_{2m-1}A_{2p-1} = \Big\{ (2m-1)^{2p-1} - (2m-1)(2m-3)^{2p-1} + \frac{(2m-1)(2m-2)}{2!}(2m-5)^{2p-1} \ldots + (-)^{m-1}\frac{(2m-1)!}{(m-1)!\,m!}1^{2p-1}\Big\} \Big/ \{2^{2m-2}(2m-1)!\}, \quad (16)
$$

$$
{}_{2m}A_{2p} = \Big\{ (2m)^{2p} - 2m\,(2m-2)^{2p} + \frac{2m\,(2m-1)}{2!}(2m-4)^{2p} - \ldots \ldots + (-)^{m-1}\frac{(2m)!}{(m-1)!\,(m+1)!}2^{2p}\Big\} \Big/ \{2^{2m-1}(2m)!\}, \quad (17)
$$

both of which may be included in the formula

$$
{}_nA_q = \left[\{\tfrac{1}{2}(E-E^{-1})\}^n/n!\right] 0^q, \quad (18)
$$

where, as usual, $$E^r f(0) = f(r). \quad (19)$$

4. *Functions of Multiple Argument.*—By expanding $\cosh n\phi$ and $\sinh n\phi$ (where n has any value, integral or fractional) in powers of $n\phi$, by (1) or (2), and then taking out the terms corresponding to the successive even or odd powers of $\sinh\phi$, as given by (10), we obtain $\cosh n\phi$ and $\sinh n\phi$ in ascending powers of $\sinh\phi$, the series commencing with 1 and $n\sinh\phi$ respectively. To find the general formula, let us write

$$
\cosh n\phi = a_0 + a_2\sinh^2\phi/2! + a_4\sinh^4\phi/4! + \ldots, \quad (20)
$$

$$
\sinh n\phi = n\,\{{}_1B_1\sinh\phi/1! + {}_1B_3\sinh^3\phi/3! + {}_1B_5\sinh^5\phi/5! + \ldots\}, \quad (21)
$$

where $$a_0 = 1, \quad {}_1B_1 = 1. \quad (22)$$

Differentiating (20) twice, writing $1+\sinh^2\phi$ for $\cosh^2\phi$, and comparing with (20), we find

$$
a_{2r+2} = \{n^2 - (2r)^2\}\,a_{2r}, \quad (23)
$$

and similarly, from (21),

$$
{}_1B_{2r+1} = \{n^2 - (2r-1)^2\}\,{}_1B_{2r-1}. \quad (24)
$$

Hence $\cosh n\phi = 1 + \frac{n^2}{2!}\sinh^2\phi + \frac{n^2(n^2-2^2)}{4!}\sinh^4\phi + \dots,$ (25)

$$\sinh n\phi = \frac{n}{1!}\sinh\phi + \frac{n(n^2-1^2)}{3!}\sinh^3\phi + \frac{n(n^2-1^2)(n^2-3^2)}{5!}\sinh^5\phi + \dots. \tag{26}$$

If n is an even integer the series (25) terminates, and if n is an odd integer the series (26) terminates. In other cases the terms in each series ultimately become alternately positive and negative, each coefficient being numerically less than the preceding. The series are therefore convergent so long as $\sinh^2\phi \leqq 1$.*

From (25) and (26), by differentiation,

$$\sinh n\phi = \left\{\frac{n}{1!}\sinh\phi + \frac{n(n^2-2^2)}{3!}\sinh^3\phi + \frac{n(n^2-2^2)(n^2-4^2)}{5!}\sinh^5\phi + \dots\right\}\cosh\phi, \tag{27}$$

$$\cosh n\phi = \left\{1 + \frac{n^2-1^2}{2!}\sinh^2\phi + \frac{(n^2-1^2)(n^2-3^2)}{4!}\sinh^4\phi + \dots\right\}\cosh\phi, \tag{28}$$

which, like (25) and (26), are convergent so long as $\sinh^2\phi \leqq 1$.

From (26) we can obtain $\sinh^m n\phi$ in ascending powers of $\sinh\phi$, where m is an integer. Writing

$$\sinh^m n\phi/m! = n^m\left\{{}_mB_m\sinh^m\phi/m! + {}_mB_{m+2}\sinh^{m+2}\phi/(m+2)! + \dots\right\}, \tag{29}$$

where $${}_mB_m = 1, \tag{30}$$

we have, by differentiation,

$$\sinh^{m-1} n\phi\cosh n\phi/(m-1)!$$
$$= n^{m-1}\left\{{}_mB_m\sinh^{m-1}\phi/(m-1)! + {}_mB_{m+2}\sinh^{m+1}\phi/(m+1)! + \dots\right\}\cosh\phi. \tag{31}$$

Differentiating again, substituting as before, and equating coefficients,

$${}_mB_{m+2r+2} = \left\{m^2n^2 - (m+2r)^2\right\}{}_mB_{m+2r} + {}_{m-2}B_{m+2r}, \tag{32}$$

which gives the successive values of ${}_mB_{m+2r}$.

---

* As to the corresponding series for trigonometrical functions, see Chrystal, *Algebra*, Pt. II., pp. 304, 305; Hobson, *Trigonometry*, p. 265.

Thus we have

$$
\left.\begin{aligned}
\sinh^2 n\phi &= \frac{n^2}{2!}\, 2\sinh^2\phi + \frac{n^2(n^2-1^2)}{4!}\, 2^3 \sinh^4\phi \\
&\qquad + \frac{n^2(n^2-1^2)(n^2-2^2)}{6!}\, 2^5 \sinh^6\phi + \dots \\
\sinh^3 n\phi &= n^3 \sinh^3\phi + \frac{n^3(n^2-1)}{2}\sinh^5\phi + \frac{n^3(n^2-1)(13n^2-37)}{120}\sinh^7\phi \\
&\qquad + \frac{n^3(n^2-1)(205n^4-1706n^2+3229)}{15120}\sinh^9\phi - \dots \\
\sinh^4 n\phi &= n^4\sinh^4\phi + \frac{2n^4(n^2-1)}{3}\sinh^6\phi \\
&\qquad + \frac{n^4(n^2-1)(3n^2-7)}{15}\sinh^8\phi + \dots \\
\sinh^5 n\phi &= n^5\sinh^5\phi + \frac{5n^5(n^2-1)}{6}\sinh^7\phi \\
&\qquad + \frac{n^5(n^2-1)(23n^2-47)}{72}\sinh^9\phi + \dots \\
\sinh^6 n\phi &= n^6\sinh^6\phi + n^6(n^2-1)\sinh^8\phi + \dots \\
\sinh^7 n\phi &= n^7\sinh^7\phi + \frac{7n^7(n^2-1)}{6}\sinh^9\phi + \dots \\
\sinh^8 n\phi &= n^8\sinh^8\phi + \dots \\
\sinh^9 n\phi &= n^9\sinh^9\phi + \dots \\
\vdots \quad & \quad \vdots
\end{aligned}\right\} \tag{33}
$$

with the corresponding series obtained by differentiation of these with regard to ϕ.

All the above series are convergent so long as $\sinh^2\phi \leqq 1$.

5. *Inverse Expansions* (*positive powers of argument*). — If we write $n = m, m+2, m+4, \dots$, in (10), and eliminate $\phi^{m+2}, \phi^{m+4}, \dots$, we obtain ϕ^m in ascending powers of $\sinh\phi$. Similarly, from (12), we could obtain $\phi^m/\cosh\phi$. The coefficients in these expansions are most simply found from formulæ (25)–(28), by expanding $\cosh n\phi$ and $\sinh n\phi$ in powers of $n\phi$, and equating coefficients of powers of n.

Thus we find, for the first two pairs of formulæ,

$$1 = \left\{1 - \frac{1^2}{1.2}\sinh^2\phi + \frac{1^2.3^2}{1.2.3.4}\sinh^4\phi - \dots\right\}\cosh\phi, \tag{34}$$

$$\phi = \sinh\phi - \frac{1^2}{2.3}\sinh^3\phi + \frac{1^2.3^2}{2.3.4.5}\sinh^5\phi - \dots, \tag{35}$$

$$\phi = \left\{\sinh\phi - \frac{2^2}{2.3}\sinh^3\phi + \frac{2^2.4^2}{2.3.4.5}\sinh^5\phi - \dots\right\}\cosh\phi, \tag{36}$$

$$\phi^2 = \sinh^2\phi - \frac{2^2}{3.4}\sinh^4\phi + \frac{2^2.4^2}{3.4.5.6}\sinh^6\phi - \dots. \tag{37}$$

For the general formula, let us write

$$\phi^m = {}_mH_m \sinh^m\phi - {}_mH_{m+2}\sinh^{m+2}\phi/(m+1)(m+2)$$
$$+ {}_mH_{m+4}\sinh^{m+4}\phi/(m+1)\dots(m+4) - \dots. \tag{38}$$

Then we have also, by differentiation,

$$\phi^{m-1} = \left\{{}_mH_m \sinh^{m-1}\phi - {}_mH_{m+2}\sinh^{m+1}\phi/m(m+1)\right.$$
$$\left. + {}_mH_{m+4}\sinh^{m+3}\phi/m\dots(m+3) - \dots\right\}\cosh\phi. \tag{39}$$

The values of the coefficients are given by

$$ {}_mH_m = 1, \tag{40}$$

$${}_{2n-1}H_{2n+2r-1} = \text{sum of products of } 1^2,\ 3^2,\ \dots\ (2n+2r-3)^2, \text{ taken } r \text{ at a time}, \tag{41}$$

$${}_{2n}H_{2n+2r} = \text{sum of products of } 2^2,\ 4^2,\ \dots\ (2n+2r-2)^2, \text{ taken } r \text{ at a time}, \tag{42}$$

all of which are included in the formula

$$\exp\left\{\frac{E^{p-1}-E^{-(p-1)}}{E-E^{-1}}\log\sqrt{\theta^2+0^2}\right\} = \Sigma\, {}_mH_p\theta^{m-1}. \tag{43}$$

To obtain the relation between the successive coefficients in (38), we have

$$E^{p-1}-E^{-(p-1)} = \left\{E^{p-2}+E^{-(p-2)}\right\}(E-E^{-1}) + E^{p-3}-E^{-(p-3)}$$

and $\quad \left\{E^{p-2}+E^{-(p-2)}\right\}\log\sqrt{\theta^2+0^2} = \log\left\{\theta^2+(p-2)^2\right\}.$

Hence, by (43),

$$\Sigma\, {}_mH_p\theta^{m-1} = \left\{\theta^2+(p-2)^2\right\}\Sigma\, {}_mH_{p-2}\theta^{m-1}.$$

Equating coefficients, we find

$$ {}_mH_p = (p-2)^2\,{}_mH_{p-2} + {}_{m-2}H_{p-2}, \tag{44}$$

which gives the successive values of ${}_mH_p$.

If all the terms in the series (25)–(28) are taken positively, the resulting series are convergent so long as $\sinh^2\phi < 1$. The series (38) and (39) are therefore also convergent for this range of values.

6. *Inverse Expansions* (*negative powers of argument*).—From (38) and (39), we have

$$\sinh^m\phi = \Big[\big\{1 - {}_mH_{m+2}\sinh^2\phi/(m+1)(m+2) + {}_mH_{m+4}\sinh^4\phi/(m+1)\dots(m+4) - \dots\big\}^{-1}\Big]\phi^m, \tag{45}$$

$$\sinh^{m-1}\phi = \Big[(1+\sinh^2\phi)^{-1}\big\{1 - {}_mH_{m+2}\sinh^2\phi/m(m+1) + {}_mH_{m+4}\sinh^4\phi/m\dots(m+3) - \dots\big\}^{-1}\Big]\phi^m\cosh\phi. \tag{46}$$

The expressions in square brackets can be expanded in ascending powers of $\sinh^2\phi$, but there is no simple relation between the successive coefficients in the resulting series. The most important cases are:—

$$\begin{aligned}\sinh\phi &= \phi\Big/\Big(1 - \frac{1^2}{2.3}\sinh^2\phi + \frac{1^2.3^2}{2.3.4.5}\sinh^4\phi - \frac{1^2.3^2.5^2}{2.3.4.5.6.7}\sinh^6\phi + \dots\Big)\\ &= \Big(1 + \frac{1}{6}\sinh^2\phi - \frac{17}{360}\sinh^4\phi + \frac{367}{15120}\sinh^6\phi - \frac{27859}{1814400}\sinh^8\phi + \dots\Big)\phi,\end{aligned} \tag{47}$$

$$\begin{aligned}\sinh^2\phi &= \phi^2\Big/\Big(1 - \frac{2^2}{3.4}\sinh^2\phi + \frac{2^2.4^2}{3.4.5.6}\sinh^4\phi - \frac{2^2.4^2.6^2}{3.4.5.6.7.8}\sinh^6\phi + \dots\Big)\\ &= \Big(1 + \frac{1}{3}\sinh^2\phi - \frac{1}{15}\sinh^4\phi + \frac{31}{945}\sinh^6\phi - \frac{289}{14175}\sinh^8\phi + \dots\Big)\phi^2,\end{aligned} \tag{48}$$

$$\sinh\phi = \phi\cosh\phi \Big/ \left(1+\frac{1}{3}\sinh^2\phi - \frac{2}{3.5}\sinh^4\phi + \frac{2.4}{3.5.7}\sinh^6\phi - \dots\right)$$

$$= \left(1 - \frac{1}{3}\sinh^2\phi + \frac{11}{45}\sinh^4\phi - \frac{191}{945}\sinh^6\phi + \frac{2497}{14175}\sinh^8\phi - \dots\right)\cosh\phi\,.\,\phi, \quad (49)$$

$$\sinh^2\phi = \phi^2\cosh\phi \Big/ \left\{1 + \frac{1}{4}\left(\frac{1}{1^2} - \frac{1}{3}\right)\sinh^2\phi - \frac{1.3}{4.6}\left(\frac{1}{1^2} + \frac{1}{3^2} - \frac{1}{5}\right)\sinh^4\phi + \frac{1.3.5}{4.6.8}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7}\right)\sinh^6\phi - \dots\right\}$$

$$= \left(1 - \frac{1}{6}\sinh^2\phi + \frac{17}{120}\sinh^4\phi - \frac{367}{3024}\sinh^6\phi + \frac{27859}{259200}\sinh^8\phi - \dots\right)\cosh\phi\,.\,\phi^2. \quad (50)$$

I do not know of any method by which the convergence of the general series given by (45) and (46) may be established; but it is not difficult to show* that the series (47)-(50) are convergent so

---

* Let $$f(x) \equiv 1 + a_1x + a_2x^2 + \dots$$
be a series in which every coefficient is positive, and the ratio of each coefficient to the preceding one continually increases, but is always less than unity. Then $f(x)$ is convergent so long as $1 > x > -1$. Now, let
$$1/f(x) = 1 - b_1x - b_2x^2 - \dots.$$
Then, if we write $$\lambda_r = a_r/a_{r-1},$$
it is easily shown that
$$b_n = (\lambda_n - \lambda_1)\,b_{n-1} + a_1(\lambda_n - \lambda_2)\,b_{n-2} + a_2(\lambda_n - \lambda_3)\,b_{n-3} + \dots + a_{n-2}(\lambda_n - \lambda_{n-1})\,b_1.$$
Hence, since λ_n is greater than any of the preceding ratios $\lambda_1, \lambda_2, \dots, \lambda_{n-1}$, and b_1 is positive, it follows, by induction, that $b_2, b_3, \dots$ are all positive. Also we find
$$a_1x + a_2x^2 + a_3x^3 + \dots = (1 + a_1x + a_2x^2 + \dots)(b_1x + b_2x^2 + b_3x^3 + \dots)$$
$$= b_1x + (b_2 + a_1b_1)\,x^2 + (b_3 + a_1b_2 + a_2b_1)\,x^3 + \dots.$$
The series on the left-hand side is convergent so long as $1 > x > 0$, and therefore, *a fortiori*, the series $b_1x + b_2x^2 + b_3x^3 + \dots$ is convergent within the same limits. It is therefore also convergent if $0 > x > -1$.

The above method applies to formulæ (47) and (48). For (49) and (50) we are dealing with a series
$$f(x) \equiv 1 + c_1x - c_2x^2 + c_3x^3 - \dots,$$
in which the coefficients after the first are alternately positive and negative. If we write
$$1/f(x) = 1 - d_1x + d_2x^2 - d_3x^3 + \dots,$$

long as $\sinh^2\phi < 1$. The question of the convergence of the series is not, however, an important one for our present purpose. They are used for determining finite-difference formulæ; and the validity of these formulæ depends on the *initial* convergence of the numerical series derived from them.

PART II.—*Central-Difference Formulæ.*

7. *Definitions.*—We consider that we are dealing with a function

$$u \equiv f(x),$$

whose values are tabulated by equal increments h in x. Taking x_0 to denote any value of x which appears in the table, and writing

$$u_0 = f(x_0),$$

$$x_n = x_0 + nh,$$

$$u_n = f(x_n),$$

where n is not necessarily integral, the table showing the values of u and of its differences will take this form:—

x	u	1st diff.	2nd diff.	3rd diff.	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
x_{-2}	u_{-2}		$u_{-1}-2u_{-2}+u_{-3}$		...
		$u_{-1}-u_{-2}$		$u_0-3u_{-1}+3u_{-2}-u_{-3}$	
x_{-1}	u_{-1}		$u_0 \;-2u_{-1}+u_{-2}$		...
		$u_0 \;-u_{-1}$		$u_1-3u_0 \;+3u_{-1}-u_{-2}$	
x_0	u_0		$u_1 \;-2u_0 \;+u_{-1}$		...
		$u_1 \;-u_0$		$u_2-3u_1 \;+3u_0 \;-u_{-1}$	
x_1	u_1		$u_2 \;-2u_1 \;+u_0$		...
		$u_2 \;-u_1$		$u_3-3u_2 \;+3u_1 \;-u_0$	
x_2	u_2		$u_3 \;-2u_2 \;+u_1$		...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

it may be shown that $d_1, d_2, d_3, \ldots$ are all positive. The coefficients in the latter series [for (49) and (50)] are connected with those in the former series [for (47) and (48)] by a relation of the form

$$1 + d_1x + d_2x^2 + \ldots = (1-b_1x-b_2x^2-\ldots)(1-x)^{-\frac{1}{2}}$$

$$= 1 + \left(\frac{1}{2}-b_1\right)x + \left(\frac{1\cdot3}{2\cdot4}-\frac{1}{2}b_1-b_2\right)x^2+\ldots.$$

The series $1+\frac{1}{2}x+\frac{1\cdot3}{2\cdot4}x^2+\ldots$ is convergent so long as $1>x>0$; and therefore, the coefficients $d_1, d_2, \ldots$ having been shown to be positive, the series $1+d_1x+d_2x^2+\ldots$ is convergent within the same limits. It is therefore also convergent so long as $0>x>-1$.

The quantities shown in this table are supposed to be the exact values of u and its differences, not the approximate values as they would appear in a numerical table. Each quantity in the second column is a definite function of a definite value of x, these values increasing successively by h. Each quantity in the third column is also a function of a value of x, the functions being all of the same form, and the values of x increasing by h; and similarly for the fourth and other columns. But in any one column the form of the function will depend on the successive values of x of which the successive differences are taken to be functions; and the assignment of these is quite arbitrary. We might, for instance, regard u_1-u_0 as a function of x_{-2}; but, if we do this, u_2-u_1 will be the same function of x_{-1}, and so on.

In the ordinary advancing-difference notation, $f(x+h)-f(x)$ is written $\Delta f(x)$; *i.e.*, it is (until the Δ is separated from the $f(\;)$, and treated as an operator) regarded as a function of x. But this is only the case if the table is read downwards; if it is read upwards, the difference $f(x+h)-f(x)$ is written $-\Delta f(x+h)$, *i.e.*, it is regarded as a function of $x+h$. This might be considered as justified by the change of sign. But the second difference $f(x+h)-2f(x)+f(x-h)$ has the same sign in whichever direction we read the table; it is, however, regarded as a function of $x-h$ if we read the table one way, and as a function of $x+h$ if we read it the other.

It seems more natural to regard $f(x+h)+f(x)$ as a function of the value of x about which $f(x)$ and $f(x+h)$ are symmetrically placed. We therefore write

$$f(x+h)-f(x) = \delta f(x+\tfrac{1}{2}h),$$

or, for the definition of $\delta f(x)$,

$$f(x+\tfrac{1}{2}h)-f(x-\tfrac{1}{2}h) = \delta f(x). \tag{51}$$

Similarly we have

$$\begin{aligned}\delta\delta f(x) &= \delta f(x+\tfrac{1}{2}h)-\delta f(x-\tfrac{1}{2}h)\\ &= f(x+h)-2f(x)+f(x-h),\end{aligned} \tag{52}$$

and generally, denoting the existence of n successive δ's by δ^n,

$$\delta^n f(x) = f(x+\tfrac{1}{2}nh)-nf(x+\tfrac{1}{2}nh-h)+\ldots+(-)^n f(x-\tfrac{1}{2}nh), \tag{53}$$

the coefficients being those of the binomial expansion. Our table of differences then becomes

x	u	1st diff.	2nd diff.	3rd diff.	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
x_{-2}	u_{-2}		$\delta^2 u_{-2}$		...
		$\delta u_{-\frac{3}{2}}$		$\delta^3 u_{-\frac{3}{2}}$	
x_{-1}	u_{-1}		$\delta^2 u_{-1}$		...
		$\delta u_{-\frac{1}{2}}$		$\delta^3 u_{-\frac{1}{2}}$	
x_0	u_0		$\delta^2 u_0$		...
		$\delta u_{\frac{1}{2}}$		$\delta^3 u_{\frac{1}{2}}$	
x_1	u_1		$\delta^2 u_1$		...
		$\delta u_{\frac{3}{2}}$		$\delta^3 u_{\frac{3}{2}}$	
x_2	u_2		$\delta^2 u_2$		...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

where it is to be observed that $\delta^n u_r$ is merely an abbreviation for $\delta\delta\delta\ldots$ (n times) $f(x+rh)$.

Again, let us write

$$\mu f(x) = \tfrac{1}{2}\left\{f(x+\tfrac{1}{2}h)+f(x-\tfrac{1}{2}h)\right\}. \tag{54}$$

Then we shall have

$$\begin{aligned}\mu\mu f(x) &= \tfrac{1}{2}\left\{\mu f(x+\tfrac{1}{2}h)+\mu f(x-\tfrac{1}{2}h)\right\}\\ &= \tfrac{1}{4}\left\{f(x+h)+2f(x)+f(x-h)\right\},\end{aligned} \tag{55}$$

and generally, denoting the existence of n successive μ's by μ^n,

$$\mu^n f(x) = \left\{f(x+\tfrac{1}{2}nh)+nf(x+\tfrac{1}{2}nh-h)+\ldots+f(x-\tfrac{1}{2}nh)\right\}/2^n, \tag{56}$$

the coefficients inside the brackets being the same as in (53), but all positive.

If in each column of differences of u, of an odd order, we take the mean of each pair of consecutive differences, and insert it, in brackets, between the two, we get a table of this form :—

x	u	1st diff.	2nd diff.	3rd diff.	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
x_{-1}	u_{-1}	$(\mu\delta u_{-1})$	$\delta^2 u_{-1}$	$(\mu\delta^3 u_{-1})$	...
		$\delta u_{-\frac{1}{2}}$		$\delta^3 u_{-\frac{1}{2}}$	
x_0	u_0	$(\mu\delta u_0)$	$\delta^2 u_0$	$(\mu\delta^3 u_0)$	...
		$\delta u_{\frac{1}{2}}$		$\delta^3 u_{\frac{1}{2}}$	
x_1	u_1	$(\mu\delta u_1)$	$\delta^2 u_1$	$(\mu\delta^3 u_1)$	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

where $\mu\delta^n u_r$ denotes $\frac{1}{2}(\delta^n u_{r+\frac{1}{2}}+\delta^n u_{r-\frac{1}{2}})$. The quantities following any value of u, in a horizontal line, are its *central differences*.

8. *μ and δ as operators.*—It is easily shown that the prefixes μ and δ, both singly and in combination, follow the ordinary laws of algebra. Thus, if we have

$$f(x) = \phi(x)+\psi(x),$$

then

$$\begin{aligned} \mu f(x) &= \tfrac{1}{2}\{f(x+\tfrac{1}{2}h)+f(x-\tfrac{1}{2}h)\} \\ &= \tfrac{1}{2}\{\phi(x+\tfrac{1}{2}h)+\phi(x-\tfrac{1}{2}h)\} \\ &\quad +\tfrac{1}{2}\{\psi(x+\tfrac{1}{2}h)+\psi(x-\tfrac{1}{2}h)\} \\ &= \mu\phi(x)+\mu\psi(x); \end{aligned}$$

and, similarly,

$$\begin{aligned} \mu\delta f(x) &= \tfrac{1}{2}\{\delta f(x+\tfrac{1}{2}h)+\delta f(x-\tfrac{1}{2}h)\} \\ &= \tfrac{1}{2}\{f(x+h)-f(x-h)\} && (57) \\ &= \tfrac{1}{2}\{f(x+h)+f(x)\}-\tfrac{1}{2}\{f(x)+f(x-h)\} \\ &= \delta\mu f(x). && (58) \end{aligned}$$

The prefixes may therefore be detached from the $f(x)$, and regarded as operators, following algebraical laws. Also, if

$$D$$

denotes, as usual, the process of differentiation, so that

$$Df(x) = f'(x),$$

or

$$Du = u',$$

then it may be shown that D combines with powers of μ and of δ according to these laws.

We shall have to consider, later on, the effect of the operators μ and δ when the intervals, instead of being h, are nh, n not being necessarily positive or an integer. The operators in this case will be written

$$\mu_n, \quad \delta_n,$$

so that

$$\mu_n f(x) = \tfrac{1}{2}\{f(x+\tfrac{1}{2}nh)+f(x-\tfrac{1}{2}nh)\}, \tag{59}$$

$$\delta_n f(x) = f(x+\tfrac{1}{2}nh)-f(x-\tfrac{1}{2}nh). \tag{60}$$

If we take $n = -1$, so that the interval is $-h$, the effect is the same as if we read the table upwards. This obviously changes the sign of δ, and leaves μ unaltered, so that

$$\left.\begin{aligned} \mu_{-1} &= \mu \\ \delta_{-1} &= -\delta \end{aligned}\right\}. \tag{61}$$

[redacted]

9. *Relation between μ and δ.*—From (56) and (52),

$$\mu^2 f(x) = \tfrac{1}{4}\{f(x+h)+2f(x)+f(x-h)\},$$

$$\delta^2 f(x) = f(x+h)-2f(x)+f(x-h),$$

and therefore $$(\mu^2-\tfrac{1}{4}\delta^2)f(x) = f(x),$$

or $$\mu^2-\tfrac{1}{4}\delta^2 = 1. \tag{62}$$

This gives $$\mu^{-1} = (1+\tfrac{1}{4}\delta^2)^{-\frac{1}{2}},$$

and therefore

$$\begin{aligned} u_{\frac{1}{2}} &= \mu(1+\tfrac{1}{4}\delta^2)^{-\frac{1}{2}}u_{\frac{1}{2}} \\ &= (\mu-\tfrac{1}{8}\mu\delta^2+\tfrac{3}{128}\mu\delta^4-\tfrac{5}{1024}\mu\delta^6+\tfrac{35}{32768}\mu\delta^8-\ldots)\,u_{\frac{1}{2}} \qquad (63) \\ &= \tfrac{1}{2}(u_1+u_0)-\tfrac{1}{16}(\delta^2u_1+\delta^2u_0)+\tfrac{3}{256}(\delta^4u_1+\delta^4u_0) \\ &\qquad -\tfrac{5}{2048}(\delta^6u_1+\delta^6u_0)+\tfrac{35}{65536}(\delta^8u_1+\delta^8u_0)-\ldots, \qquad (63\text{A}) \end{aligned}$$

which is the formula for interpolating in the middle of an interval.

10. *Relations between μ, δ, and hD.*—By Taylor's theorem,

$$\begin{aligned} f(x+\tfrac{1}{2}h) &= f(x)+\tfrac{1}{2}hf'(x)+\frac{1}{2!}(\tfrac{1}{2}h)^2f''(x)+\ldots \\ &= e^{\frac{1}{2}hD}f(x), \end{aligned}$$

and, similarly, $$f(x-\tfrac{1}{2}h) = e^{-\frac{1}{2}hD}f(x).$$

Hence $$\mu f(x) = \tfrac{1}{2}(e^{\frac{1}{2}hD}+e^{-\frac{1}{2}hD})f(x),$$

$$\delta f(x) = (e^{\frac{1}{2}hD}-e^{-\frac{1}{2}hD})f(x),$$

or $$\mu = \cosh\tfrac{1}{2}hD, \tag{64}$$

$$\delta = 2\sinh\tfrac{1}{2}hD. \tag{65}$$

These relations may be obtained directly, in the same way as Taylor's theorem was first obtained. For suppose we write

$$\theta = 1/(2m+1),$$

where m is a positive integer. Then, by (53),

$$\left.\begin{aligned} \delta_\theta &= \delta_\theta \\ \delta_\theta^3 &= \delta_{3\theta}-3\delta_\theta \\ \vdots\ & \quad \vdots \\ \delta_\theta^{2m+1} &= \delta_{(2m+1)\theta}-(2m+1)\,\delta_{(2m-1)\theta}+\frac{(2m+1)\,2m}{2!}\,\delta_{(2m-3)\theta}-\ldots \\ &\qquad \ldots+(-)^m\frac{(2m+1)!}{m!\,(m+1)!}\,\delta_\theta \end{aligned}\right\}. \tag{66}$$

Eliminating $\delta_{3e}, \delta_{5e}, \ldots, \delta_{(2m-1)e}$, we have*

$$\delta_{(2m+1)e} = (2m+1)\left\{\delta_e + \frac{4.5\ldots(m+1)}{(m-1)!}\delta_e^3 + \frac{6.7\ldots(m+2)}{(m-2)!}\delta_e^5 + \ldots \right.$$

$$\left. \ldots + \frac{2m-2}{2!}\delta_e^{2m-3} + \delta_e^{2m-1}\right\} + \delta_e^{2m+1},$$

or, if we write $$2m+1 = n,$$

then $$\delta = n\delta_e + \frac{n(n^2-1^2)}{2^2.3!}\delta_e^3 + \frac{n(n^2-1^2)(n^2-3^2)}{2^4.5!}\delta_e^5 + \ldots$$

$$\ldots + \frac{n(n^2-1^2)(n^2-3^2)\ldots(n^2-\overline{n-4}^2)}{2^{n-3}(n-2)!}\delta_e^{n-2} + \delta_e^n. \quad (67)$$

Now increase n indefinitely. Then the limit of

$$n^r \delta_e^r u$$

is $$h^r D^r u,$$

and we get $$\delta = hD + \frac{1}{2^2.3!}h^3D^3 + \frac{1}{2^4.5!}h^5D^5 + \ldots$$
$$= 2\sinh \tfrac{1}{2}hD.$$

Similarly, or by means of (62), we get (64).

This latter method has some advantages, as it shows that the relations between differencing and differentiation on the one hand,

---

* It is not absolutely necessary to find the coefficients in this formula. We see that (66) gives δ as a linear function of $\delta_e, \delta_e^3, \ldots$, so that, increasing m without limit, we have

$$f(x+\tfrac{1}{2}h) - f(x-\tfrac{1}{2}h) = \phi(h, d/dx)f(x).$$

Hence, writing $\xi = x/n$,

$$\begin{aligned}\phi(h, nd/dx)f(x) &= \phi(h, d/d\xi)f(n\xi) \\ &= f\{n(\xi+\tfrac{1}{2}h)\} - f\{n(\xi-\tfrac{1}{2}h)\} \\ &= f(x+\tfrac{1}{2}nh) - f(x-\tfrac{1}{2}nh) \\ &= \phi(nh, d/dx)f(x).\end{aligned}$$

The powers of d/dx in $\phi(h, d/dx)$ can therefore only occur in connexion with like powers of h, and we may write

$$\delta f(x) = \psi(hD)f(x).$$

Differentiating twice with regard to h, we find successively

$$\mu f'(x) = \psi'(hD)f'(x),$$
$$\delta f''(x) = 4\psi''(hD)f''(x).$$

This gives $$\psi(hD) = 4\psi''(hD),$$

and $$\mu = \psi'(hD),$$

whence, since $\mu = 1$ when $hD = 0$,

$$\psi(hD) = 2\sinh \tfrac{1}{2}hD.$$

and between summing and integration on the other, are really based on the relations between various sets of differences obtained by taking various intervals in x.

11. *Differences in terms of Differential Coefficients.*—From (65) and (10),

$$\begin{aligned} \delta^n &= (2\sinh \tfrac{1}{2}hD)^n \\ &= {}_nA_n h^n D^n + \tfrac{1}{4}\,{}_nA_{n+2} h^{n+2} D^{n+2}/(n+1)(n+2) + \ldots, \end{aligned} \tag{68}$$

and, from (64) and (12),

$$\begin{aligned} \mu\delta^{n-1} &= (2\sinh \tfrac{1}{2}hD)^{n-1} \cosh \tfrac{1}{2}hD \\ &= {}_nA_n h^{n-1} D^{n-1} + \tfrac{1}{4}\,{}_nA_{n+2} h^{n+1} D^{n+1}/n(n+1) + \ldots, \end{aligned} \tag{69}$$

where ${}_nA_n$, ${}_nA_{n+2}$, ... have the values given by (11) and (13). Thus we find

$$\left.\begin{aligned} u_0 &= u_0 \\ \mu\delta u_0 &= (hD + \tfrac{1}{6}h^3D^3 + \tfrac{1}{120}h^5D^5 + \tfrac{1}{5040}h^7D^7 + \ldots)\, u_0 \\ \delta^2 u_0 &= (h^2D^2 + \tfrac{1}{12}h^4D^4 + \tfrac{1}{360}h^6D^6 + \tfrac{1}{20160}h^8D^8 + \ldots)\, u_0 \\ \mu\delta^3 u_0 &= (h^3D^3 + \tfrac{1}{4}h^5D^5 + \tfrac{1}{45}h^7D^7 + \ldots)\, u_0 \\ \delta^4 u_0 &= (h^4D^4 + \tfrac{1}{6}h^6D^6 + \tfrac{1}{90}h^8D^8 + \ldots)\, u_0 \\ \mu\delta^5 u_0 &= (h^5D^5 + \tfrac{1}{3}h^7D^7 + \ldots)\, u_0 \\ \delta^6 u_0 &= (h^6D^6 + \tfrac{1}{4}h^8D^8 + \ldots)\, u_0 \\ \mu\delta^7 u_0 &= (h^7D^7 + \ldots)\, u_0 \\ \delta^8 u_0 &= (h^8D^8 + \ldots)\, u_0 \\ \vdots\quad & \quad\vdots \end{aligned}\right\}, \tag{70}$$

and

$$\left.\begin{aligned} \mu u_{\frac{1}{2}} &= (1 + \tfrac{1}{8}h^2D^2 + \tfrac{1}{384}h^4D^4 + \tfrac{1}{46080}h^6D^6 + \tfrac{1}{10321920}h^8D^8 + \ldots)\, u_{\frac{1}{2}} \\ \delta u_{\frac{1}{2}} &= (hD + \tfrac{1}{24}h^3D^3 + \tfrac{1}{1920}h^5D^5 + \tfrac{1}{322560}h^7D^7 + \ldots)\, u_{\frac{1}{2}} \\ \mu\delta^2 u_{\frac{1}{2}} &= (h^2D^2 + \tfrac{5}{24}h^4D^4 + \tfrac{91}{5760}h^6D^6 + \tfrac{41}{64512}h^8D^8 + \ldots)\, u_{\frac{1}{2}} \\ \delta^3 u_{\frac{1}{2}} &= (h^3D^3 + \tfrac{1}{8}h^5D^5 + \tfrac{13}{1920}h^7D^7 + \ldots)\, u_{\frac{1}{2}} \\ \mu\delta^4 u_{\frac{1}{2}} &= (h^4D^4 + \tfrac{7}{24}h^6D^6 + \tfrac{23}{640}h^8D^8 + \ldots)\, u_{\frac{1}{2}} \\ \delta^5 u_{\frac{1}{2}} &= (h^5D^5 + \tfrac{5}{24}h^7D^7 + \ldots)\, u_{\frac{1}{2}} \\ \mu\delta^6 u_{\frac{1}{2}} &= (h^6D^6 + \tfrac{3}{8}h^8D^8 + \ldots)\, u_{\frac{1}{2}} \\ \delta^7 u_{\frac{1}{2}} &= (h^7D^7 + \ldots)\, u_{\frac{1}{2}} \\ \mu\delta^8 u_{\frac{1}{2}} &= (h^8D^8 + \ldots)\, u \end{aligned}\right\}. \tag{71}$$

12. *Differential Coefficients in terms of Differences.*—From (38) and (39),

$$h^m D^m = {}_mH_m\delta^m - \tfrac{1}{4}\,{}_mH_{m+2}\delta^{m+2}/(m+1)(m+2) + \ldots, \tag{72}$$

$$h^{m-1}D^{m-1} = {}_mH_m\mu\delta^{m-1} - \tfrac{1}{4}\,{}_mH_{m+2}\mu\delta^{m+1}/m\,(m+1) + \ldots. \tag{73}$$

Hence, by (40) and (44),

$$\left.\begin{aligned}
hDu_0 &= (\mu\delta - \tfrac{1}{6}\mu\delta^3 + \tfrac{1}{30}\mu\delta^5 - \tfrac{1}{140}\mu\delta^7 + \ldots)\,u_0\\
h^2D^2u_0 &= (\delta^2 - \tfrac{1}{12}\delta^4 + \tfrac{1}{90}\delta^6 - \tfrac{1}{560}\delta^8 + \ldots)\,u_0\\
h^3D^3u_0 &= (\mu\delta^3 - \tfrac{1}{4}\mu\delta^5 + \tfrac{7}{120}\mu\delta^7 - \ldots)\,u_0\\
h^4D^4u_0 &= (\delta^4 - \tfrac{1}{6}\delta^6 + \tfrac{7}{240}\delta^8 - \ldots)\,u_0\\
h^5D^5u_0 &= (\mu\delta^5 - \tfrac{1}{3}\mu\delta^7 + \ldots)\,u_0\\
h^6D^6u_0 &= (\delta^6 - \tfrac{1}{4}\delta^8 + \ldots)\,u_0\\
h^7D^7u_0 &= (\mu\delta^7 - \ldots)\,u_0\\
h^8D^8u_0 &= (\delta^8 - \ldots)\,u_0\\
\vdots \quad & \quad \vdots
\end{aligned}\right\}, \tag{74}$$

$$\left.\begin{aligned}
u_{\frac{1}{2}} &= (\mu - \tfrac{1}{8}\mu\delta^2 + \tfrac{3}{128}\mu\delta^4 - \tfrac{5}{1024}\mu\delta^6 + \tfrac{35}{32768}\mu\delta^8 - \ldots)\,u_{\frac{1}{2}}\\
hDu_{\frac{1}{2}} &= (\delta - \tfrac{1}{24}\delta^3 + \tfrac{3}{640}\delta^5 - \tfrac{5}{7168}\delta^7 + \ldots)\,u_{\frac{1}{2}}\\
h^2D^2u_{\frac{1}{2}} &= (\mu\delta^2 - \tfrac{5}{24}\mu\delta^4 + \tfrac{259}{5760}\mu\delta^6 - \tfrac{3229}{322560}\mu\delta^8 + \ldots)\,u_{\frac{1}{2}}\\
h^3D^3u_{\frac{1}{2}} &= (\delta^3 - \tfrac{1}{8}\delta^5 + \tfrac{37}{1920}\delta^7 + \ldots)\,u_{\frac{1}{2}}\\
h^4D^4u_{\frac{1}{2}} &= (\mu\delta^4 - \tfrac{7}{24}\mu\delta^6 + \tfrac{47}{640}\mu\delta^8 - \ldots)\,u_{\frac{1}{2}}\\
h^5D^5u_{\frac{1}{2}} &= (\delta^5 - \tfrac{5}{24}\delta^7 + \ldots)\,u_{\frac{1}{2}}\\
h^6D^6u_{\frac{1}{2}} &= (\mu\delta^6 - \tfrac{3}{8}\mu\delta^8 + \ldots)\,u_{\frac{1}{2}}\\
h^7D^7u_{\frac{1}{2}} &= (\delta^7 - \ldots)\,u_{\frac{1}{2}}\\
h^8D^8u_{\frac{1}{2}} &= (\mu\delta^8 - \ldots)\,u_{\frac{1}{2}}\\
\vdots \quad & \quad \vdots
\end{aligned}\right\}. \tag{75}$$

13. *Accuracy of preceding Results.*—In applying the formulæ in the last section to any actual table of u, it must be borne in mind that the table does not give the exact value of u, but only its value to the nearest multiple of some particular unit. If this unit is ρ, the difference between the tabulated value of u and the true value may be any quantity between $-\tfrac{1}{2}\rho$ and $+\tfrac{1}{2}\rho$. Denote this difference, or "error," by a, so that the tabulated values of $u_0, u_1, \ldots$ are $u_0+a_0, u_1+a_1, \ldots$.

[redacted]

Then, if we denote the greatest possible error in $\delta^n u$ by

$$\pm\,\theta_n,$$

it is easily seen that $$\theta_n = 2^{n-1}\rho. \tag{76}$$

By substitution from (53), we have

$$\begin{aligned}
(p-q\delta^2+r\delta^4-s\delta^6+\dots)\,u_0
&= pu_0-q\,(u_1-2u_0+u_{-1})+r\,(u_2-4u_1+6u_0+4u_{-1}-u_{-2})\\
&\qquad -s\,(u_3-6u_2+15u_1-20u_0+15u_{-1}-6u_{-2}+u_{-3})+\dots\\
&= (p+2q+6r+20s+\dots)\,u_0\\
&\qquad -(q+4r+15s+\dots)(u_1+u_{-1})\\
&\qquad\quad +(r+\;6s+\dots)(u_2+u_{-2})\\
&\qquad\qquad -(s+\dots)(u_3+u_{-3})\\
&\qquad\qquad +\&c.
\end{aligned}$$

Hence, *if $p, q, r, s, \dots$ are all of the same sign*, we have

$$\begin{aligned}
&\text{greatest possible error in } (p-q\delta^2+r\delta^4-s\delta^6+\dots)\,u\\
&= (p+2q+6r+20s+\dots)\,\tfrac{1}{2}\rho\\
&\qquad +(q+4r+15s+\dots)\,\rho\\
&\qquad\quad +(r+6s+\dots)\,\rho\\
&\qquad\qquad +(s+\dots)\,\rho\\
&\qquad\qquad +\&c.\\
&= p\,\tfrac{1}{2}\rho+q\,(1+2+1)\,\tfrac{1}{2}\rho+r\,(1+4+6+4+1)\,\tfrac{1}{2}\rho\\
&\qquad\qquad +s\,(1+6+15+20+15+6+1)\,\tfrac{1}{2}\rho+\dots\\
&= p\,\tfrac{1}{2}\rho+q\,2\rho+r\,2^3\rho+s\,2^5\rho+\dots\\
&= p\theta_0+q\theta_2+r\theta_4+s\theta_6+\dots\,.
\end{aligned} \tag{77}$$

Similarly, it may be shown that, on the same condition,

$$\begin{aligned}
&\text{greatest possible error in } (p\delta-q\delta^3+r\delta^5-s\delta^7+\dots)\,u\\
&\qquad = p\theta_1+q\theta_3+r\theta_5+s\theta_7+\dots\,.
\end{aligned} \tag{78}$$

If, for instance, we were finding h^2D^2u by the second formula in (74), and differences after the fifth were negligible, the greatest possible error would be

$$(2+\tfrac{1}{12}\,.\,2^3)\,\rho = \tfrac{8}{3}\rho.$$

If we were finding it by the ordinary method, which gives

$$h^2D^2u = (\Delta - \tfrac{1}{2}\Delta^2 + \tfrac{1}{3}\Delta^3 - \dots)^2\, u$$
$$= (\Delta^2 - \Delta^3 + \tfrac{11}{12}\Delta^4 - \tfrac{5}{6}\Delta^5 + \dots)\, u,$$

the greatest possible error, even if we did not go beyond fifth differences, would be

$$(2 + 2^2 + \tfrac{11}{12} \cdot 2^3 + \tfrac{5}{6} \cdot 2^4)\, \rho = \tfrac{80}{3}\rho,$$

which is ten times as great. And, as a matter of fact, we should usually have to go to differences of a higher order with this latter formula than with the central-difference formula.

Again, let us denote the greatest possible error in $\mu\delta^n u$ by

$$\pm \phi_n.$$

Then we have

$$\delta^{2m}u_0 = u_m - 2mu_{m-1} + \dots + (-)^{m-1}\frac{(2m)!}{(m-1)!\,(m+1)!}\,u_1$$
$$+ (-)^m \frac{(2m)!}{m!\,m!}\,u_0 + \dots + u_{-m},$$

$$\delta^{2m}u_1 = u_{m+1} - 2mu_m + \dots + (-)^m \frac{(2m)!}{m!\,m!}\,u_1$$
$$+ (-)^{m+1}\frac{(2m)!}{(m+1)!\,(m-1)!}\,u_0 + \dots + u_{-m+1},$$

whence, by addition,

$$\mu\delta^{2m}u_{\frac{1}{2}} = \tfrac{1}{2}\left\{u_{m+1} - A_1u_m + A_2u_{m-1} - \dots + (-)^m A_m u_1\right.$$
$$\left. + (-)^m A_m u_0 + \dots + A_2 u_{-m+2} - A_1 u_{-m+1} + u_{-m}\right\},$$

where $A_1, A_2, \dots, A_m$ are coefficients whose values need not be written down. We have therefore

$$\phi_{2m} = \tfrac{1}{2}\,(1 + A_1 + A_2 + \dots + A_m + A_m + \dots + A_2 + A_1 + 1)\,\tfrac{1}{2}\rho.$$

But, observing how $A_1, A_2, \dots, A_m$ are obtained, we see that

$$1 + A_1 + A_2 + \dots + A_m + A_m + \dots + A_2 + A_1 + 1$$
$$= 1 + 2m + \dots + \frac{(2m)!}{m!\,m!} - \frac{(2m)!}{(m+1)!\,(m-1)!} - \dots - 1$$
$$-1 - \dots - \frac{(2m)!}{(m-1)!\,(m+1)!} + \frac{(2m)!}{m!\,m!} + \dots + 2m + 1$$
$$= 2\,\frac{(2m)!}{m!\,m!},$$

and therefore

$$\phi_{2m} = \frac{(2m)!}{m!\,m!}\,\tfrac{1}{2}\rho. \tag{79}$$

[redacted]

In the same way it may be shown that

$$\phi_{2m+1} = \frac{(2m+1)!}{m!\,(m+1)!}\,\tfrac{1}{2}\rho. \tag{80}$$

Now

$$(p\mu - q\mu\delta^2 + r\mu\delta^4 - \dots)\,u_{\frac{1}{2}} = \tfrac{1}{2}p\,(u_1+u_0) - \tfrac{1}{2}q\,(u_2-u_1-u_0+u_{-1})$$
$$+\tfrac{1}{2}r\,(u_3-3u_2+2u_1+2u_0-3u_{-1}+u_{-2}) - \dots,$$

and, if we collect coefficients, we shall find that, as before, the signs attached to each u are either all positive or all negative. Hence, if $p, q, r, s, \dots$ are all of the same sign,

greatest possible error in $(p\mu - q\mu\delta^2 + r\mu\delta^4 - s\mu\delta^6 + \dots)\,u$

$$= p\phi_0 + q\phi_2 + r\phi_4 + s\phi_6 + \dots \tag{81}$$

$$= \left(p + q\frac{2!}{1!\,1!} + r\frac{4!}{2!\,2!} + s\frac{6!}{3!\,3!} + \dots\right)\tfrac{1}{2}\rho, \tag{81A}$$

and, similarly,

greatest possible error in $(p\mu\delta - q\mu\delta^3 + r\mu\delta^5 - s\mu\delta^7 + \dots)\,u$

$$= p\phi_1 + q\phi_3 + r\phi_5 + s\phi_7 + \dots \tag{82}$$

$$= \left(p + q\frac{3!}{1!\,2!} + r\frac{5!}{2!\,3!} + s\frac{7!}{3!\,4!} + \dots\right)\tfrac{1}{2}\rho. \tag{82A}$$

Thus, in finding hDu_0 by the first formula in (74), the greatest possible error, if we do not go beyond sixth differences, is

$$(1 + \tfrac{1}{6}\,.\,3 + \tfrac{1}{30}\,.\,10)\,\tfrac{1}{2}\rho = \tfrac{11}{12}\rho,$$

while, if we used the ordinary formula

$$hDu = (\Delta - \tfrac{1}{2}\Delta^2 + \tfrac{1}{3}\Delta^3 - \tfrac{1}{4}\Delta^4 + \tfrac{1}{5}\Delta^5 - \tfrac{1}{6}\Delta^6 + \dots)\,u,$$

the greatest possible error, to the same order of differences, would be

$$(1 + \tfrac{1}{2}\,.\,2 + \tfrac{1}{3}\,.\,2^2 + \tfrac{1}{4}\,.\,2^3 + \tfrac{1}{5}\,.\,2^4 + \tfrac{1}{6}\,.\,2^5)\,\rho = \tfrac{208}{15}\rho,$$

which is more than fifteen times as great as the former.

14. *Sub-differences in terms of Differences.*—Suppose that, starting in both directions from x_0, we take x by intervals of

$$nh$$

instead of by intervals of h; where n has any value, integral or not.

Let the new values of μ, δ, be denoted by

$$\mu_n,\ \delta_n.$$

Then we have
$$\left.\begin{aligned}\mu_n &= \cosh \tfrac{1}{2}nhD \\ \delta_n &= 2\sinh \tfrac{1}{2}nhD\end{aligned}\right\}, \tag{83}$$

from which we notice that
$$\left.\begin{aligned}\mu_{-n} &= \mu_n \\ \delta_{-n} &= -\delta_n\end{aligned}\right\}, \tag{84}$$

$$\left.\begin{aligned}\mu_{m\pm n} &= \mu_m\mu_n \pm \tfrac{1}{4}\delta_m\delta_n \\ \delta_{m\pm n} &= \mu_n\delta_m \pm \mu_m\delta_n\end{aligned}\right\}. \tag{85}$$

From (83), by (25)-(28),

$$\mu_n = 1 + \frac{n^2}{2.4}\delta^2 + \frac{n^2(n^2-2^2)}{2.4.6.8}\delta^4 + \frac{n^2(n^2-2^2)(n^2-4^2)}{2.4.6.8.10.12}\delta^6 + \dots \tag{86}$$

$$\mu_n = \mu + \frac{n^2-1^2}{2.4}\mu\delta^2 + \frac{(n^2-1^2)(n^2-3^2)}{2.4.6.8}\mu\delta^4 + \frac{(n^2-1^2)(n^2-3^2)(n^2-5^2)}{2.4.6.8.10.12}\mu\delta^6 + \dots, \tag{87}$$

$$\delta_n = n\delta + \frac{n(n^2-1^2)}{4.6}\delta^3 + \frac{n(n^2-1^2)(n^2-3^2)}{4.6.8.10}\delta^5 + \frac{n(n^2-1^2)(n^2-3^2)(n^2-5^2)}{4.6.8.10.12.14}\delta^7 + \dots \tag{88}$$

$$\delta_n = n\mu\delta + \frac{n(n^2-2^2)}{4.6}\mu\delta^3 + \frac{n(n^2-2^2)(n^2-4^2)}{4.6.8.10}\mu\delta^5 + \frac{n(n^2-2^2)(n^2-4^2)(n^2-6^2)}{4.6.8.10.12.14}\mu\delta^7 + \dots \tag{89}$$

Also, by (29) and (31),

$$\delta_n^m = n^m\left\{{}_mB_m\delta^m + {}_mB_{m+2}\delta^{m+2}/(2m+2)(2m+4) + {}_mB_{m+4}\delta^{m+4}/(2m+2)\dots(2m+8) + \dots\right\}, \tag{90}$$

$$\mu_n\delta_n^{m-1} = n^{m-1}\left\{{}_mB_m\mu\delta^{m-1} + {}_mB_{m+2}\mu\delta^{m+1}/2m(2m+2) + {}_mB_{m+4}\mu\delta^{m+3}/2m\dots(2m+6) + \dots\right\}, \tag{91}$$

where ${}_mB_m$, ${}_mB_{m+2}$, ${}_mB_{m+4}$, ... have the values given by (30) and (32).

The important case is that in which

$$n = 1/N,$$

where N is an integer. If, for instance,

$$N = 10, \quad n = \cdot 1,$$

we find

$$\left.\begin{aligned}
\mu_n \delta_n u_0 &= \cdot 1\,(\mu\delta - \cdot 165\mu\delta^3 + \cdot 0329175\mu\delta^5 - \cdot 0070459125\mu\delta^7 + \ldots)\,u_0 \\
\delta_n^2 u_0 &= \cdot 01\,(\delta^2 - \cdot 0825\delta^4 + \cdot 0109725\delta^6 - \cdot 001761478125\delta^8 + \ldots)\,u_0 \\
\mu_n \delta_n^3 u_0 &= \cdot 001\,(\mu\delta^3 - \cdot 2475\mu\delta^5 + \cdot 0575025\mu\delta^7 - \ldots)\,u_0 \\
\delta_n^4 u_0 &= \cdot 0001\,(\delta^4 - \cdot 165\delta^6 + \cdot 02875125\delta^8 - \ldots)\,u_0 \\
u_n \delta_n^5 u_0 &= \cdot 00001\,(\mu\delta^5 - \cdot 33\mu\delta^7 + \ldots)\,u_0 \\
\delta_n^6 u_0 &= \cdot 000001\,(\delta^6 - \cdot 2475\delta^8 + \ldots)\,u_0 \\
\mu_n \delta_n^7 u_0 &= \cdot 0000001\,(\delta^7 - \ldots)\,u_0 \\
\delta_n^8 u_0 &= \cdot 00000001\,(\delta^8 - \ldots)\,u_0 \\
\vdots \quad & \quad \vdots
\end{aligned}\right\} \tag{92}$$

If $N \equiv 1/n$ is an odd integer, it is not usually necessary to make use of $\mu_n \delta_n u_0$, $\mu_n \delta_n^3 u_0$, ... For, if we suppose a table to be constructed by intervals of nh, this table will show the following differences at the middle of the interval between x_0 and x_1:—

x	u	δ_n	δ_n^2	δ_n^3	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$x_{\frac{1}{2}(1-n)}$	$u_{\frac{1}{2}(1-n)}$		$\delta_n^2 u_{\frac{1}{2}(1-n)}$		
		$\delta_n u_{\frac{1}{2}}$		$\delta_n^3 u_{\frac{1}{2}}$	...
$x_{\frac{1}{2}(1+n)}$	$u_{\frac{1}{2}(1+n)}$		$\delta_n^2 u_{\frac{1}{2}(1+n)}$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Any purpose, therefore, for which we might require $\mu_n \delta_n u_0$, $\mu_n \delta_n^3 u_0$, ..., can be quite as well served by $\delta_n u_{\frac{1}{2}}$, $\delta_n^3 u_{\frac{1}{2}}$, ... Suppose, for instance, that

$$N = 5, \quad n = \cdot 2.$$

Then

$$\left.\begin{aligned} \delta_n^2 u_0 &= \cdot 04\,(\delta^2 - \cdot 08\delta^4 + \cdot 01056\delta^6 - \cdot 0016896\delta^8 + \dots)\,u_0 \\ \delta_n^4 u_0 &= \cdot 0016\,(\delta^4 - \cdot 16\delta^6 + \cdot 02752\delta^8 - \,..)\,u_0 \\ \delta_n^6 u_0 &= \cdot 000064\,(\delta^6 - \cdot 24\delta^8 + \dots)\,u_0 \\ \delta_n^8 u_0 &= \cdot 00000256\,(\delta^8 - \dots)\,u_0 \\ \vdots \qquad & \qquad \vdots \end{aligned}\right\}, \tag{93}$$

and

$$\left.\begin{aligned} \delta_n u_{\frac{1}{2}} &= \cdot 2\,(\delta - \cdot 04\delta^3 + \cdot 00448\delta^5 - \cdot 0006656\delta^7 + \dots)\,u_{\frac{1}{2}} \\ \delta_n^3 u_{\frac{1}{2}} &= \cdot 008\,(\delta^3 - \cdot 12\delta^5 + \cdot 01824\delta^7 - \dots)\,u_{\frac{1}{2}} \\ \delta_n^5 u_{\frac{1}{2}} &= \cdot 00032\,(\delta^5 - \cdot 2\delta^7 + \dots)\,u_{\frac{1}{2}} \\ \delta_n^7 u_{\frac{1}{2}} &= .\,0000128\,(\delta^7 - \dots)\,u_{\frac{1}{2}} \\ \vdots \qquad & \qquad \vdots \end{aligned}\right\}. \tag{94}$$

15. *Interpolation at regular intervals.*— We have, by definition.

$$\tfrac{1}{2}\{f(x+nh)+f(x-nh)\} = \mu_{2n} f(x),$$

$$\tfrac{1}{2}\{f(x+nh)-f(x-nh)\} = \tfrac{1}{2}\delta_{2n} f(x);$$

and therefore $$f(x+nh) = (\mu_{2n} + \tfrac{1}{2}\delta_{2n}) f(x). \tag{95}$$

Suppose now that, the values of u being tabulated by intervals h, we require to form a new table in which the intervals are

$$h/N,$$

where N is an integer. We can do this either by interpolating on each side of the given values $x_0, x_1, x_2, \dots$, or by interpolating through the intervals x_0 to x_1, x_1 to x_2, ... For the first method we have

$$u_n = \mu_{2n} u_0 + \tfrac{1}{2}\delta_{2n} u_0, \tag{96}$$

where $$n = \pm p/N,$$

and p has values ranging from 0 to $\frac{1}{2}(N+1)$ or $\frac{1}{2}N$; and for the second method

$$u_{\frac{1}{2}+n} = \mu_{2n} u_{\frac{1}{2}} + \tfrac{1}{2}\delta_{2n} u_{\frac{1}{2}}, \tag{97}$$

where $$n = \pm(2p+1)/2N \text{ or } \pm p/N,$$

according as N is odd or even, and p has values ranging from 0 to $\frac{1}{2}(N-1)$ or $\frac{1}{2}N$. By (86)-(89), these give respectively

$$u_n = u_0 + \frac{n}{1!}\mu\delta u_0 + \frac{n^2}{2!}\delta^2 u_0 + \frac{n(n^2-1^2)}{3!}\mu\delta^3 u_0 + \frac{n^2(n^2-1^2)}{4!}\delta^4 u_0 + \dots, \tag{98}$$

and

$$u_{\frac{1}{2}+n} = \mu u_{\frac{1}{2}} + \frac{2n}{2.1!}\delta u_{\frac{1}{2}} + \frac{4n^2-1^2}{2^2.2!}\mu\delta^2 u_{\frac{1}{2}} + \frac{2n\,(4n^2-1^2)}{2^3.3!}\delta^3 u_{\frac{1}{2}}$$
$$+\frac{(4n^2-1^2)(4n^2-3^2)}{2^4.4!}\mu\delta^4 u_{\frac{1}{2}} + \dots \quad (99)$$

The usual subdivision is into fifths or tenths. For subdivision into fifths, we can write $n = \mp\frac{3}{10}$ and $n = \mp\frac{1}{10}$ in the latter formula, and we have

$$\left.\begin{matrix} u_{\frac{1}{5}} \\ u_{\frac{4}{5}} \end{matrix}\right\} = \mu u_{\frac{1}{2}} - \cdot 08\mu\delta^2 u_{\frac{1}{2}} + \cdot 0144\mu\delta^4 u_{\frac{1}{2}} - \cdot 0029568\mu\delta^6 u_{\frac{1}{2}} + \cdot 000642048\mu\delta^8 u_{\frac{1}{2}} - \dots$$
$$\mp\left\{\cdot 3\delta u_{\frac{1}{2}} - \cdot 008\delta^3 u_{\frac{1}{2}} + \cdot 000864\delta^5 u_{\frac{1}{2}} - \cdot 00012672\delta^7 u_{\frac{1}{2}} + \dots\right\}$$

$$\left.\begin{matrix} u_{\frac{2}{5}} \\ u_{\frac{3}{5}} \end{matrix}\right\} = \mu u_{\frac{1}{2}} - \cdot 12\mu\delta^2 u_{\frac{1}{2}} + \cdot 0224\mu\delta^4 u_{\frac{1}{2}} - \cdot 0046952\mu\delta^6 u_{\frac{1}{2}} + \cdot 001018368\mu\delta^8 u_{\frac{1}{2}} - \dots$$
$$\mp\left\{\cdot 1\delta u_{\frac{1}{2}} - \cdot 004\delta^3 u_{\frac{1}{2}} + \cdot 000448\delta^5 u_{\frac{1}{2}} - \cdot 00006656\delta^7 u_{\frac{1}{2}} + \dots\right\}, \quad (100)$$

the upper sign being taken for $u_{\frac{1}{5}}$ and $u_{\frac{2}{5}}$, and the lower for $u_{\frac{4}{5}}$ and $u_{\frac{3}{5}}$. For subdivision into tenths we might obtain similar formulæ, but it is simpler to bisect the intervals by means of (63), and then apply (100).

16. *Interpolation generally.*—The preceding formulæ involve the calculation of $\mu\delta u_0$, $\mu\delta^3 u_0$, ..., or of $\mu u_{\frac{1}{2}}$, $\mu\delta^2 u_{\frac{1}{2}}$, ..., and therefore they are not convenient when we only require u for a single value of x. Let x_0 be the nearest value of x given in the table. Then we require a formula giving u in terms of

$$u_0,\ \delta^2 u_0,\ \delta^4 u_0,\ \dots,$$

and either $\qquad \delta u_{-\frac{1}{2}},\ \delta^3 u_{-\frac{1}{2}},\ \delta^5 u_{-\frac{1}{2}},\ \dots$

or $\qquad \delta u_{\frac{1}{2}},\ \delta^3 u_{\frac{1}{2}},\ \delta^5 u_{\frac{1}{2}},\ \dots,$

according as x is between $x_0 - \frac{1}{2}h$ and x_0, or between x_0 and $x_0 + \frac{1}{2}h$.

Suppose that x is between x_0 and $x_0 + \frac{1}{2}h$, so that we may write

$$x = x_0 + nh,$$

where n is between 0 and $\frac{1}{2}$. Then we require to express

$$u_n$$

in the form $\qquad f_1(\delta^2)\,u_0 + f_2(\delta^2)\,\delta u_{\frac{1}{2}};$

i.e., we require to express $\quad \mu_{2n}+\frac{1}{2}\delta_{2n}$

in the form $\quad f_1(\delta^2)+f_2(\delta^2 \quad \mu+\frac{1}{2}\delta)\,\delta.$

Now

$$\mu_{2n}+\tfrac{1}{2}\delta_{2n} = \cosh nhD+\sinh nhD$$

$$=\left\{\cosh(n-\tfrac{1}{2})hD+\sinh nhD(\cosh\tfrac{1}{2}hD+\sinh\tfrac{1}{2}hD)\right\}/\cosh\tfrac{1}{2}hD,$$

and therefore

$$u_n = \cosh(n-\tfrac{1}{2})hD/\cosh\tfrac{1}{2}hD\,.\,u_0+\sinh nhD/\cosh\tfrac{1}{2}hD\,.\,u_{\frac{1}{2}}.$$

Substituting from (28) and (27),

$$u_n = \left\{1+\frac{n(n-1)}{2!}\delta^2+\frac{(n+1)\,n\,(n-1)(n-2)}{4!}\delta^4+\dots\right\}u_0$$

$$+\left\{\frac{n}{1!}\delta+\frac{(n+1)n(n-1)}{3!}\delta^3+\frac{(n+2)(n+1)n(n-1)(n-2)}{5!}\delta^5+\dots\right\}u_{\frac{1}{2}}. \tag{101}$$

Similarly, when x is between $x_0-\frac{1}{2}h$ and x_0, we have

$$u_{-n} = \left\{1+\frac{n(n-1)}{2!}\delta^2+\frac{(n+1)\,n\,(n-1)(n-2)}{4!}\delta^4+\dots\right\}u_0$$

$$-\left\{\frac{n}{1!}\delta+\frac{(n+1)n(n-1)}{3!}\delta^3+\frac{(n+2)(n+1)n(n-1)(n-2)}{5!}\delta^5+\dots\right\}u_{-\frac{1}{2}}. \tag{102}$$

The coefficients in (101) or (102) are appreciably smaller than the coefficients in the ordinary interpolation formula.

Part III.—*Central-Sum Formulæ.*

Central-Sum Notation.

17. The justification of the use of δ as an operator lies in the fact that we can regard any column of differences, such as δu or $\delta^2 u$, as a series of quantities which are the same function of successive values of x, increasing by differences h, and that we can then regard the following columns as giving the first, second, ... differences of these functions. In the same way we can regard the values of u as being themselves differences of some other function, so that the

successive values of this function are found by successive additions of the values of u. Denoting the function by

$$\sigma u \equiv \sigma f(x),$$

we have, for our definition of the symbol σ,

$$\sigma f(x+\tfrac{1}{2}h)-\sigma f(x-\tfrac{1}{2}h)=f(x), \tag{103}$$

which may also be written

$$\delta\sigma f(x)=f(x). \tag{103A}$$

Changing x in (103) into $x-h$, $x-2h$, ..., and adding the results so obtained, we have

$$\sigma f(x+\tfrac{1}{2}h) = \ldots+f(x-h)+f(x)$$

and, similarly,

$$\left.\begin{aligned}\sigma f(x+\tfrac{3}{2}h) &= \ldots+f(x-h)+f(x)+f(x+h)\\ \sigma f(x+\tfrac{5}{2}h) &= \ldots+f(x-h)+f(x)+f(x+h)+f(x+2h)\\ \vdots\quad &\qquad\qquad \vdots\end{aligned}\right\}. \tag{104}$$

Since the differences of $\phi(x)$ are the same as those of $C+\phi(x)$, where C is any quantity independent of x, the solution of (103), considered as a functional equation in $\sigma f(x)$, contains an arbitrary constant. But, if the series on the right-hand side of any one of the equations (104) is made to start from any particular value of $f(x)$, and has any constant added, the other series will start from the same value, and will contain the same constant.

It is clear from (104) that, provided the constants are properly chosen,

$$\sigma\{\phi(x)+\psi(x)\} = \sigma\phi(x)+\sigma\psi(x),$$

so that σ, regarded as an operator, follows the laws of algebra. Also, replacing $f(x)$ by u, we have, from (103A),

$$\delta\sigma u = u,$$

while, if we apply (104) to $\delta f(x)$ instead of to $f(x)$, we have, if the constants are properly chosen,

$$\begin{aligned}\sigma\delta f(x) &= \ldots+\delta f(x-h)+\delta f(x)\\ &= f(x),\end{aligned}$$

or

$$\sigma\delta u = u.$$

Thus σ combines with δ according to the laws of algebra, in the same way as if

$$\sigma = \delta^{-1}. \tag{105}$$

The process represented by σ may be repeated, so that we shall have

$$\left.\begin{aligned} \sigma^2 u_0 &= \sigma . \sigma u_0 \\ &= \sigma\left(\ldots + u_{-\frac{3}{2}} + u_{-\frac{1}{2}}\right) \\ &= \ldots + \sigma u_{-\frac{3}{2}} + \sigma u_{-\frac{1}{2}} \\ \sigma^2 u_1 &= \ldots + \sigma u_{-\frac{3}{2}} + \sigma u_{-\frac{1}{2}} + \sigma u_{\frac{1}{2}} \\ \vdots \quad & \quad \vdots \end{aligned}\right\}, \tag{106}$$

and so on. Each repetition of the operation involves the introduction of an arbitrary constant, so that $\sigma^m u$ will contain m of these constants.

Again, by our original definition of μ,

$$\mu\sigma u_0 = \tfrac{1}{2}\left(\sigma u_{\frac{1}{2}} + \sigma u_{-\frac{1}{2}}\right).$$

Hence, if the arbitrary constant in σu is such that

$$\left.\begin{aligned} \sigma u_{-\frac{1}{2}} &= C + u_{-2} + u_{-1} \\ \sigma u_{\frac{1}{2}} &= C + u_{-2} + u_{-1} + u_0 \\ \vdots \quad & \quad \vdots \\ \sigma u_{n+\frac{1}{2}} &= C + u_{-2} + u_{-1} + u_0 + \ldots + u_n \end{aligned}\right\}, \tag{107}$$

we have $$\mu\sigma u_0 = C + u_{-2} + u_{-1} + \tfrac{1}{2}u_0. \tag{108}$$

Also $$\begin{aligned} \sigma\mu u_0 &= \ldots + \tfrac{1}{2}(u_{-3} + u_{-2}) + \tfrac{1}{2}(u_{-2} + u_{-1}) + \tfrac{1}{2}(u_{-1} + u_0) \\ &= \ldots + u_{-2} + u_{-1} + \tfrac{1}{2}u_0. \end{aligned} \tag{109}$$

If therefore the arbitrary constant is properly chosen, we have

$$\mu\sigma u = \sigma\mu u.$$

Thus the operators μ, δ, σ can be combined according to the ordinary laws of algebra, provided attention is given to the arbitrary constants introduced by σ; and it is clear that the same applies to D, since the differential coefficient of a sum is the same as the sum of the differential coefficients.

It should be noticed that, in all cases,

$$\sigma(u_{n+\frac{1}{2}}-u_{-\frac{1}{2}}) = \sigma u_{n+\frac{1}{2}}-\sigma u_{-\frac{1}{2}}$$
$$= u_0+u_1+u_2+\ldots+u_{n-1}+u_n, \tag{110}$$
$$\mu\sigma(u_n-u_0) = \mu\sigma u_n-\mu\sigma u_0$$
$$= \tfrac{1}{2}u_0+u_1+u_2+\ldots+u_{n-1}+\tfrac{1}{2}u_n. \tag{111}$$

Integration- and Summation-Formulæ (Differential Coefficients).

18. By (65), and (1),

$$u_1-u_0 = \delta u_{\frac{1}{2}}$$
$$= 2\sinh\tfrac{1}{2}hD\,.\,u_{\frac{1}{2}}$$
$$= (1+\tfrac{1}{24}h^2D^2+\tfrac{1}{1920}h^4D^4+\tfrac{1}{322560}h^6D^6$$
$$+\tfrac{1}{92897280}h^8D^8+\ldots)\,hDu_{\frac{1}{2}}, \tag{112}$$

and, by (64) and (65), and (3),

$$u_1-u_0 = \mu\,.\,\delta/\mu\,.\,u_{\frac{1}{2}}$$
$$= \mu\,.\,2\tanh\tfrac{1}{2}hD\,.\,u_{\frac{1}{2}}$$
$$= \mu\,(1-\tfrac{1}{12}h^2D^2+\tfrac{1}{120}h^4D^4-\tfrac{17}{20160}h^6D^6$$
$$+\tfrac{31}{362880}h^8D^8-\ldots)\,hDu_{\frac{1}{2}}. \tag{113}$$

Hence, if
$$u = \int^x v\,dx, \tag{114}$$

we have
$$\int_{x_0}^{x_1} v\,dx = (1+\tfrac{1}{24}h^2D^2+\tfrac{1}{1920}h^4D^4+\ldots)\,hv_{\frac{1}{2}}, \tag{115}$$

$$\int_{x_0}^{x_1} v\,dx = \mu\,(1-\tfrac{1}{12}h^2D^2+\tfrac{1}{120}h^4D^4-\ldots)\,hv_{\frac{1}{2}}. \tag{116}$$

By means of either of these formulæ we can build up a table of values of

$$\int^x v\,dx,$$

provided that v and its differential coefficients can be calculated. Of the two formulæ, the first requires the values of v, D^2v, ... for the intermediate values of x, while the second requires them for the values ... x_0, x_1, ... entered in the table. The first formula converges

more rapidly than the second, but either of them is much better than the ordinary formula

$$\int_{x_0}^{x_1} v\,dx = \left(\frac{1}{1!}+\frac{1}{2!}hD+\frac{1}{3!}h^2D^2+\dots\right)hv_0.$$

19. To adapt the preceding formulæ for purposes of summation, the portion containing differential coefficients must in each case be expressed as the difference of functions of x_0 and x_1. Thus we have

$$\begin{aligned}\delta u_{\frac12} &= hDu_{\frac12}+(\delta-hD)\,u_{\frac12}\\ &= hDu_{\frac12}+\delta\,.\,(\delta-hD)/\delta\,.\,u_{\frac12}\\ &= hDu_{\frac12}+\delta\,(1-\tfrac12 hD\operatorname{cosech}\tfrac12 hD)\,u_{\frac12},\end{aligned}$$

and

$$\begin{aligned}\delta u_{\frac12} &= \mu hDu_{\frac12}+\delta\,.\,(\delta-\mu hD)/\delta\,.\,u_{\frac12}\\ &= \mu hDu_{\frac12}+\delta\,(1-\tfrac12 hD\coth\tfrac12 hD)\,u_{\frac12}.\end{aligned}$$

Hence, by (5) and (4),

$$\delta u_{\frac12} = hDu_{\frac12}\;+\delta U_{\frac12}, \tag{117}$$

$$\delta u_{\frac12} = \mu hDu_{\frac12}+\delta U'_{\frac12}, \tag{118}$$

where

$$U = \left(\tfrac{1}{24}hD-\tfrac{7}{5760}h^3D^3+\tfrac{31}{967680}h^5D^5-\tfrac{127}{154828800}h^7D^7+\dots\right)hDu, \tag{119}$$

$$U' = \left(-\tfrac{1}{12}hD+\tfrac{1}{720}h^3D^3-\tfrac{1}{30240}h^5D^5+\tfrac{1}{1209600}h^7D^7-\dots\right)hDu. \tag{120}$$

Substituting from (114), and adding for n consecutive intervals,

$$\int_{x_0}^{x_n} v\,dx = h\sigma\,(v_n-v_0)+h\,(V_n-V_0), \tag{121}$$

$$\int_{x_0}^{x_n} v\,dx = h\mu\sigma\,(v_n-v_0)+h\,(V'_n-V'_0), \tag{122}$$

where V and V' are what U and U' become by writing v for hDu.

By transposing the terms in (121) and (122), we have

$$\begin{aligned}v_{\frac12}+v_{\frac32}+\dots+v_{n-\frac12} &= \sigma\,(v_n-v_0)\\ &= h^{-1}\int_{x_0}^{x_n} v\,dx-\left[\left(\tfrac{1}{24}hD-\tfrac{7}{5760}h^3D^3+\tfrac{31}{967680}h^5D^5-\dots\right)v\right]_{x=x_0}^{x=x_n},\end{aligned} \tag{123}$$

$$\begin{aligned}\tfrac12 v_0+v_1+v_2+\dots+v_{n-1}+\tfrac12 v_n &= \mu\sigma\,(v_n-v_0)\\ &= h^{-1}\int_{x_0}^{x_n} v\,dx+\left[\left(\tfrac{1}{12}hD-\tfrac{1}{720}h^3D^3+\tfrac{1}{30240}h^5D^5-\dots\right)v\right]_{x=x_0}^{x=x_n}.\end{aligned} \tag{124}$$

The second of these is the Euler-Maclaurin formula for the sum of n consecutive values of a function, and first is an alternative form which is sometimes useful.

20. The preceding formulæ may be expressed in other ways. If, for instance, it is more convenient to use the even than the odd differential coefficients of v, we have

$$\delta u_{\frac{1}{2}} = hDu_{\frac{1}{2}} + \delta^2 . (\delta - hD)/\delta^2 . u_{\frac{1}{2}}$$
$$= hDu_{\frac{1}{2}} + \delta^2 \left(\tfrac{1}{2} \operatorname{cosech} \tfrac{1}{2}hD - \tfrac{1}{4}hD \operatorname{cosech}^2 \tfrac{1}{2}hD\right) u_{\frac{1}{2}},$$

and, similarly,

$$\delta u_{\frac{1}{2}} = \mu hDu_{\frac{1}{2}} + \delta^2 \left(\tfrac{1}{2} \operatorname{cosech} \tfrac{1}{2}hD - \tfrac{1}{4}hD \operatorname{cosech} \tfrac{1}{2}hD \coth \tfrac{1}{2}hD\right) u_{\frac{1}{2}}.$$

Substituting from (5), (8), and (9),

$$\delta u_{\frac{1}{2}} = hDu_{\frac{1}{2}} + \delta^2 W_{\frac{1}{2}}, \tag{125}$$

$$\delta u_{\frac{1}{2}} = \mu hDu_{\frac{1}{2}} + \delta^2 W'_{\frac{1}{2}}, \tag{126}$$

where

$$W = \left(\tfrac{1}{24} - \tfrac{17}{5760}h^2D^2 + \tfrac{43}{322560}h^4D^4 - \tfrac{769}{154828800}h^6D^6 + \dots\right) hDu, \tag{127}$$

$$W' = \left(-\tfrac{1}{12} + \tfrac{7}{1440}h^2D^2 - \tfrac{31}{161280}h^4D^4 + \tfrac{127}{19353600}h^6D^6 - \dots\right) hDu; \tag{128}$$

and therefore

$$\int_{x_0}^{x_n} v\,dx = h\sigma\,(v_n - v_0) + h\,(\delta X_n - \delta X_0), \tag{129}$$

$$\int_{x_0}^{x_n} v\,dx = h\mu\sigma\,(v_n - v_0) + h\,(\delta X'_n - \delta X'_0), \tag{130}$$

where X and X' are what W and W' become by writing v for hDu in (127) and (128).

21. Again, by (65) and (15),

$$\delta^2 u = 4 \sinh^2 \tfrac{1}{2}hD . u$$
$$= \left(1 + \tfrac{1}{12}h^2D^2 + \tfrac{1}{360}h^4D^4 + \tfrac{1}{20160}h^6D^6 + \tfrac{1}{1814400}h^8D^8 + \dots\right) h^2D^2u, \tag{131}$$

and therefore, if
$$\frac{d^2u}{dx^2} = w, \tag{132}$$

so that
$$u = \int^x \int^x w\,(dx)^2, \tag{132A}$$

we have $\quad u = \sigma^2 \delta^2 u$

$$= \sigma^2 h^2 \left(w + \tfrac{1}{12} h^2 D^2 w + \tfrac{1}{360} h^4 D^4 w + \ldots\right). \qquad (133)$$

The arbitrary constants in σ^2 must be adjusted according to the circumstances of the case. Thus, if we know the values of u_0 and u_1, the expression following σ^2 in (133) has to be calculated for $x = x_1$, x_2, ..., and the values of u_2, u_3, ... are found by successive additions.

To adapt the formula for summation, we have

$$\delta^2 u = h^2 D^2 u + \delta^2 \left(1 - \tfrac{1}{4} h^2 D^2 \operatorname{cosech}^2 \tfrac{1}{2} hD\right) u,$$

and therefore, by (8),

$$\delta^2 u = h^2 D^2 u + \delta^2 \left(\tfrac{1}{12} - \tfrac{1}{240} h^2 D^2 + \tfrac{1}{6048} h^4 D^4 - \tfrac{1}{172800} h^6 D^6 + \ldots\right) h^2 D^2 u. \qquad (134)$$

Substituting from (132A), and performing the successive additions up to $x = x_p$, we find

$$\int_{x_0}^{x_p} \int^x w\,(dx)^2 = A + pB + h^2 \left\{p w_0 + (p-1) w_1 + (p-2) w_2 + \ldots + w_{p-1}\right\} + h^2 \left\{Z_p - Z_0\right\},$$

where A and B are constants, and

$$Z = \left(\tfrac{1}{12} - \tfrac{1}{240} h^2 D^2 + \tfrac{1}{6048} h^4 D^4 - \tfrac{1}{172800} h^6 D^6 + \ldots\right) w. \qquad (135)$$

Taking $p = 0$ and $p = 1$ in the above relation, we find, C being an arbitrary value of x,

$$\int_{x_0}^{x_p} \int_C^x w\,(dx)^2 = p \int_{x_0}^{x_1} \int_C^x w\,(dx)^2 + h^2 \left\{(p-1) w_1 + (p-2) w_2 + \ldots + w_{p-1}\right\}$$
$$+ h^2 \left\{Z_p - p Z_1 + (p-1) Z_0\right\}. \qquad (136)$$

The above methods might be extended to any number of successive summations. But the resulting formulæ are not of practical value.

Integration-Formulæ (Differences).

22. In order to apply (115) or (116) to the tabulation of

$$\int^x v\,dx$$

by intervals of h in x, we have to calculate the successive values not only of hv, but also of $h^3 D^2 v$, $h^5 D^4 v$, ... The calculation is avoided by

expressing the differential coefficients in terms of the differences. We have then only to calculate the values of hv.

By (47),

$$\begin{aligned}\delta u_{\frac{1}{2}} &= 2\sinh\tfrac{1}{2}hD\,.\,u_{\frac{1}{2}} \\ &= \left(1+\tfrac{1}{24}\delta^2-\tfrac{17}{5760}\delta^4+\tfrac{367}{967680}\delta^6-\tfrac{27859}{464486400}\delta^8+\ldots\right)hDu_{\frac{1}{2}} \qquad (137) \\ &= \left\{1+\tfrac{1}{24}\delta^2-\left(\tfrac{1}{360}+\tfrac{1}{16}\cdot\tfrac{1}{360}\right)\delta^4+\cdot 000\ 379\ 258\ \ldots\ \delta^6 \right. \\ &\qquad \left. -\cdot 000\ 059\ 978\ \ldots\ \delta^8+\ldots\right\}hDu_{\frac{1}{2}},\end{aligned}$$

and similarly, by (49),

$$\begin{aligned}\delta u_{\frac{1}{2}} &= \mu\left(1-\tfrac{1}{12}\delta^2+\tfrac{11}{720}\delta^4-\tfrac{191}{60480}\delta^6+\tfrac{2497}{3628800}\delta^8-\ldots\right)hDu_{\frac{1}{2}} \qquad (138) \\ &= \mu\left\{1-\tfrac{1}{12}\delta^2+\left(\tfrac{1}{60}-\tfrac{1}{12}\cdot\tfrac{1}{60}\right)\delta^4-\cdot 003\ 158\ 069\ \ldots\ \delta^6 \right. \\ &\qquad \left. +\cdot 000\ 688\ 106\ \ldots\ \delta^8-\ldots\right\}hDu_{\frac{1}{2}}.\end{aligned}$$

Hence

$$\int_{x_0}^{x_1} v\,dx = \left(1+\tfrac{1}{24}\delta^2-\tfrac{17}{5760}\delta^4+\ldots\right)hv_{\frac{1}{2}}, \qquad (139)$$

$$\int_{x_0}^{x_1} v\,dx = \mu\left(1-\tfrac{1}{12}\delta^2+\tfrac{11}{720}\delta^4-\ldots\right)hv_{\frac{1}{2}}. \qquad (140)$$

By taking the successive intervals x_0 to x_1, x_1 to x_2, ..., we get the differences in the table of

$$u \equiv \int^x v\,dx,$$

and by successive additions of these differences we get the values of u.

If the values of hv are accurate within $\pm\frac{1}{2}\rho$, and if the expression given in (139) is taken to the same number of places as hv, a further error of $\pm\frac{1}{2}\rho$ may be introduced by the portion

$$\left(\tfrac{1}{24}\delta^2-\tfrac{17}{5760}\delta^4+\ldots\right)hv,$$

so that after p additions, if the initial value of u is correct within $\frac{1}{2}\rho$, there may be a total error of $\pm(p+\frac{1}{2})\,\rho$. On account of the accumulation of errors, a table formed by the above method must always be checked at intervals by direct calculation of the integral. The amount of the accumulated error may be kept down by retaining one or two extra figures in the differences of the integral, so that the error in u after p additions will lie within the limits $\pm(p+2)\,\frac{1}{2}\rho$. But the retention of these extra figures is avoided, and the values of u at intervals are more easily checked, by expressing the sums of the

differences δ^2, δ^4, ... in (139) in terms of δ, δ^3, ... Thus by n successive additions of (139) we find*

$$u_n = \int^{x_n} v\,dx = \left(\sigma + \tfrac{1}{24}\delta - \tfrac{17}{5760}\delta^3 + \tfrac{367}{967680}\delta^5 - \tfrac{27859}{464486400}\delta^7 + \dots\right) hv_n, \tag{141}$$

and similarly, from (140),

$$u_n = \int^{x_n} v\,dx = \mu\left(\sigma - \tfrac{1}{12}\delta + \tfrac{11}{720}\delta^3 - \tfrac{191}{60480}\delta^5 + \tfrac{2497}{3628800}\delta^7 - \dots\right) hv_n. \tag{142}$$

The arbitrary constant in σ or $\mu\sigma$ in either formula depends on the lower limit of integration; but, if we take σhv to be such that

$$u_0 = \left(\sigma + \tfrac{1}{24}\delta - \tfrac{17}{5760}\delta^3 + \dots\right) hv_0, \tag{143}$$

i.e., if we tabulate the values of hv, and then take σhv so that

$$\sigma hv_0 = u_0 - \tfrac{1}{24}\delta hv_0 + \tfrac{17}{5760}\delta^3 hv_0 - \dots, \tag{143A}$$

then we shall have always

$$u_n = \left(\sigma + \tfrac{1}{24}\delta - \tfrac{17}{5760}\delta^3 + \dots\right) hv_n. \tag{143B}$$

Similarly, if we take $\mu\sigma hv$ so that

$$u_0 = \mu\left(\sigma - \tfrac{1}{12}\delta + \tfrac{11}{720}\delta^3 - \dots\right) hv_0, \tag{144}$$

i.e., if we take

$$\sigma hv_{\frac{1}{2}} = u_0 + \tfrac{1}{2}hv_0 + \tfrac{1}{12}\mu\delta hv_0 - \tfrac{11}{720}\mu\,\delta^3 hv_0 + \dots, \tag{144A}$$

then we shall have always

$$u_n = \mu\left(\sigma - \tfrac{1}{12}\delta + \tfrac{11}{720}\delta^3 - \dots\right) hv_n. \tag{144B}$$

Suppose that σhv_0 is fixed by (143A), its value being taken to the number of decimal places to which we require u_n. Then for checking the table after an interval ph, we ought to have

$$\sigma hv_p = u_p - \tfrac{1}{24}\delta\, hv_p + \tfrac{17}{5760}\delta^3\, hv_p - \dots .$$

If $\sigma hv_0 + hv_{\frac{1}{2}} + hv_{\frac{3}{2}} + \dots + hv_{p-\frac{1}{2}}$ does not agree with this value of σhv_p, one or more of the values of hv must be altered, by inspection

* This modification of the formula is taken from Hansen. I do not know to whom the original formula (137) is due, but it was used by Legendre in calculating his table of elliptic functions.

of the sequence of differences. But the coefficients of δhv, $\delta^3 hv$, ... in (143) are so small that it is not usually necessary to go through the labour of making the corresponding alterations in the differences of hv. The same remark applies to the use of formula (144A).

23. In the last section we have only considered the tabulation of u at intervals of h in x, by means either of (141) or of (142); the values of hv being calculated for the tabulated values of x in the case of the latter, and for the intermediate values of x in the case of the former. From the table so formed we could construct a table at intervals of $\frac{1}{2}h$ by taking the differences of u, and applying (63). But it should be noticed that the arbitrary constants in (141) and (142) are the same, so that by taking hv at intervals of h we get u at intervals of $\frac{1}{2}h$ with very little trouble. For (141) may be written

$$u_n = (1+\tfrac{1}{24}\delta^2-\tfrac{17}{5760}\delta^4+\dots)\,\sigma h v_n.$$

Taking this with

$$u_{n+1} = (1+\tfrac{1}{24}\delta^2-\tfrac{17}{5760}\delta^4+\dots)\,\sigma h v_{n+1},$$

we have

$$\begin{aligned} u_{n+\frac{1}{2}} &= \mu^{-1}(1+\tfrac{1}{24}\delta^2-\tfrac{17}{5760}\delta^4+\dots)\,\mu\sigma h v_{n+\frac{1}{2}} \\ &= (1-\tfrac{1}{12}\delta^2+\tfrac{11}{720}\delta^4-\dots)\,\mu\sigma h v_{n+\frac{1}{2}} \\ &= (\mu\sigma-\tfrac{1}{12}\mu\delta+\tfrac{11}{720}\mu\delta^3-\dots)\,h v_{n+\frac{1}{2}} \\ &= \mu\,(\sigma-\tfrac{1}{12}\delta+\tfrac{11}{720}\delta^3-\dots)\,h v_{n+\frac{1}{2}}, \end{aligned}$$

the constant in σ being unaltered, by (108), and the coefficients in the series just obtained being the same as the coefficients in (142), since they are obtained by the development of

$$\mu^{-1}.\,2\sinh\tfrac{1}{2}hD = 2\tanh\tfrac{1}{2}hD.$$

In applying this method, it is simplest to calculate first the values of

$$\tfrac{1}{12}\delta hv,\quad \tfrac{11}{720}\delta^3 hv,\quad \dots,$$

for the tabulated values of x, and thence write down the values of

$$a \equiv (\sigma-\tfrac{1}{12}\delta+\tfrac{11}{720}\delta^3+\dots)\,hv,$$

keeping in one or two extra figures. The intermediate values of u

are then written down first, by taking the arithmetic means of successive values of a; and the terms in

$$(\sigma+\tfrac{1}{24}\delta-\tfrac{17}{5760}\delta^3+\dots)\,hv$$

are found from those already used, by the relations

$$\tfrac{1}{24}=\tfrac{1}{2} \text{ of } \tfrac{1}{12},$$

$$\tfrac{17}{5760}=\tfrac{17}{88} \text{ of } \tfrac{11}{720}=\cdot 193\dot{1}\dot{8} \text{ of } \tfrac{11}{720},$$

&c.

24. Again, by (48),

$$\begin{aligned}\delta^2 u &= 4\sinh^2 \tfrac{1}{2}hD.u\\ &= (1+\tfrac{1}{12}\delta^2-\tfrac{1}{240}\delta^4+\tfrac{31}{60480}\delta^6-\tfrac{289}{3628800}\delta^8+\dots)\,h^2D^2u \qquad (145)\\ &= (1+\tfrac{1}{12}\delta^2-\tfrac{1}{240}\delta^4+\cdot 000\ 512\ 566\dots\delta^6\\ &\qquad -\cdot 000\ 079\ 641\dots\delta^8+\dots)\,h^2D^2u.\end{aligned}$$

Hence, if the second differential coefficients of u are known, we can tabulate the values of u by a double summation. Thus, taking

$$u=\int^x\!\!\int^x w\,(dx)^2,$$

we find

$$u_n=\int^{x_n}\!\!\int^x w\,(dx)^2=(\sigma^2+\tfrac{1}{12}-\tfrac{1}{240}\delta^2+\tfrac{31}{60480}\delta^4-\tfrac{289}{3628800}\delta^6+\dots)\,h^2w_n. \tag{146}$$

To determine the two arbitrary constants in σ^2, we usually know the initial values u_0 and u_1. We must then fix the value of $\sigma h^2 w_{\frac{1}{2}}$ so that

$$u_1-u_0=\delta u_{\frac{1}{2}}=(\sigma+\tfrac{1}{12}\delta-\tfrac{1}{240}\delta^3+\tfrac{31}{60480}\delta^5-\dots)\,h^2w_{\frac{1}{2}}, \tag{147}$$

and we shall have always

$$\delta u_{n+\frac{1}{2}}=(\sigma+\tfrac{1}{12}\delta-\tfrac{1}{240}\delta^3+\tfrac{31}{60480}\delta^5-\dots)\,h^2w_{n+\frac{1}{2}}. \tag{147A}$$

Then, choosing $\sigma^2 h^2 w_0$ so that

$$u_0=(\sigma^2+\tfrac{1}{12}-\tfrac{1}{240}\delta^2+\tfrac{31}{60480}\delta^4-\dots)\,h^2w_0, \tag{148}$$

we shall have always

$$u_n=(\sigma^2+\tfrac{1}{12}-\tfrac{1}{240}\delta^2+\tfrac{31}{60480}\delta^4-\dots)\,h^2w_n. \tag{148A}$$

... to be tested. Calling this w, we

$$u_p = u_0 + p\delta u_{\frac{1}{2}} + (p-1)\, h^2 w_1 + (p-2)\, h$$

whence

$$\delta u_{\frac{1}{2}} = \{u_p - u_0 - (p-1)\, h^2 w_1 - (p-2)\, h^2 w_2 -$$

To check the table at intervals ph, when u_0, also either $u_1, u_{p+1}, \ldots$ or $Du_0, Du_p, \ldots$, we ca two ways. We may first check the table of pendently, altering values of $h^2 w$ where nec their differences; and then check the table values of $\sigma h^2 w$. Or we may perform the taneously, by altering values of $h^2 w$; the effec values at different points being given by (15 $u_0, u_p, u_{2p}, \ldots$, we can of course only use (151).

If we wish to use the table of $\sigma^2 h^2 w$ to const of u by intervals of $\frac{1}{2}h$ in x, we have, by (50),

$$u_{n+\frac{1}{2}} = \mu\,(\sigma^2 - \tfrac{1}{24} + \tfrac{17}{1920}\delta^2 - \tfrac{367}{193536}\delta^4 + \tfrac{27859}{663552}$$

$$= \mu\,\{\sigma^2 - \tfrac{1}{24} + (\tfrac{1}{120} + \tfrac{1}{16}\cdot\tfrac{1}{120})\,\delta^2 - \cdot 001\ 8$$

$$+ \cdot 000\ 419$$

25. The above methods can be extended to a tions. In practice, if we wish to make more generally arrange to make an ...

$$\left.\begin{aligned} u^4 &= \left(\sigma^4+\tfrac{1}{6}\sigma^2-\tfrac{1}{720}+\tfrac{1}{30240}\delta^2-\tfrac{41}{7257600}\delta^4+\ldots\right)h^4D^4u \\ hDu &= \mu\left(\sigma^3+0\,.\,\sigma+\tfrac{1}{240}\delta-\tfrac{31}{302400}\delta^3+\tfrac{289}{12096000}\delta^5-\ldots\right)h^4D^4u \end{aligned}\right\}, \tag{154}$$

$$\left.\begin{aligned} u &= \left(\sigma^6+\tfrac{1}{4}\sigma^4+\tfrac{1}{120}\sigma^2+\tfrac{1}{30240}-\tfrac{1}{57600}\delta^2+\ldots\right)h^6D^6u \\ hDu &= \mu\left(\sigma^5+\tfrac{1}{12}\sigma^3+0\,.\,\sigma-\tfrac{1}{6048}\delta+\tfrac{41}{7257600}\delta^3-\ldots\right)h^6D^6u \end{aligned}\right\}, \tag{155}$$

$$\left.\begin{aligned} u &= \left(\sigma^8+\tfrac{1}{3}\sigma^6+\tfrac{1}{40}\sigma^4+\tfrac{1}{50400}\sigma^2-\tfrac{1}{1209600}+\ldots\right)h^8D^8u \\ hDu &= \mu\left(\sigma^7+\tfrac{1}{6}\sigma^5+\tfrac{1}{360}\sigma^3+0\,.\,\sigma+\tfrac{1}{172800}\delta-\ldots\right)h^8D^8u \end{aligned}\right\}. \tag{156}$$

In using these formulæ, it is assumed that each column representing summation can be checked by its own guiding values; otherwise the small errors in the values of $h^{2n}D^{2n}u$ might accumulate very rapidly. The formulæ to be used for checking the several columns are different according as the initial sums are determined by the accurate calculation of a few successive values of u, or by calculation of u_0 and its differential coefficients. For the former case it is easily shown that the effect of successive summations is to multiply the initial sums, and also the added values of $h^{2n}D^{2n}u$, by the coefficients appearing in the expansion of different powers of

$$(1-x)^{-1}.$$

Thus, denoting by U the quantity $h^{2n}D^{2n}u$, whose values for successive values of x are calculated and tabulated, and supposing that we perform m summations, we have

$$\begin{aligned} \sigma^m U_{\frac{1}{2}m+p} = {} & \Big\{\sigma^m U_{\frac{1}{2}m}+\frac{p}{1!}\sigma^{m-1}U_{\frac{1}{2}m-\frac{1}{2}}+\frac{p\,(p+1)}{2!}\sigma^{m-2}U_{\frac{1}{2}m-1}+\ldots \\ & \qquad\qquad \ldots+\frac{p\,(p+1)\ldots(p+m-2)}{(m-1)!}\sigma U_{\frac{1}{2}}\Big\} \\ & +\Big\{\frac{p\,(p+1)\ldots(p+m-2)}{(m-1)!}U_1+\frac{(p-1)\,p\ldots(p+m-3)}{(m-1)!}U_2+\ldots \\ & \qquad\qquad \ldots+\frac{1\,.\,2\ldots(m-1)}{(m-1)!}U_p\Big\}, \end{aligned} \tag{157}$$

the second term in which may also be written

$$\left\{U_p+\frac{m}{1!}U_{p-1}+\frac{m\,(m+1)}{2!}U_{p-2}+\ldots+\frac{m\,(m+1)\ldots(m+p-2)}{(p-1)!}U_1\right\}. \tag{157A}$$

The coefficients in (157) for values of m from 1 to 8, and for values of p from 1 to 20, are given by the following table, the coefficients

being read horizontally from the value of p, and then upwards to the value of m:—

p	m = 1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7	8
3	1	3	6	10	15	21	28	36
4	1	4	10	20	35	56	84	120
5	1	5	15	35	70	126	210	330
6	1	6	21	56	126	252	462	792
7	1	7	28	84	210	462	924	1716
8	1	8	36	120	330	792	1716	3432
9	1	9	45	165	495	1287	3003	6435
10	1	10	55	220	715	2002	5005	11440
11	1	11	66	286	1001	3003	8008	19448
12	1	12	78	364	1365	4368	12376	31824
13	1	13	91	455	1820	6188	18564	50388
14	1	14	105	560	2380	8568	27132	77520
15	1	15	120	680	3060	11628	38760	116280
16	1	16	136	816	3876	15504	54264	170544
17	1	17	153	969	4845	20349	74613	245157
18	1	18	171	1140	5985	26334	100947	346104
19	1	19	190	1330	7315	33649	134596	480700
20	1	20	210	1540	8855	42504	177100	657800

. . (158)

If the initial sums, instead of being $\sigma U_{\frac{1}{2}}$, $\sigma^2 U_1$, $\sigma^3 U_{\frac{3}{2}}$, ..., are $\sigma U_{-\frac{1}{2}}$, $\sigma^2 U_0$, $\sigma^3 U_{\frac{1}{2}}$, ..., these being determined by the formulæ (153)–(156), the coefficients are practically given by (101), and are got from the table (158) by going alternately horizontally and diagonally upwards until we arrive at the coefficient of $\sigma U_{\frac{1}{2}}$, and then going vertically upwards.

26. The various formulæ given in the preceding sections may be applied to the calculation of integrals involving two or more variables. The only important cases, in practice, are those of

$$u = \int_0^x \int_0^y f(x, y)\, dx\, dy,$$

and

$$u = \int_0^x \int_0^y \int_0^z f(x, y, z)\, dx\, dy\, dz.$$

The method is the same in both cases, and it will be sufficient to

consider the former. Suppose that we require our double table, giving u for different values of x and y, to proceed by intervals of h in x, and of k in y, the values of x being

$$\ldots, x_0, x_1, \ldots,$$

and those of y being $\ldots, y_0, y_1, \ldots.$

In order to use (141) we should calculate the values of

$$hkw_{p,q} = hkf(x_p, y_q)$$

for each of the intermediate values

$$\ldots, x_{-\frac{1}{2}}, x_{\frac{1}{2}}, x_{\frac{3}{2}}, \ldots$$

of x, taken with each of the intermediate values

$$\ldots, y_{-\frac{1}{2}}, y_{\frac{1}{2}}, y_{\frac{3}{2}}, \ldots$$

of y, and arrange them in a double table, thus :—

y	x				
	…	$x_{-\frac{1}{2}}$	$x_{\frac{1}{2}}$	$x_{\frac{3}{2}}$	…
⋮		⋮	⋮	⋮	
$y_{-\frac{1}{2}}$	…	$hkw_{-\frac{1}{2}, -\frac{1}{2}}$	$hkw_{\frac{1}{2}, -\frac{1}{2}}$	$hkw_{\frac{3}{2}, -\frac{1}{2}}$	…
$y_{\frac{1}{2}}$	…	$hkw_{-\frac{1}{2}, \frac{1}{2}}$	$hkw_{\frac{1}{2}, \frac{1}{2}}$	$hkw_{\frac{3}{2}, \frac{1}{2}}$	…
$y_{\frac{3}{2}}$	…	$hkw_{-\frac{1}{2}, \frac{3}{2}}$	$hkw_{\frac{1}{2}, \frac{3}{2}}$	$hkw_{\frac{3}{2}, \frac{3}{2}}$	…
⋮	…	⋮	⋮	⋮	

Applying (141) to the column marked

$$x_{\frac{1}{2}},$$

we get the successive values of

$$h\int^y f(x_{\frac{1}{2}}, y)\, dx\, dy.$$

This must be done for each column, and the results arranged in a double table in which the values of x are, as before,

$$\ldots, x_{-\frac{1}{2}}, x_{\frac{1}{2}}, x_{\frac{3}{2}}, \ldots,$$

those of y being $\ldots, y_0, y_1, \ldots.$

Then, repeating the process on each row, we finally get the value of

$$\int^{x}\int^{y} f(x, y)\,dx\,dy$$

for each of the values $\ldots, x_0, x_1, \ldots$

of x, taken with each of the values

$$\ldots, y_0, y_1, \ldots$$

of y. Or we can use (142) in both cases, or (141) in one case and (142) in the other, with the necessary modifications.

APPENDIX (*a*).

(Session 1899–1900.)

It should have been stated on p. 283 that the date of Mr. S. O. Roberts' death was May 31st, 1899. A fuller account of his life-work is given in *The Taylorian*, Vol. XXI., No. 6, July, 1899.

Major-General Frederick Close, whose death took place on November 23rd, 1899, at his residence, Cranleigh, Shanklin, Isle of Wight, was son of the late Captain Close, of the Royal Artillery. He was born April 15th, 1830, and received his first commission in the Royal Artillery in December, 1847. During the earlier portion of his career he was employed in purely military work. About the year 1865 he passed the "advance class" (afterwards called the "Artillery College"), and subsequently received an appointment as Assistant Superintendent of the Royal Carriage Department of the Woolwich Arsenal. He then held the post of Professor of Artillery, from November, 1873, to March, 1875, at the Royal Military Academy; from that date until the end of March, 1880, he was Superintendent of the Royal Small Arms Factory at Enfield. Finally, from 1881 to 1886, he was Superintendent of the Royal Carriage Department at Woolwich.

In 1880–1881 General Close was a member of the Ordnance Committee. He was placed on half-pay in 1888, and retired from the service in 1890.

His mathematical work was mainly connected with artillery matters, and he carried out many experiments on traction, the strength of materials, &c., the results of which appear in much of the service equipment. After his retirement he took little interest in mathematics and wrote nothing. The only pamphlet we have come across is one on "The Constrained Motion of Conical Wheels" (*Proceedings*, *Royal Artillery Institution*, No. 7, Vol. VIII.).

He was elected a member of the Society April 13th, 1871.*

The following lists of Corrigenda have been received :—

VOL. XXX.

Page 81, line 11, *for* D^k_{ij}, *read* D^{kl}_{ij}.
,, 83, bottom, formula (1), *for* $j+$, *read* $j+2$.
,, 92, line 9, *for* λa_{ji}, *read* λa_{j1}.
,, 94, line 7, *for* β^j_i, *read* β_{ij}.
,, 98, line 11, *for* δ, *read* δ_{j1}.
,, 203, line 2, *for* ξ_k, *read* ξ_{kl}.
,, 204, line 15, *for the first* $a^{i}_{k_{q-1}}{}^{q-1}_{l_{q-1}}$, *read* $a^{i}_{k_1}{}^{q-1}_{l_1}$.
,, 256, line 8 from foot, *for* C^q, *read* Q^q.
,, 309, line 15, *for* $\kappa_1 \mathrm{K}\mu$, *read* $\kappa_1^2 \mathrm{K}\mu$.
,, 312, equation (18), *for* $\frac{R^2}{2n+1}$, *read* $\frac{R^2}{2n+3}$.
,, 317, equation (34), *for* $\frac{R^2}{3}$, *read* $\frac{R^2}{5}$.
,, 317, equation (34), *omit* $(1+\frac{2}{45}\kappa^2R^2)$.
,, 317, line 15, *omit* $+\frac{2}{45}\kappa^2R^2$.
,, 318, equations (42) and (43), *for* $\frac{12}{15}$, *read* $\frac{11}{15}$.

VOL. XXXI.

,, 31, line 27, *for* ξ'_1, *read* ξ'_i.
,, 32, line 21, *read* $\xi_1^{p^n+1}+\xi_2^{p^n+1}$.
,, 32, line 26, *for* exponent p, *read* p^n.
,, 33, formula for S^{-1}, *for* ξ, *read* ξ_{2j}.
,, 35, formula (5), *for* a_r, *read* a_{jr}.

---

* General Close has left six sons, all in the Engineers or Artillery. We are indebted for personal information to Lieut. A. J. Close, R.E.

Page 36, formula (6), *for* a_s, *read* a_{js}.
,, 37, line 25, *read* $a^{2m} \mid \lambda_U \mid$.
,, 39, line 9, *for* A, *read* A_1.
,, 47, line 3, *for* a, *read* a_{22}.
,, 50, matrix (23), third row, *for* γ_{12}, *read* γ_{12}^2.
,, 57, line 12, *delete* "(26) becomes S'' of case (1)," and *insert* "the transformed of (26) by $Q_{2,1,1}$ will have $a_{11} \neq 0$."

The theorem at the top of p. 40 may be completed to include the case m odd. In this case, it can be shown that $q = 1$.

INDEX.

INDEX OF PAPERS, &c.

[*The mark * indicates that the paper either has been printed* in extenso, *or an abstract of it given, in the "Proceedings."*]

GENERAL INDEX.

Printed by C. F. Hodgson & Son, 2 Newton Street, High Holborn, W.C.

VOLUMES ALREADY ISSUED.

Vol.	I.	(from January 1865 to November 1866)	10*s.*
,,	II.	(from November 1866 to November 1869)	16*s.*
,,	III.	(from November 1869 to November 1871)	20*s.*
,,	IV.	(from November 1871 to November 1873)	31*s.* 6*d.*
,,	V.	(from November 1873 to November 1874)	15*s.*
,,	VI.	(from November 1874 to November 1875)	21*s.*
,,	VII.	(from November 1875 to November 1876)	21*s.*
,,	VIII.	(from November 1876 to November 1877)	21*s.*
,,	IX.	(from November 1877 to November 1878)	21*s.*
,,	X.	(from November 1878 to November 1879)	18*s.*
,,	XI.	(from November 1879 to November 1880)	12*s.* 6*d.*
,,	XII.	(from November 1880 to November 1881)	16*s.*
,,	XIII.	(from November 1881 to November 1882)	18*s.*
,,	XIV.	(from November 1882 to November 1883)	20*s.*
,,	XV.	(from November 1883 to November 1884)	20*s.*
,,	XVI.	(from November 1884 to November 1885)	20*s.*
,,	XVII.	(from November 1885 to November 1886)	25*s.*
,,	XVIII.	(from November 1886 to November 1887)	25*s.*
,,	XIX.	(from November 1887 to November 1888)	30*s.*
,,	XX.	(from November 1888 to November 1889)	25*s.*
,,	XXI.	(from November 1889 to November 1890)	25*s.*
,,	XXII.	(from November 1890 to November 1891)	27*s.* 6*d.*
,,	XXIII.	(from November 1891 to November 1892)	20*s.*
,,	XXIV.	(from November 1892 to November 1893)	22*s.*
,,	XXV.	(from November 1893 to November 1894)	22*s.*
,,	XXVI.	(from November 1894 to November 1895)	30*s.*
,,	XXVII.	(from November 1895 to November 1896)	35*s.*
,,	XXVIII.	(from November 1896 to November 1897)	32*s.* 6*d.*
,,	XXIX.	(from November 1897 to November 1898) Part I.	20*s.*
		Part II.	20*s.*
,,	XXX.	(from November 1898 to March 1899)	20*s.*
,,	XXXI.	(from April 1899 to December 1899)	20*s.*

[redacted]

[redacted]